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AN ISOMORPHIC CHARACTERIZATION OF PROPERTY (β) OF ROLEWICZ

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Abstract. *In the paper it is shown that in a separable Banach space there is a norm with property (β) of S. Rolewicz if and only if there is a norm which is simultaneously nearly uniformly convex and nearly uniformly smooth.*

The Kuratowski measure of noncompactness of a set A in a Banach space is the infimum $\alpha(A)$ of those $\epsilon > 0$ for which there is a covering of A by a finite number of sets A_i with $\text{diam}(A_i) < \epsilon$.

Let X be a Banach space with closed unit ball B. By the **drop** $D(x, B)$ defined by an element $x \in X \setminus B$, we mean $\text{conv}(\{x\} \cup B)$ and we let $R(x, B) = D(x, B) \setminus B$. Rolewicz [16] has proved that X is uniformly convex if and only if for each $\epsilon > 0$ there is a $\delta > 0$ such that $1 <\|x\|< 1 + \delta$ implies $\text{diam}(R(x, b)) < \epsilon$. In connection with this he has introduced [17] the following property.

A Banach space X is said to have **property** (β) if for each $\epsilon > 0$ there is a $\delta > 0$ such that $1 <\|x\|< 1 + \delta$ implies $\alpha(R(x, B)) < \epsilon$.

The notation of **nearly uniform convexity** (NUC) has been introduced by Huff [4]. Rolewicz [17] has given the following equivalent definition.

A Banach space X is said to be (NUC) if for each $\epsilon > 0$ there is a $\delta, 0 < \delta < 1$, such that the measure of non-compactness of the slice $S(f, \delta) = \{x \in X : \|x\| \leqq 1, f(x) \geqq 1 - \delta\}$ is smaller than ϵ for each continuous linear functional f with $\|f\|= 1$.

A Banach space X is **uniformly Kadec-Klee** (UKK) if for every $\epsilon > 0$ there is a $\delta > 0$ such that $\| x \| \leqq 1 - \delta$ whenever x is a weak limit of some sequence $\{x_n\}$ in B with $\text{sep}(x_n) = \inf\{\|x_n - x_m\|: n \neq m\} > \epsilon$.

Huff [4] has proved that X is NUC if and only if X is reflexive and UKK.

Rolewicz [17] has shown that $UC \Rightarrow (\beta) \Rightarrow NUC$. The class of Banach spaces with an equivalent norm with property (β) coincides neither with that of superreflexive spaces (independently proved by Montesinos and Torregrosa [13] and the author [5]), nor with the class of nearly uniformly convexifiable spaces (cf. [6] and [7]).

An isometric characterization of (β) in terms of «crescents» instead of drops is given in [11].

In [8] and [9] we have defined the notions $k - \beta$, $k \geqq 1$, and $k - NUC$, $k \geqq 2$, where $1 - \beta$ coincides with property (β). All of these properties imply NUC and they are even isomorphically stronger. Moreover, we have shown that Schachermayer's space [18] is an example of a $k - NUC$ space with $k = 8$, which fails to have an equivalent $1 - \beta$ norm (i.e. with property (β)). In [9] we have also given some equivalent formulations of the notations $k - \beta$ and $k - NUC$; in particular, we shall use in the sequel the following characterization of property (β).

Proposition 1. *A Banach space X has the property (β) if and only if for each $\epsilon > 0$ there exists a $\delta, 0 < \delta < 1$, such that for every element $x \in B$ and every sequence $\{x_n\} \subset B$ with* sep $(x_n) > \epsilon$, *there is an index i so that $\|x + x_i\|/2 \leqq 1 - \delta$.*

Sekowski and Stachura [19] and Prus [15] have independently defined the notion of **nearly uniform smoothness** (NUS) (see below). They have proved that a Banach space X (resp. X^*) is NUS if and only if X^* (resp. X) is NUC. We shall use also the equivalent definition given by Prus [15].

A Banach space X is said to be **nearly uniformly smooth** (NUS) if for every $\epsilon > 0$ there exists an $\eta > 0$ such that for each $t \in [0, \eta)$ and each basic sequence $\{u_n\}$ in B there is an $i > 1$ such that

$$\|u_1 + tu_i\| \leqq 1 + \epsilon t.$$

Prus has investigated finite dimensional decompositions of Banach spaces with (p, q)-estimates [14], and in [15] he has given a nice isomorphic characterization of NUS and NUC for Banach spaces with a countable basis in terms of (p, q)-estimates. (He has also mentioned that, using total biorthogonal systems instead of bases, the isomorphic characterization of NUS can be easily generalized to the case of separable spaces).

