

ON A CLASS OF *SP*-SASAKIAN MANIFOLD

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1. THE OBJECT OF THE PAPER

The Riemannian manifold satisfying curvature condition $R \cdot R = 0$ or $R \cdot R = L Q(g, R)$ have been studied by various authors ([6], [7], [8], [9]). Manifolds satisfying these conditions are called semi-symmetric and pseudo-symmetric respectively.

In the present paper we consider *SP*-Sasakian manifold M satisfying, in a neighborhood \mathcal{U} of $x \in M$, the condition

$$(1.1) \quad R \cdot R = L Q(a, R)$$

where $a = g$ or $a = S_p$, $S_p = S^a_{p ij} = S^{a-1}_{i-1 aj}$, g is the metric tensor of M , $S_{ij} = S_{ij}$ is the Ricci tensor, L is some function on \mathcal{U} , while $R \cdot R$ and $Q(a, R)$ are defined as in (2.4) and (2.5) respectively. Our main result is the following.

If an *SP*-Sasakian manifold satisfies the condition

$$(1.2) \quad R \cdot R = L Q(S_p, R)$$

for some $p = 1, 2, \dots$, then

- a) it is a space of quasi-constant curvature;
- b) it can be transformed into a manifold of constant curvature -1 by a D -conircular change; and
- c) it satisfies the condition (1.2) for every $p = 0, 1, 2, \dots$, (except for some special cases when (1.2) may not be satisfied for one specific p).

2. PRELIMINARIES

Let M be a connected n -dimensional ($n \geq 4$) Riemannian manifold of class C^∞ with a positive definite Riemannian metric g which admits a unit 1-form η satisfying

$$\begin{aligned} \nabla_j \eta_i - \nabla_i \eta_j &= 0, \\ \nabla_k \nabla_j \eta_i &= -g_{kj} \eta_i - g_{ki} \eta_j + 2 \eta_k \eta_j \eta_i, \end{aligned}$$

where ∇ is the operator of the covariant differentiation with respect to Levi-Civita connection.

If we put

$$\xi^i = g^{ia} \eta_a \quad (g^{ia} g_{aj} = \delta^i_j), \quad \varphi^i_j = \nabla_j \xi^i, \quad \varphi_{ij} = g_{ik} \varphi^k_j,$$

we get

$$\eta_a \xi^a = 1, \varphi_j^a \varphi_a^i = \delta_j^i - \eta_j \xi^i, \xi^a \varphi_a^i = 0, \eta_a \varphi_a^i = 0, \\ g_{ij} \varphi_p^i \varphi_q^j = g_{pq} - \eta_p \eta_q, \varphi_{ij} = \varphi_{ji}, \text{rank}(\varphi_j^i) = n - 1.$$

These relations show that M is a special paracontact Riemannian manifold with a structure (φ, ξ, η, g) . Such a manifold is called a P -Sasakian (para-Sasakian) manifold. The notion of a manifold with P -Sasakian structures was introduced by I. Sato [10] (who also gave examples) and has been studied by Sato himself and other authors (see, for example [1], [2], [5]). P -Sasakian structure is closely related to the almost product structure and may be considered as analogous to the almost contact structure (which is closely related to the almost complex structure).

In the P -Sasakian manifold we have the equations

$$(2.1) \quad R_{kji}{}^a \eta_a = g_{ki} \eta_j - g_{ji} \eta_k,$$

$$(2.2) \quad S_{ia} \xi^a = -(n-1) \eta_j,$$

where $R_{kji}{}^h$ denotes the Riemannian curvature tensor.

If in a P -Sasakian manifold M the unit 1-form η satisfies the equation

$$(2.3) \quad \nabla_j \eta_i = \varepsilon(-g_{ji} + \eta_j \eta_i), \quad \varepsilon = \pm 1$$

then M is called an SP -Sasakian (special para-Sasakian) manifold.

Let a be a symmetric tensor of the type (0,2). Then we define the tensors $R \cdot R$, $Q(a, R)$ and $Q(\eta\eta, R(1))$ by the formulas

$$(2.4) \quad (R \cdot R)_{kjihrs} = \nabla_s \nabla_r R_{kjih} - \nabla_r \nabla_s R_{kjih},$$

$$(2.5) \quad Q(a, R)_{kjihrs} = \\ = a_{kr} R_{sjih} + a_{jr} R_{ksih} + a_{ir} R_{kjsih} + a_{hr} R_{kjis} - \\ - a_{ks} R_{rjih} - a_{js} R_{krih} - a_{is} R_{kjrth} - a_{hs} R_{kjir},$$

