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ON CERTAIN TRANSLATION COVERS

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Let $\mathscr{C}_1$ be the collection of closed curves of length ≤ 1 in the d-dimensional Euclidean space E^d. We say a set $X \subset E^d$ is a translation cover for $\mathscr{C}_1$ if every curve in $\mathscr{C}_1$ can be covered by a translate of X.

A hyperplane is called a supporting hyperplane for a convex body in E^d if the hyperplane has a non-empty intersection with the body and the body is contained in one of the closed half-spaces called a supporting half-space with the hyperplane as boundary. The minimal width w of a convex body B of E^d is the distance between a pair of distinct parallel supporting hyperplanes for B with the property that the distance between any pair of distinct parallel supporting hyperplanes of B is $\geq w$.

We start with

Problem 1. *Determine the smallest positive real number $w(\mathrm{d})$ for which it is true that any convex body of E^d (i.e. any compact convex set of E^d with a non-empty interior) of minimal width $\geq w(\mathrm{d})$ $(\mathrm{d} \geq 2)$ is a translation cover for $\mathscr{C}_1$.*

The results we are going to prove below reduce Problem 1 to a special problem of simplices, which is also unsolved but seems to be simpler than the original problem (see Problem 2 and Theorem 2 below). First we have

Remark 1. It is obvious that $w(2) \leq w(3) \leq \ldots \leq w(\mathrm{d}) \leq w(\mathrm{d}+1) \leq \ldots$.

Definition. *$\mathcal{O}$ is called a $(\mathrm{d}+1)$-gon of E^d if it is the union of the segments $\overline{A_1A_2}$, $\overline{A_2A_3}, \ldots, \overline{A_{s-1}A_s}, \overline{A_sA_1}$ where $A_1A_2, \ldots, A_{s-1}A_s$ are pairwise different points in E^d with $2 \leq s \leq \mathrm{d}+1$. We suppose that the angles of $(\mathrm{d}+1)$-gons of E^d are distinct from the straight angle.*

For our investigations the following definition is the most important.

Definition. *$\mathcal{O}_b = \overline{A_1A_2} \cup \ldots \cup \overline{A_{s-1}A_s} \cup \overline{A_sA_1}$ $(2 \leq s \leq \mathrm{d}+1)$ is called a billiard $(\mathrm{d}+1)$-gon of the convex body B of $E^d (\mathrm{d} \geq 2)$ if*

(1) $\mathcal{O}_b$ is a $(\mathrm{d}+1)$-gon in E^d,

(2) $A_i \in \mathrm{bd}\, B$ (=the boundary of B) for any $i \in \{1,2,\ldots,s\}$ and there exists a supporting hyperplane H_i of B which passes through the point A_i having inner normal vector $\underline{n}_i$ such that,

(3) the ray emanating from A_i having direction vector $\underline{n}_i$ is the inner angle bisector of the angle $A_{i-1}A_iA_{i+1}$ (with $A_{s+1} = A_1$ and $A_0 = A_s$, see Fig. 1).