Let $\{x_n\}$ be a basis of a Banach space X with coefficient functionals $x_n^* \in X^*$. An element $x \in X$ is said to be a **block of** $\{x_n\}$ if either $x = 0$ or the set supp $x = \{n : x_n^*(x) \neq 0\}$ is finite. A family $\{X_n\}$ of finite dimensional subspaces of X is a **blocking** of $\{x_n\}$ provided there exists an increasing sequence of integers $\{n_k\}$, $n_1 = 1$, such that $X_k = [x_i]_{i=n_k}^{n_{k+1}-1}$ for each k. We say that blocks $y_1, \ldots, y_n$ are disjoint (with respect to the blocking $\{X_k\}$) if

$$\min\{m : y_i \in \sum_{j=1}^{m} X_j\} < \max\{m : y_{i+1} \in \sum_{j=m}^{\infty} X_j\} \text{ for } i = 1, \ldots, n-1.$$

Next, if $1 < q \leqq p < \infty$, then the blocking $\{X_k\}$ is said to satisfy (p, q)-estimates provided there exist positive constants c, C such that

$$c\left(\sum_{i=1}^{n} \|y_i\|^p\right)^{1/p} \leqq \|\sum_{i=1}^{n} y_i\| \leqq C\left(\sum_{i=1}^{n} \|y_i\|^q\right)^{1/q}$$

for all disjoint blocks $y_1, \ldots, y_n$.

Moreover, if only the left hand side of the above inequalities holds, then we say that $\{X_k\}$ satisfies $(p, 1)$-estimates, and if only the right hand side inequality holds, then we say that $\{X_k\}$ satisfies (∞, q)-estimates.

Proposition 2. *[15] Let X be a NUS space. Then there exist constants $q > 1$ and C such that each basic sequence $\{x_n\}$ in X has a blocking $\{X_n\}$ which satisfies (∞, q)-estimates with the constant C.*

Proposition 3. *[15] If X is a NUC space, then there exists a constant $p > 1$ such that each basic sequence $\{x_n\}$ in X has a blocking $\{X_n\}$ which satisfies $(p, 1)$-estimates.*

Prus has proved counterparts of the above two results, i.e. about the existence of an equivalent NUS (NUC) norm for Banach spaces with countable basis. Moreover, he has given a result in the spirit of the averaged norms of Asplund [1].

We shall first prove the following.

Theorem 4. *Let X be a Banach space. If the norm is both NUS and NUC, then it possesses property (β).*

Proof. Let $\epsilon > 0$. Since the norm is NUC, it is also UKK and we may find a corresponding $\delta_1 > 0$ such that

$$\|x\| \leqq 1 - \delta_1, \tag{1}$$

whenever x is a weak limit of some sequence $\{x_n\}$ in the closed unit ball B with sep $(x_n) > \epsilon$.

Applying the definition of NUS, given by Prus, for $\epsilon_1 = \delta_1/4$ there exists a corresponding $\eta > 0$ such that for each $t \in [0, \eta)$ and each basic sequence $\{u_n\}$ in B there is an index $i > 1$ so that

$$\|u_1 + tu_i\| \leqq 1 + \epsilon_1 t. \tag{2}$$

We may choose $\lambda, 0 < \lambda < 1$, small enough so that $\lambda/(1 - \lambda\delta_1) < \eta/2$. Put

$$t = 2\lambda/(1 - \lambda\delta_1).$$

Thus, we have $t \in [0, \eta)$.

We shall show that for the given $\epsilon > 0$ the equivalent definition of (β) is satisfied for

$$\delta = \lambda\delta_1/4(1 - \lambda).$$

Let $x \in B$ and $\{x_n\} \subset B$ with sep $(x_n) > \epsilon$ be arbitrary. By reflexivity, passing to a subsequence, we may suppose without loss of generality that $\{x_n\}$ is weakly convergent, say to an element v, i.e. $x_n = v_n + v$, where $\{v_n\}$ tends weakly to zero. Clearly, $\|v_n\| \leqq 2$. Moreover, we get by (1) that $\|v\| \leqq 1 - \delta_1$. Therefore,

$$\|(1 - \lambda)x + \lambda v\| \leqq 1 - \lambda\delta_1.$$

Denote

$$u_1 = [(1-\lambda)x + \lambda v]/(1-\lambda\delta_1) \text{ and } u_n = v_n/2 \text{ for } n > 1.$$