$$(2.6) \quad Q(\eta\eta, R(1))_{kjihrs} = \\ = \eta_k \eta_r (g_{sh} g_{ji} - g_{si} g_{jh}) + \eta_j \eta_r (g_{kh} g_{si} - g_{ki} g_{sh}) + \\ + \eta_i \eta_r (g_{kh} g_{js} - g_{ks} g_{jh}) + \eta_h \eta_r (g_{ks} g_{ji} - g_{ki} g_{js}) - \\ - \eta_k \eta_s (g_{rh} g_{ji} - g_{ri} g_{jh}) - \eta_j \eta_s (g_{kh} g_{ri} - g_{ki} g_{rh}) - \\ - \eta_i \eta_s (g_{kh} g_{jr} - g_{kr} g_{jh}) - \eta_h \eta_s (g_{kr} g_{ji} - g_{ki} g_{jr}).$$

Using (2.1) and (2.3), we get

$$(2.7) \quad (\nabla_r R_{kji}{}^a) \eta_a = \varepsilon R_{kji r} + \varepsilon (g_{kr} g_{ji} - g_{ki} g_{jr}).$$

Applying the operator ∇_s to (2.7) and using (2.3) and (2.7), we find

$$\begin{aligned} & (\nabla_s \nabla_r R_{kji}{}^a) \eta_a + \eta_s R_{kji r} + \eta_s (g_{ji} g_{kr} - g_{ki} g_{jr}) = \\ & = \varepsilon (\nabla_r R_{kjis} + \nabla_s R_{kji r}), \end{aligned}$$

from which we get

$$\begin{aligned} & (\nabla_s \nabla_r R_{kji}{}^a - \nabla_r \nabla_s R_{kji}{}^a) \eta_a = \eta_r R_{kjis} - \eta_s R_{kji r} + \\ & + \eta_r (g_{ji} g_{sk} - g_{ki} g_{js}) - \eta_s (g_{kr} g_{ji} - g_{ki} g_{jr}). \end{aligned}$$

Now, we use the condition (1.1). Substituting it into the preceding relation and taking into account (2.1), we obtain

$$\begin{aligned} & L [a_{rk} (g_{si} \eta_i - g_{ji} \eta_s) + a_{jr} (g_{ki} \eta_s - g_{si} \eta_k) + a_{ir} (g_{ks} \eta_j - g_{js} \eta_k) - \\ & - a_{ks} (g_{ri} \eta_j - g_{ji} \eta_r) - a_{js} (g_{ki} \eta_r - g_{ri} \eta_k) - a_{is} (g_{kr} \eta_j - g_{jr} \eta_k) + \\ (2.8) \quad & + a_{tr} \xi^t R_{kjis} - a_{ts} \xi^t R_{kji r}] = \\ & = \eta_r R_{kjis} - \eta_s R_{kji r} + \eta_r (g_{ji} g_{sk} - g_{ki} g_{sj}) - \eta_s (g_{ji} g_{rk} - g_{ki} g_{rj}). \end{aligned}$$

Transvecting (2.8) with ξ^r and using (2.1), we have

$$\begin{aligned} & (1 - L a_{tr} \xi^t \xi^r) R_{kjis} = -(g_{ji} g_{sk} - g_{ki} g_{sj}) + \\ (2.9) \quad & + L [a_{kr} \xi^r (g_{si} \eta_j - g_{ji} \eta_s) + a_{jr} \xi^r (g_{ki} \eta_s - g_{si} \eta_k) + a_{ir} \xi^r (g_{ks} \eta_j - g_{js} \eta_k) + \\ & + a_{ts} \xi^t (g_{ji} \eta_k - g_{ki} \eta_j) + a_{ks} (g_{ji} - \eta_j \eta_i) - a_{js} (g_{ki} - \eta_k \eta_i)]. \end{aligned}$$

3. *SP*-SASAKIAN MANIFOLD SATISFYING $R \cdot R = L Q(g, R)$

In this case $a_{tr} \xi^t = g_{tr} \xi^t = \eta_r$ and (2.9) reduces to

$$(1 - L) R_{kjis} = -(1 - L) (g_{ji} g_{sk} - g_{ki} g_{sj}),$$

from which, if $L \neq 1$, we obtain

$$R_{kjis} = -(g_{ks} g_{ji} - g_{ki} g_{js}).$$

Thus, we have

Theorem 1. *If an SP-Sasakian manifold satisfies the condition $R \cdot R = L Q(g, R)$ with $L \neq 1$, it is the space of constant curvature -1 .*

If $L = 0$, the considered manifold is semi-symmetric. Thus

Corollary. *If an SP-Sasakian manifold is semi-symmetric, it is a space of constant curvature -1 .*