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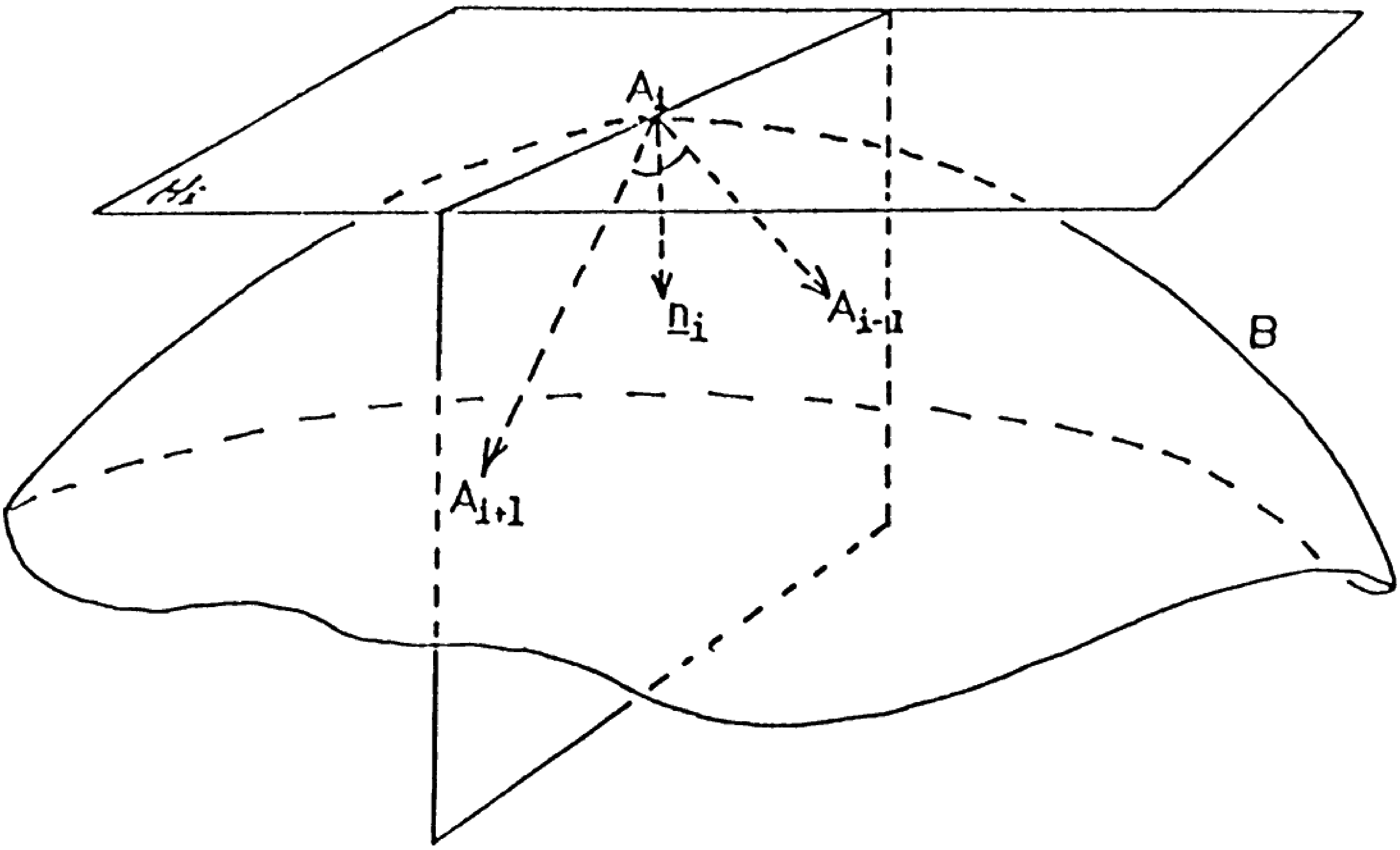

Figure 1

Remark 2. If $\mathcal{O}_b = \overline{A_1A_2} \cup \ldots \cup \overline{A_{s-1}A_s} \cup \overline{A_sA_1}$ is a billiard (d + 1)-gon of a convex body of E^d, then either for all $i \in \{1,2,\ldots,s\}$ we have $0 < \sphericalangle' A_{i-1}A_iA_{i+1} < \pi$ or $s = 2$ and $\mathcal{O}_b = \overline{A_1A_2} \cup \overline{A_2A_1}$ and $\overline{A_1A_2} \| \underline{n}_1$ and $\overline{A_2A_1} \| \underline{n}_2$ (where $\|$ stands for the parallelism).

Before we prove some lemmas we need three more definitions.

Definition. *The billiard* (d + 1)*-gon* $\mathcal{O}_b$ *of the previous definition is called a real billiard* (d+ 1)*-gon of the convex body* B *of* E^d $(d \geq 2)$ *if for all* $i \in \{1,2,\ldots,s\}$ H_i *is the unique supporting hyperplane of* B *at* A_i.

Definition. *The billiard* (d+ 1)*-gon* $\mathcal{O}_b = \overline{A_1A_2} \cup \ldots \cup \overline{A_{s-1}A_s} \cup \overline{A_sA_1}$ $(2 \leq s \leq d+1)$ *is called a simplicial billiard path (of dimension* $s-1$ *) of the convex body* B *of* E^d $(d \geq 2)$ *if* $\dim(\mathrm{aff}(\mathcal{O}_b)) = s-1$ *where* aff (...) *means the affine hull of the corresponding set in* E^d. *(Among other things this means that* $A_1, A_2, \ldots, A_s$ *form the vertices of an* $(s-1)$*-simplex inscribed in* B *).*

The next definition is taken from [1] and [2].