Thus, $\{u_n\} \subset B$ and $\{u_n\}$ tends weakly to zero. By sep $(x_n) > \epsilon$ we obtain that lim inf $\|u_n\| > 0$ and we may pass to a basic subsequence with first element u_1. Then, it follows from (2) that there exists an index $i > 1$ so that

$$\|u_1 + u_i\| \leqq 1 + \epsilon_1 t,$$

i.e.

$$\left\|\frac{(1-\lambda)x + \lambda v}{1-\lambda\delta_1} + \frac{\lambda}{1-\lambda\delta_1} v_i\right\| \leqq 1 + \frac{\lambda\delta_1}{2(1-\lambda\delta_1)}.$$

Therefore,

$$\|(1-\lambda)x + \lambda x_i\| \leqq 1 - \lambda\delta_1 + \lambda\delta_1/2 = 1 - \lambda\delta_1/2.$$

Then, by the triangle inequality,

$$\begin{aligned}\left\|\frac{x + x_i}{2}\right\| &= \left\|\frac{1}{2(1-\lambda)}[(1-\lambda)x + \lambda x_i] + \frac{1-2\lambda}{2(1-\lambda)} x_i\right\| \\ &\leqq 1 - \lambda\delta_1/4(1-\lambda) = 1 - \delta,\end{aligned}$$

which completes the proof.

The converse of Theorem 4 is not true isometrically. Property (β) implies NUC but not necessarily NUS, as we see in the following.

Example 5. There exists a Banach space which is uniformly convex but fails to be NUS.
Proof. Consider the function $M : \mathbf{R}^2 \to \mathbf{R}$, defined by

$$M(a,b) = \frac{1}{2}(a^2 + b^2)^{\frac{1}{2}} + \frac{1}{2}(|x| + |y|).$$

Let X be the direct sum of ℓ_2 with itself with the following norm

$$|||(x,y)||| = M(\|x\|, \|y\|),$$

where $\|\cdot\|$ is the usual norm of ℓ_2. Obviously, X is uniformly convex. On the other hand, since M is not partially differentiable at the point $(1,0)$, it follows from [10] that X is not NUS.

Yet, one can obtain for (β) a statement which is similar to the result of Prus, cited here as Proposition 1. Actually, we shall prove this under the following **condition** (β, ϵ): In the spirit of Proposition 1, by (β, ϵ) for some fixed $0 < \epsilon < 1$ we mean that there is $0 < \delta < 1$ so that for every element $x \in B$ and every sequence $\{x_n\} \subset B$ with sep $(x_n) > \epsilon$, there exists an i such that $\|x + x_i\| / 2 \leqq 1 - \delta$, which implies in particular that

$$\|x + x_i/2\| \leqq 3/2 - \delta. \tag{3}$$

We shall first prove the following.

Lemma 6. *Let X be a Banach space with the property (β, ϵ) for some $0 < \epsilon < 1$ and a corresponding δ as above. Then, for every basic sequence $\{x_n\}$ in X and every integer n there is an $i > n$ such that*

$$\|b_1 + b_2/2\| \leqq 3/2 - \delta/3$$

whenever $b_1 \in [x_j]_{j=1}^n$, $b_2 \in [x_j]_{j=i}^\infty$, and $\|b_1\| = \|b_2\| = 1$.

Proof. Assume the contrary, i.e. that there exists a basic sequence $\{x_n\}$ for which there is an integer n and two sequences of blocks $\{b_{1,m}\}$ and $\{b_{2,m}\}$ of norm one with $b_{1,m} \in [x_j]_{j=1}^n$, $b_{2,m} \in [x_j]_{j=m}^\infty$ for all m and such that

$$\|b_{1,m} + b_{2,m}/2\| > 3/2 - \delta/3.$$

Passing to a subsequence, there is no loss of generality in assuming the $\{b_{2,m}\}$ is a normalized block basic sequence of $\{x_n\}$ and moreover that $\{b_{1,m}\}$ tends in norm to some $b \in [x_j]_{j=1}^n$, $\|b\| = 1$. We may also suppose that for all m,

$$\|b + b_{2,m}/2\| > 3/2 - \delta/2. \tag{4}$$

Since (β, ϵ) implies (NUC, ϵ), which in turn implies reflexivity if $0 < \epsilon < 1$ (cf. e.g. [12]), then every basic sequence in X converges weakly to zero. Thus, $b_{2,m} \to 0$ weakly. Therefore, passing once more to a subsequence, we may assume that the basic constant K of $\{b_{2,m}\}$ is less than $1/\epsilon$. For the **inclination** k of a basic sequence $\{u_m\}$, i.e.