4. SP-SASAKIAN MANIFOLD SATISFYING (1.2)

In the case $\alpha_{ij} = S_{P ij}$, we find, in view of (2.2)

$$(4.1) \quad \begin{aligned} \alpha_{ij}\xi^j &= S_{P ij} \xi^j = S_{P^{-1} ia} S_j^a \xi^j = (1-n) S_{P^{-1} ia} \xi^a = (1-n)^P \eta_i \\ \alpha_{ij}\xi^i \xi^j &= (1-n)^P. \end{aligned}$$

Therefore, the relation (2.9) reduces to

$$(4.2) \quad \begin{aligned} [1 - (1-n)^P L] R_{kjis} &= -(g_{ji}g_{ks} - g_{ki}g_{js}) + (1-n)^P L(g_{ks}\eta_j\eta_i - g_{js}\eta_k\eta_i) + \\ &+ L[S_{P ks} (g_{ji} - \eta_j\eta_i) - S_{P js} (g_{ki} - \eta_k\eta_i)]. \end{aligned}$$

Interchanging in (4.2) the indices i and s and adding the obtained relation to (4.2), we get

$$(4.3) \quad \begin{aligned} L\{(1-n)^P(g_{ks}\eta_j\eta_i - g_{js}\eta_k\eta_i + g_{ki}\eta_j\eta_s - g_{ji}\eta_k\eta_s) + S_{P ks} (g_{ji} - \eta_j\eta_i) - \\ - S_{P js} (g_{ki} - \eta_k\eta_i) + S_{P ki} (g_{js} - \eta_j\eta_s) - S_{P ji} (g_{ks} - \eta_k\eta_s)\} = 0. \end{aligned}$$

Now, we suppose $L \neq 0$ (the case $L = 0$ has been considered in § 3). Then, transvecting (4.3) with g^{ij} , taking into account (4.1) and denoting by τ the expression $S_{P ij} g^{ij}$, we find

$$S_{P ks} = \frac{\tau - (1-n)^P}{n-1} g_{ks} + \frac{n(1-n)^P - \tau}{n-1} \eta_k\eta_s.$$

Substituting this into (4.2), we obtain

$$(4.4) \quad \begin{aligned} [1 - (1-n)^P L] R_{kjis} &= \left(\frac{\tau - (1-n)^P}{n-1} L - 1 \right) (g_{ks}g_{ji} - g_{ki}g_{js}) + \\ &+ \frac{n(1-n)^P - \tau}{n-1} L (g_{ks}\eta_j\eta_i + g_{ji}\eta_k\eta_s - g_{ki}\eta_j\eta_s - g_{js}\eta_k\eta_i). \end{aligned}$$

If $L \neq \frac{1}{(1-n)^p}$, (4.4) can be written in the form

$$(4.5) \quad R_{kjis} = A(g_{ks}g_{ji} - g_{ki}g_{js}) + B((g_{ks}\eta_j\eta_i + g_{ji}\eta_k\eta_s - g_{ki}\eta_j\eta_s - g_{js}\eta_k\eta_i),$$

where

$$A = \frac{1}{1 - (1 - n)^p L} \left(\frac{\tau - (1 - n)^p}{n - 1} L - 1 \right),$$

$$B = \frac{n(1 - n)^p - \tau_p}{(n - 1)[1 - (1 - n)^p L]} L.$$

It is easy to see that

$$(4.6) \quad A + B = -1.$$

On the other hand, from (4.5) we have

$$(4.7) \quad S_{ij} = [(n - 1)A + B]g_{ij} + (n - 2)B\eta_i\eta_j,$$

$$(4.8) \quad \tau = \tau_1 = (n - 1)(nA + 2B).$$

The relations (4.6) and (4.8) imply

$$(4.9) \quad A = \frac{\tau + 2(n - 1)}{(n - 1)(n - 2)}, \quad B = -\frac{\tau + n(n - 1)}{(n - 1)(n - 2)}.$$

The relation (4.5) shows that the considered manifold is the space of quasi-constant curvature ([3], [4]). Thus

Theorem 2. *If an SP -Sasakian manifold M satisfies (1.2) for any $p = 1, 2, \dots$, with $L \neq \frac{1}{(1-n)^p}$ it is a space of quasi-constant curvature; the functions A and B , given by (4.9) are independent of the number p .*