Definition. *A set* $\{H_i^+\}_{i=1}^s$ *of closed half-spaces of* E^d $(d \geq 2)$ *is called nearly bounded if* $\cap_{i=1}^s H_i^+$ *is contained in the set between two parallel hyperplanes of* E^d. *(Further on let*

$\underline{n}_i$ *be the inner normal vector of the hyperplane* $H_i (1 \leq i \leq s)$ *which bounds* H_i^+. *(So* $\underline{n}_i$ *points into the interior of* H_i^+.*)*

Lemma 1. *A set* $\{H_i^+\}_{i=1}^s$ *of closed half-spaces of* E^d *whose origin is say, 0 is nearly bounded if and only if any of the following three properties holds:*

(a) $\cup_{i=1}^s(\underline{t}_i + H_i^+) = E^d$ *where* $\underline{t}_i + H_i^+$ *means the translate of* H_i^+ *by the vector* $\underline{t}_i$ *with* $0 \in \underline{t}_i + H_i$;

(b) $0 \in \operatorname{conv}\{\underline{n}_1, \underline{n}_2, \ldots, \underline{n}_s\}$ *where of course the vectors emanating from 0 mean the points into which they point;*

(c) $\cap_{i=1}^s H_i^+$ *cannot be covered by a translate of* $\operatorname{int}(\cap_{i=1}^s H_i^+)$. *(Here and further on also* $\operatorname{int}(\ldots)$ *means the interior of a set in* E^d.*)*

Proof. Suppose that:

(a*) The set $\{H_i^+\}_{i=1}^s$ of closed half-spaces of E^d is nearly bounded i.e. $\cap_{i=1}^s H_i^+$ is contained in the set between two parallel hyperplanes of E^d.

Then it is easy to show that $(a) \Rightarrow (b) \Rightarrow (c) \Rightarrow (a)$ and $(a^*) \Longleftrightarrow (b)$ which proves our assertion. ∎

Lemma 2. *If* $0_b = \overline{A_1A_2} \cup \ldots \cup \overline{A_{s-1}A_s} \cup \overline{A_sA_1}$ $(2 \leq s \leq d+1, d \geq 2)$ *is a billiard* $(d+1)$*-gon of the convex body* B *of* E^d, *then it cannot be covered by a translate of* $\operatorname{int} B$.

Proof. Let H_i be the supporting hyperplane of B going through $A_i \in bd\,B$ for any $i \in \{1,2,\ldots,s\}$. Also, let H_i^+ be the supporting half-space of B which is bounded by H_i, and which is closed by definition (see page 1), and suppose that $\underline{n}_i$ is the inner normal vector of H_i (i.e. $\underline{n}_i$ points into the interior of H_i^+). Because of (c) of Lemma 1 it is enough to show that the set $\{H_i^+\}_{i=1}^s$ of closed half-spaces of E^d $(d \geq 2)$ is nearly bounded. Without loss of generality we may suppose that $\underline{n}_i$ emanates from A_i and the vector $\underline{n}_i^*$ emanating from 0 (i.e. from the origin of E^d) is equal to $\underline{n}_i$ for any $i \in \{1,2,\ldots,s\}$. In this way $\underline{n}_i^*$ corresponds to a unique point of E^d for any $i \in \{1,2,\ldots,s\}$. It is enough to prove that $0 \in \operatorname{conv}\{\underline{n}_1^*, \underline{n}_2^*, \ldots, \underline{n}_s^*\}$ (see (b) of Lemma 1). In order to prove it consider an arbitrary hyperplane H of E^d passing through 0. Let H^+, H^- be the two closed half-spaces of E^d bounded by H. We have to prove that $H^+ \cap \{\underline{n}_1^*, \underline{n}_2^*, \ldots, \underline{n}_s^*\} \neq \emptyset$ and $H^- \cap \{\underline{n}_1^*, \underline{n}_2^*, \ldots, \underline{n}_s^*\} \neq \emptyset$. Because of the «symmetry» we prove only that $H^+ \cap \{\underline{n}_1^*, \underline{n}_2^*, \ldots, \underline{n}_s^*\} \neq \emptyset$. But that follows from the simple fact that for the supporting half-space H_t^+ of $\operatorname{conv}\{A_1, A_2, \ldots, A_s\}$ which is in addition a translate of H^+ we can find at least one point say, A_{i_o} $(i_o \in \{1,2,\ldots,s\})$ with the property that $A_{i_o} \in bd\,H_t^+$ and $\underline{n}_{i_o} \subseteq H_t^+$ which means here that the endpoint of $\underline{n}_{i_o}$ different from A_{i_o} belongs to H_t^+. ∎

As an easy corollary of Helly's theorem [1] we get the following

Lemma 3. *Let Y be any set of E^d $(d \geq 2)$ with at least $(d+1)$ points. A compact convex set X of E^d has a translate which covers Y if and only if X has a translate which covers every $(d+1)$-point subset of Y.*

