Note di Matematica Vol. X - n. 2, 363-370 (1990)

NATURALLY REDUCTIVE S-MANIFOLDS

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Abstract. *We study naturally reductive Riemannian S-manifolds when the endomorphism field S has no eigenvalue -1. We prove that in dimension four all of them are Kählerian. Further, we determine for general dimensions all such manifolds of constant holomorphic sectional curvature.*

1. INTRODUCTION

Naturally reductive Riemannian S-manifolds have been introduced in [8] as natural generalizations of naturally reductive s-regular manifolds. This last class of manifolds generalizes symmetric spaces. Also all nearly Kähler manifolds are naturally reductive S-manifolds and these last manifolds share a lot of properties with the nearly Kähler spaces, in particular when -1 is not an eigenvalue of the endomorphism field S. (In this case the naturally reductive Riemannian S-manifolds are almost Hermitian manifolds.) For example, in [9] we proved a Schur-like theorem similar to the one for nearly Kähler manifolds proved in [10]. See [8] for more examples.

In this paper we continue our exploration of these proprieties. More specifically, we determine the four-dimensional ones and prove, just as for the nearly Kähler manifolds, that they are all Kähler manifolds. Further, we prove that the ones of constant holomorphic sectional curvature are precisely the nearly Kähler manifolds with the same property.

2. PRELIMINARIES

Let (M, g) be a connected, smooth, n-dimensional Riemannian manifold with Levi-Civita connection ∇ and Riemann curvature tensor R defined by

$$R_{XY} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

where $X, Y \in \mathscr{X}(M)$, the Lie algbra of smooth vector fields on M.

A *Riemannian S-manifolds* (M, g, S) is a Riemannian manifold (M, g) together with a (1, 1)-tensor field S such that g and ∇S are S-invariant, that is,

$$g(SX, SY) = g(X, Y), \qquad (\nabla_{SX}S)SY = S(\nabla_X S)Y$$

for all $X, Y \in \mathscr{X}(M)$, and $I - S$ is non-singular [8].

Locally s-regular manifolds (M, g, s) are endowed with a so-called s-structure and they are S-manifolds where $S_m = s_{m*}|M_m$ for all $m \in M$ (see [1], [7] for more details). Further, a quasi-Kähler manifold is an almost Hermitian manifold (M, g, J) such that

$$(\nabla_X J)Y + (\nabla_{JX} J)JY = 0,$$

$X, Y \in \mathscr{X}(M)$, and hence, any quasi-Kähler manifold is a Riemannian $\left(-\frac{1}{2}I + \frac{\sqrt{3}}{2}J\right)$-manifold. Moreover, it is clear that any element of the subclass of Kähler manifolds ($\nabla J = 0$) is an S-manifold for

$$S = I\cos\theta + J\sin\theta \tag{1}$$

where θ is an arbitrary non-zero constant.

Motivated by the study of locally s-regular manifolds, on any Riemannian S-manifold (M, g, S) we consider the tensor field of type (1, 2) defined by

$$D(X,Y) = D_X Y = (\nabla_{(I-S)^{-1}X} S)S^{-1}Y$$

for all $X, Y \in \mathscr{X}(M)$. With this tensor field we define a connection $\overline{\nabla}$ by

$$\overline{\nabla}_X Y = \nabla_X Y - D_X Y,$$

$X, Y \in \mathscr{X}(M)$. It follows easily that $\overline{\nabla}$ is a metric connection, or equivalently,

$$g(D_X Y, Z) + g(Y, D_X Z) = 0$$

for all $X, Y, Z \in \mathscr{X}(M)$. Moreover, $\overline{\nabla} S = 0$. Hence, the eigenvalues of S, regarded as a field of orthogonal endomorphisms, are constant. These eigenvalues have the form

$$e^{\pm i\theta_1} = c_1 \pm i s_1, \ldots, e^{\pm i\theta_r} = c_r \pm i s_r$$

where $0 < \theta_1, \ldots, \theta_r < \pi$, together with -1 as the only possible real eigenvalue.

Assumption. In the rest of this note we assume that -1 does not occur as an eigenvalue for any (M, g, S) under consideration.