There exists in a P -Sasakian manifold M an $(n - 1)$ -dimensional distribution D defined by a Pfaffian equation $\eta = 0$ and called the D -distribution. Assume that in M are given two P -Sasakian structures (φ, ξ, η, g) and $(^*\varphi, ^*\xi, ^*\eta, ^*g)$ satisfying

$$(4.10) \quad \begin{aligned} ^*g_{ij} &= e^{2\sigma}g_{ij} + (e^{2\sigma} - e^{2\alpha})\eta_i\eta_j, \\ ^*\eta_j &= \varepsilon e^\sigma\eta_j, \quad ^*\xi^i = \varepsilon e^{-\sigma}\xi^i, \quad ^*\varphi_j^i = \varepsilon\varphi_j^i, \end{aligned}$$

with functions α and σ . Then (φ, ξ, η, g) and $(^*\varphi, ^*\xi, ^*\eta, ^*g)$ have the same D -distribution in common, because of which the relation (4.10) is called by G. Chuman [5] a D -conformal change of (φ, ξ, η, g) . The D -conformal change (4.10) satisfying

$$\alpha_{ji} = \theta(g_{ji} - \eta_j \eta_i),$$

where θ is a function and

$$\begin{aligned} \alpha_{ji} &= \nabla_j \alpha_i - \alpha_j \alpha_i - e^\sigma (\alpha_j \eta_i + \alpha_i \eta_j) + \frac{1}{2} (\alpha_h \alpha^h - e^{2\sigma} + 1) (g_{ji} - \eta_j \eta_i) + \\ &+ (e^\sigma \sigma_h \xi^h - e^{2\sigma} + 1) \eta_j \eta_i, \\ \alpha_i &= \frac{\partial \alpha}{\partial x^i}, \quad \alpha^i = g^{ih} \alpha_h, \quad \sigma_i = \frac{\partial \sigma}{\partial x^i}, \end{aligned}$$

is called a D -conircular change. T. Adti and G. Chuman proved [1] that an SP -Sasakian manifold is transformed into a manifold of constant curvature -1 by a D -conircular change if and only if it is a space of quasi-constant curvature (4.5) satisfying (4.9). Thus

Theorem 3. *If an SP -Sasakian manifold satisfies (1.2) for any $p = 1, 2, \dots$, with $L \neq \frac{1}{(1-n)^p}$, it can be transformed into a manifold of constant curvature -1 by a D -conircular change.*

5. SOME CURVATURE CONDITIONS ON A SPACE OF QUASI-CONSTANT CURVATURE

In this section, we consider a space of quasi-constant curvature (4.5). We suppose that it is not a space of constant curvature, i.e. we suppose that $B \neq 0$.

First, let us calculate the tensor $Q(S, R)_p$. To do that, we note that (4.7) can be expressed in the form

$$S_{ij} = S_{1ij} = ag_{ij} + b\eta_i \eta_j,$$

where

$$a = (n - 1)A + B, \quad b = (n - 2)B.$$

Then

$$S_{p ij} = a^p g_{ij} + [(a + b)^p - a^p] \eta_i \eta_j,$$

i.e.

$$(5.1) \quad S_{p ij} = \gamma g_{ij} + \beta \eta_i \eta_j$$

where

$$(5.2) \quad \gamma = [(n - 1)A + B]^p, \quad \beta = (n - 1)^p(A + B)^p - [(n - 1)A + B]^p.$$

We note that if $p = 0$, then $\gamma = 1$ and $\beta = 0$. Thus $S_0 = g$.

In view of (2.5), (2.6), (4.5) and (5.1), we have

$$Q(S, R) = -(\gamma B - \beta A)Q(\eta\eta, R(1)).$$

The relation (5.2) implies

$$\gamma B - \beta A = [(n - 1)A + B]^p(A + B) - (n - 1)^p(A + B)^p A.$$

Therefore

$$(5.3) \quad Q(S, R) = -(A + B)\{[(n - 1)A + B]^p - (n - 1)^p(A + B)^{p-1}A\}Q(\eta\eta, R(1)).$$

In the case $p = 0$, if $A + B \neq 0$, (5.3) reduces to

$$(5.4) \quad Q(g, R) = -BQ(\eta\eta, R(1)).$$

On the other hand, we have from (4.5) that

$$\begin{aligned} \nabla_r R_{kjih} = & A_r(g_{kh}g_{ji} - g_{ki}g_{jh}) + \\ & + B_r(g_{kh}\eta_j\eta_i + g_{ji}\eta_k\eta_h - g_{ki}\eta_j\eta_h - g_{jh}\eta_k\eta_i) + \\ & + B[g_{kh}(\eta_i\nabla_r\eta_j + \eta_j\nabla_r\eta_i) + g_{ji}(\eta_h\nabla_r\eta_k + \eta_k\nabla_r\eta_h) - \\ & - g_{ki}(\eta_h\nabla_r\eta_i + \eta_j\nabla_r\eta_h) - g_{jh}(\eta_i\nabla_r\eta_k + \eta_k\nabla_r\eta_i)], \end{aligned}$$