$$k = \inf_m \operatorname{dist}(S_{[u_1,\dots,u_m]}, [u_j : j > m]),$$

where S stands for the unit sphere of the corresponding space, we know that $kK = 1$ (see [2, p. 134]), where K is the basic constant of $\{u_m\}$. Thus, if we denote by k the inclination of $\{b_{2,m}\}$, we obtain that $k > \epsilon$. Therefore, by the definition of k, we get immediately that sep $(b_{2,m}) \geqq k > \epsilon$. Then, according to (3), there is an index i such that

$$\|b + b_{2,i}/2\| \leqq 3/2 - \delta,$$

which contradicts (4). This ends the proof of the claim.

Theorem 7. *Let X be a Banach space with the property (β, ϵ) with $0 < \epsilon < 1$. Then there exist constants $q > 1$ and C such that each basic sequence $\{x_n\}$ in X has a blocking $\{X_n\}$ which satisfies (∞, q)-estimates with constant C.*

Proof. Having proved Lemma 6, we may proceed as in [15]. Let $\{x_n\}$ be an arbitrary basic sequence in X. According to Lemma 6, we can construct inductively a sequence $1 = n_1 < n_2 < \dots$ for which

$$\|y_1 + y_2/2\| \leqq 3/2 - \delta/3$$

whenever $y_1 \in [x_i]_{i=1}^{n_k - 1}$, $y_2 \in [x_i]_{i=n_{k+1}}^{\infty}$, and $\|y_1\| = \|y_2\| = 1$. In particular, for every λ with $|1 - \lambda| < 1/2$, we have

$$\|y_1 + \lambda y_2\| \leqq \|y_1 + y_2/2\| + (\lambda - 1/2)\, \|y_2\| \leqq 1 + \lambda - \delta/3.$$

Since $2 - \delta/3 < 2$, there exists a $q > 1$ such that $(2 - \delta/3)^q < 2$. Then by the continuity of the functions $\lambda \to (1 + \lambda - \delta/3)^q$ and $\lambda \to 1 + \lambda^q$, there exists a ν with $0 < \nu < 1/2$ such that

$$(1 + \lambda - \delta/3)^q < 1 + \lambda^q$$

for all λ with $|1 - \lambda| < \nu$. For such λ we also have

$$\|y_1 + \lambda y_2\| < (1 + \lambda^q)^{1/q}.$$

In light of the theorem of N. and V. Gurarii (cf. [3] or [2, p. 135]), this implies that there exists a constant K such that if $X_k = [x_i]_{i=n_k}^{n_{k+1}-1}$, then each of the sequences $\{X_{2k-1}\}$ and $\{X_{2k}\}$ satisfies (∞, q)-estimates with constant K.

Thus, if the blocks $y_1, \dots, y_n$ are disjoint with respect to the blocking $\{X_k\}$, then

$$\|\sum_{i=1}^{n} y_i\| \leqq 2^{1/q'}(\|\sum y_{2j-1}\|^q + \|\sum y_{2j}\|^q)^{1/q} \leqq 2^{1/q'} K \left(\sum_{i=1}^{n} \|y_i\|^q \right)^{1/q},$$

where $1/q' + 1/q = 1$. Setting $C = 2^{1/q'} K$, this ends the proof.

Putting together the results of Prus and Theorems 4 and 7, we immediately obtain the following.

Corollary 8. *Let X be a separable Banach space with the property (β, ϵ) for some $0 < \epsilon < 1$. Then X has an equivalent NUS norm.*

In the next statement we repeat Theorem 4.3 [15], adding a new equivalent condition concerning (β).

Corollary 9. *Let X be a Banach space with a basis $\{e_n\}$. Then the following conditions are equivalent.*

(i) X admits an equivalent norm with property (β).

(ii) X admits an equivalent NUS norm and an equivalent NUC norm.

(iii) There are constants $p \geqq q > 1$, $C, c > 0$ such that each basic sequence $\{x_n\}$ in X has a blocking $\{X_n\}$ which satisfies (p, q)-estimates with the constants c, C.

(iv) The basis $\{e_n\}$ has a blocking $\{E_n\}$ which satisfies some (p, q)-estimates with $1 < q \leqq p < \infty$.

(d) X admits an equivalent norm which is both NUS and NUC.

Remark 10. In [9] we have shown that Schachermayer's space fails to have an equivalent NUS norm. Thus, Corollary 8 (or 9) provides another proof of the fact that this space does not admit an equivalent norm with the property (β).

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