In this case, associated with $\theta_1, \ldots, \theta_r$, smooth distribution $\mathscr{D}_i$ are defined by

$$\mathscr{D}_i = \ker(S^2 - 2c_i S + I).$$

Hence, any $X \in \mathscr{X}(M)$ has a unique decomposition into a sum of *distribution vector fields*, that is, $X = \sum_{i=1}^{r} X_i$ where $X_i \in \mathscr{D}_i, i = 1, \ldots, r$. Further, we define smooth *projection tensor fields* I_i by $I_i X = X_i$ and finally, an *almost complex structure* J is defined by

$$J = \sum_{i=1}^{r} \frac{1}{s_i}(S_i - c_i I_i),$$

where $S_i X = S X_i$ and then, g is *almost Hermitian.* We call this J *the canonical almost complex structure.*

Now, we recall a useful result for an (M, g, S). It is a consequence of the S-invariance of ∇S (see, [8, Lemma 2.5]).

Proposition 1. *For any i, j, k either*

a) $I_i D_{Y_k} X_j = 0$ for all X_j, Y_k, or

b) $\cos\theta_k = \cos(\theta_i + \alpha_{ijk}\theta_j)$ where the only possibilities are

1) $\alpha_{ijk} = 1$ if $\theta_i + \theta_j + \theta_k = 2\pi$ or $\theta_k = \theta_i + \theta_j$, and

2) $\alpha_{ijk} = -1$ if $\theta_j = \theta_k + \theta_i$ or $\theta_i = \theta_k + \theta_j$.

In case b) we have

$$I_i(J D_{Y_k} X_j + \alpha_{ijk} D_{Y_k} J X_j) = 0$$

for all X_j, Y_k.

Motivated by the study of naturally reductive homogeneous spaces (see for example [12]) and of nearly Kähler manifolds [3], we now turn to a special class of Riemannian S-manifolds. A Riemannian S-manifold is said to be *naturally reductive* if the tensor field T defined by

$$T(X, Y, Z) = g(D_X Y, Z),$$

$X, Y, Z \in \mathscr{X}(M)$, is skew-symmetric. Since $\overline{\nabla}$ is metric, this is equivalent to

$$D_X X = 0,$$

for all $X \in \mathscr{X}(M)$.

It is clear that any Kähler manifold is a naturally reductive S-manifold with S given by (1). Further, nearly Kähler manifolds are almost Hermitian manifolds (M, g, J) such that

$$(\nabla_X J)X = 0$$

for all $X \in \mathscr{X}(M)$. (See, for example, [3].) Since such manifolds are automatically quasi-Kählerian, it is easy to see that they are all naturally reductive $\left(-\frac{1}{2}I + \frac{\sqrt{3}}{2}J\right)$-manifolds.

Of course, all naturally reductive locally s-regular manifolds, in particular naturally reductive k-symmetric spaces, provide examples. Some interesting examples are given in [5]. There it is proved that the homogeneous space $SU(n)/T, n > 2$, where T is a maximal torus, admits the structure of a naturally reductive almost Hermitian n-symmetric space which is not m-symmetric for each $m < n$. The author also gives other almost Hermitian examples for $SO(n)/T, n > 6$ and for $Sp(n)/T, n > 1$. Finaly, his results are strengthened further in [6] by considering G/T where G is any connected compact semisimple Lie group.

For the sake of brevity we give the following

Definition. *A naturally reductive Riemannian S-manifold without eigenvalue -1 for S will be called an* NRS-manifold *in the rest of this paper.*

3. NRS-MANIFOLDS OF CONSTANT HOLOMORPHIC SECTIONAL CURVATURE

Since on an NRS-manifold we have an almost Hermitian structure, we consider now the NRS-manifolds of constant holomorphic sectional curvature. Just as for the real and complex space forms and for the nearly Kähler manifolds, their determination relies on a Schur-like theorem. We start by recalling this result, proved in [9].