where

$$A_r = \frac{\partial A}{\partial x^r}, \quad B_r = \frac{\partial B}{\partial x^r},$$

and

$$\begin{aligned} \nabla_s \nabla_r R_{kjih} - \nabla_r \nabla_s R_{kjih} = & \\ = & B[g_{kh}(\nabla_s \nabla_r \eta_j - \nabla_r \nabla_s \eta_j)\eta_i + g_{kh}(\nabla_s \nabla_r \eta_i - \nabla_r \nabla_s \eta_i)\eta_j + \\ (5.5) \quad & + g_{ji}(\nabla_s \nabla_r \eta_k - \nabla_r \nabla_s \eta_k)\eta_h + g_{ji}(\nabla_s \nabla_r \eta_h - \nabla_r \nabla_s \eta_h)\eta_k - \\ & - g_{ki}(\nabla_s \nabla_r \eta_j - \nabla_r \nabla_s \eta_j)\eta_h - g_{ki}(\nabla_s \nabla_r \eta_h - \nabla_r \nabla_s \eta_h)\eta_j - \\ & - g_{jh}(\nabla_s \nabla_r \eta_k - \nabla_r \nabla_s \eta_k)\eta_i - g_{jh}(\nabla_s \nabla_r \eta_i - \nabla_r \nabla_s \eta_i)\eta_k]. \end{aligned}$$

Also, from (4.5) we have

$$R_{rsj} \text{ }^a \eta_a = (A + B)(g_{sj} \eta_r - g_{rj} \eta_s),$$

because of which the Ricci identity

$$\nabla_s \nabla_r \eta_j - \nabla_r \nabla_s \eta_j = R_{rsj} \text{ }^a \eta_a$$

can be expressed as

$$\nabla_s \nabla_r \eta_j - \nabla_r \nabla_s \eta_j = (A + B)(g_{sj} \eta_r - g_{rj} \eta_s).$$

Substituting this into (5.5) and using (2.6), we get

$$(5.6) \quad R \cdot R = (A + B) B Q(\eta\eta, R(1)).$$

Thus ([3], theorem 3.1')

The space of quasi-constant curvature (4.5) is semi-symmetric if and only if $A + B = 0$.

In the sequel we suppose $A + B \neq 0$. Then (5.4) and (5.6) imply

$$(5.7) \quad R \cdot R = -(A + B) Q(g, R).$$

Thus

Theorem. *Non semi-symmetric space of quasi-constant curvature is pseudo-symmetric.*

In view of (5.3) and (5.6) we find

$$Q\left(\frac{S}{p}, R\right) = -\frac{[(n-1)A + B]^p - (n-1)^p (A + B)^{p-1} A}{B} R \cdot R.$$

Thus, if

$$[(n-1)A + B]^p - (n-1)^p (A + B)^{p-1} A \neq 0,$$

we have

$$(5.8) \quad R \cdot R = \frac{L}{p} Q\left(\frac{S}{p}, R\right)$$

where

$$(5.9) \quad \frac{L}{p} = \frac{B}{(n-1)^p (A + B)^{p-1} A - [(n-1)A + B]^p}.$$

Thus, the condition (5.8) including (5.7) for $p = 0$, we can state

Theorem 4. *A non semi-symmetric space of quasi-constant curvature satisfies (5.8), (5.9) for every $p = 0, 1, 2, \dots$, except in the case $(n - 1)^p(A + B)^{p-1}A - [(n - 1)A + B]^p = 0$.*

If $A + B = -1$, we find

$$(5.10) \quad L_p = -\frac{1 + A}{(n - 1)^p(-1)^{p-1}A - [(n - 2)A - 1]^p}.$$

Thus, taking into account Theorem 2, we have

Theorem 5. *If an SP -Sasakian manifold satisfies (1.2) for any $p = 1, 2, \dots$, with $L \neq \frac{1}{(1-n)^p}$, then it satisfies this condition for every $p = 0, 1, 2, \dots$, where $L = L_p$ has the form (5.10) and A — the form (4.9), except in the case $(n - 1)^p(-1)^{p-1}A - [(n - 2)A - 1]^p = 0$.*

In § 3, we did not consider SP -Sasakian manifold satisfying $R \cdot R = Q(g, R)$. The theorem 5 shows that, in view of $L = 1$, an SP -Sasakian manifold satisfying (1.2) for any $p = 1, 2, \dots$, satisfies $R \cdot R = Q(g, R)$, too.

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