Let $\mathscr{D}_i$ be an eigenspace distribution on (M, g, S). We say that $\mathscr{D}_i$ has constant holomorphic sectional curvature K_i if K_i is a smooth function on M such that at each point $p \in M$ the sectional curvature $K_i(P)$ of every J-invariant two-plane P at p contained in $\mathscr{D}_i$ takes the value $K_i(p)$. Then we have from [9]

Proposition 2. *Let (M, g, S) be an* NRS-*manifold and suppose $\mathscr{D}_i$ is an eigenspace distribution of dimension > 2 which has constant holomorphic sectional curvature K_i with respect to the canonical almost complex structure. Then K_i is constant on M.*

From this we get at once

Corollary 3. *Let (M, g, S) be an* NRS-*manifold with canonical almost complex structure J and suppose that at least one eigenspace distribution has dimension > 2. If (M, g, J) has constant holomorphic sectional curvature $K(p)$ at each point $p \in M$, then K is constant on M.*

Then we have

Theorem 4. *Let (M, g, S, J) be an* NRS-*space of constant holomorphic sectional curvature with respect to the canonical almost complex structure J and with at least one eigenspace distribution of dimension > 2. Then (M, g, S, J) is a nearly Kähler manifold and hence* *locally isometric to a complex space form of constant holomorphic sectional curvature K or to $S^6(K)$ where the last case arises only if S has order 3 (that is, $S^3 = I$).*

Proof. First we have from [9, (2)] and for a general NRS-space:

$$\begin{aligned} R(X_i, Y_j, Z_k, W_l) &= g(R(X_i, Y_j)W_l, Z_k) = 2g(D_{X_i}Y_j, D_{Z_k}W_l) \\ &\quad + g(D_{X_i}W_l, D_{Y_j}Z_k) - g(D_{X_i}Z_k, D_{Y_j}W_l) \end{aligned} \tag{2}$$

if i, j, k, l are not all equal.

Next, when the holomorphic sectional curvature is a constant K, we get from [9, (16)]

$$\begin{aligned} R(X_i, Y_i, Z_i, W_i) = \frac{1}{4}K\{&g(X_i, Z_i)g(Y_i, W_i) - g(X_i, W_i)g(Y_i, Z_i) \\ &+ g(X_i, JZ_i)g(Y_i, JW_i) - g(X_i, JW_i)g(Y_i, JZ_i) \\ &+ 2g(X_i, JY_i)g(Z_i, JW_i)\} \\ &+ 2g(D_{X_i}Y_i, D_{Z_i}W_i) - g(D_{X_i}W_i, D_{Y_i}Z_i) + g(D_{X_i}Z_i, D_{Y_i}W_i). \end{aligned} \tag{3}$$

Hence, by combining (2) and (3), we obtain

$$\begin{aligned} R(X, Y, Z, W) = \frac{1}{4}K\sum_{i=1}^{r}\{&g(X_i, Z_i)g(Y_i, W_i) - g(X_i, W_i)g(Y_i, Z_i) \\ &+ g(X_i, JZ_i)g(Y_i, JW_i) - g(X_i, JW_i)g(Y_i, JZ_i) \\ &+ 2g(X_i, JY_i)g(Z_i, JW_i)\} \\ &+ 2\sum_{i=1}^{r}\{g(D_{X_i}Z_i, D_{Y_i}W_i) - g(D_{X_i}W_i, D_{Y_i}Z_i)\} \\ &+ 2g(D_XY, D_ZW) - g(D_XZ, D_YW) + g(D_XW, D_YZ). \end{aligned} \tag{4}$$

Thus, we get

$$R(X, JX, X, JX) = K(g(X, X))^2 + g(D_XJX, D_XJX), \tag{5}$$

where we have used the relation $D_{X_i}JX_i = 0$ which follows at once from Proposition 1.

The hypothesis and (5) then yield $D_XJX = 0$ for all $X \in \mathscr{X}(M)$. But, since $\overline{\nabla}J = 0$, this is equivalent to $(\nabla_XJ)X = 0$ and so, (M, g, J) is a nearly Kähler manifold. A result of A. Gray [4] then implies that (M, g, J) is locally isometric to a complex space form or to S^6.

To prove the last part we note that it follows from a result in [11] that S^{2n} never admits an S-manifold structure where S has more than one eigenspace distribution. Further, for $n > 2$, only S^6 can have an almost complex structure and this is never Kählerian. So, Proposition 1 then yields $\theta = \frac{2\pi}{3}$ and hence, S has order 3 because $\theta \neq \frac{2\pi}{3}$ leads to $D = 0$, the Kähler case.

Remarks. A. The arguments in the proof imply that S^6 can never admit a naturally reductive S-structure of order $k > 3$.

B. It is still an open problem to decide what spaces we have if we suppose that the NRS-manifold has constant holomorphic sectional curvatures K_i, $1 \leq i \leq r, r > 1$, without the assumption that all K_i are equal.

C. In the case considered above we supposed all K_i equal. The most important step in the proof was to show that the manifold is nearly Kählerian and in fact, once we know that, we can avoid the use of our Schur-like theorem. Instead, we may use the similar theorem for nearly Kähler manifolds proved in [10]. As a consequence we see that *Theorem 4 still holds when we replace the condition on the dimension of at least one eigenspace distribution by the condition* dim $M \geq 4$.

4. FOUR-DIMENSIONAL NRS-MANIFOLDS

In this section we consider four-dimensional NRS-manifolds and prove

Theorem 5. *A four-dimensional NRS-manifold is Kählerian with respect to the canonical almost complex structure.*

Proof. First, we suppose that there is only one eigenspace distribution, that is, $e^{\pm i\theta}$ are the only eigenvalues of S. Then, for $\theta \neq \frac{2\pi}{3}$, Proposition 1 yields $D = 0$ and hence, since $\overline{\nabla} J = \nabla J = 0$, (M, g, J) is Kählerian. For $\theta = \frac{2\pi}{3}$ the same proposition gives

$$J D_X Y + D_X J Y = 0$$

which implies $(\nabla_X J) X = 0$. So (M, g, J) is nearly Kählerian and then, since any four-dimensional nearly Kähler space is a Kähler manifold [2], we get again the required result. In these cases $S = I \cos\theta + J \sin\theta$.

So, we are left with the case of distinct eigenvalues $e^{\pm i\theta_1}, e^{\pm i\theta_2}$ with corresponding vector fields X_1, JX_1, X_2, JX_2 which form a local basis for the distributions $\mathscr{D}_1, \mathscr{D}_2$. Since $D_{X_i} X_i = D_{X_i} J X_i = 0$ for $i = 1, 2$, we get $D_{Y_i} Z_i = 0$ for all $Y_i, Z_i \in \mathscr{D}_i$. Hence, for $i \neq j$,

$$g(D_{Y_i} W_j, Z_i) = -g(W_j, D_{Y_i} Z_i) = 0$$

and

$$g(D_{Y_i}W_j, V_j) = -g(D_{W_j}Y_i, V_j) = g(Y_i, D_{W_j}V_j) = 0.$$

So $D_{Y_i}W_j = 0$. Then we obtain again $D = 0$ and the result follows as before.

(Note that in this last case and since $\nabla S = 0$, the distribution $\mathscr{D}_1, \mathscr{D}_2$ are invariant under parallel transport. Hence, (M, g, S) is locally a product of two 2-dimensional NRS-manifolds $(M, g_1, S_1) \times (M_2, g_2, S_2)$ where S_i is obtained by restricting S to $\mathscr{D}_i$, $i = 1, 2$. We can define J_i similarly and then $S_i = I_i \cos\theta_i + J_i \sin\theta_i$. Clearly J_i determines an orientation on M_i.)

Remarks. A. It is clear that Proposition 1 implies easily that each NRS-manifold with only one eigenspace distribution is either a Kähler manifold or a nearly Kähler space. In the last case $S^3 = I$.

B. As we noted already, Proposition 1 and the definition yield that

$$D_{X_i}X_i = 0, \qquad D_{X_i}JX_i = 0$$

for all $X_i \in \mathscr{D}_i$. Hence, if $\{E_i, i = 1, \ldots, \dim M\}$ is an orthonormal basis of vectors spanning the eigenspace distributions, we see that

$$\sum_i (\nabla_{E_i}J)E_i = 0$$

and so, $\delta\Omega = 0$, where Ω is the associated Kähler form of the NRS-manifold. This implies that *each* NRS*-manifold is a semi-Kähler manifold with respect to the canonical almost complex structure.*

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