Proof. We leave the proof to the reader since it is a simple exercise. ∎

Lemma 4. *The $(d+1)$-gon $\mathcal{O} = \overline{A_1A_2} \cup \ldots \cup \overline{A_{s-1}A_s} \cup \overline{A_sA_1}$ $(2 \leq s \leq d+1, d \geq 2)$ of E^d cannot be covered by a translate of the interior of a given convex body B of E^d if and only if there exists a vector $\underline{t}$ such that for the translates $\underline{t} + A_{i_1}, t + A_{i_2}, \ldots, \underline{t} + A_{i_\ell}$ $(\{i_1, i_2, \ldots, i_\ell\} \subset \{1, 2, \ldots, s\}, 2 \leq \ell \leq s)$ of certain vertices $A_{i_1}, A_{i_2}, \ldots, A_{i_\ell}$ of $\mathcal{O}$ we can find a nearly bounded set of supporting half-spaces $H^+_{i_1}, H^+_{i_2}, \ldots, H^+_{i_\ell}$ of B with $\underline{t} + A_{i_1} \in H^-_{i_1}, \underline{t} + A_{i_2} \in H^-_{i_2}, \ldots, \underline{t} + A_{i_\ell} \in H^-_{i_\ell}$ where $H^-_{i_1} = (E^d \setminus H^+_{i_1}) \cup \mathrm{bd}\, H^+_{i_1}$, $H^-_{i_2} = (E^d \setminus H^+_{i_2}) \cup \mathrm{bd}\, H^+_{i_2}, \ldots, H^-_{i_\ell} = (E^d \setminus H^+_{i_\ell}) \cup \mathrm{bd}\, H^+_{i_\ell}$.*

Proof. It is easy to prove that the existence of the vector $\underline{t}$ of our lemma yields that the $(d+1)$-gon $\mathcal{O}$ cannot be covered by a translate of int B (see (c) of Lemma 1 again). So we have to deal with the case where we suppose that the $(d+1)$-gon $\mathcal{O}$ cannot be covered by a translate of int B and having this fact in our mind we look for a vector $\underline{t}$ with the property mentioned in the lemma. Without loss of generality we may suppose that B is strictly convex and smooth. (Since each convex body can be approximated by a sequence of smooth strictly convex bodies the general case of the lemma follows from the mentioned special case quite easily.) So let $\lambda_0 = \sup\{\lambda | \lambda > 0, \lambda \cdot \mathcal{O}$ can be covered by a translate of $B\}$ where of course $\lambda \cdot \mathcal{O} = \{\underline{x} | \underline{x} = \lambda \cdot \underline{y}$ with $\underline{y} \in \mathcal{O}\}$ i.e. $\lambda \cdot \mathcal{O}$ is a scalar multiple of $\mathcal{O}$. It is obvious that $\lambda_0 \cdot \mathcal{O}$ can be covered by a translate of B say, by B_0. Hence $0 < \lambda_0 \leq 1$. Because of the definition of λ_0 and because of the strictly convexity of B_0 some vertices of $\lambda_0 \cdot \mathcal{O}$ say, $A^0_{i_1}, A^0_{i_2}, \ldots, A^0_{i_\ell}$ and only those points of $\lambda_0 \cdot \mathcal{O}$ belong to $\mathrm{bd}\, B_0$. (Obviously $2 \leq \ell \leq s$.) Let $\overline{H}_{i_j}$ be the unique supporting hyperplane of B_0 through the vertex $A^0_{i_j} (1 \leq j \leq \ell)$. Also, let $\overrightarrow{H}^+_{i_j}$ be the supporting half-space of B_0 bounded by $\overline{H}_{i_j} (1 \leq j \leq \ell)$. Now the set $\{\overrightarrow{H}^+_{i_j}\}^\ell_{j=1}$ of closed half-spaces of E^d has to be nearly bounded since otherwise $\cap^\ell_{j=1} \overrightarrow{H}^+_{i_j}$ were covered by a translate of int $(\cap^\ell_{j=1} \overrightarrow{H}^+_{i_j})$. That would yield the existence of a translate of $\lambda_0 \cdot \mathcal{O}$ which were covered by int B_0. (Here we used the fact again that B_0 is a smooth strictly convex body.) The contradiction obtained in this way shows that the set $\{\overrightarrow{H}^+_{i_j}\}^\ell_{j=1}$ is nearly bounded indeed. Since $0 < \lambda_0 \leq 1$ a homothetic transformation will transform the vertices $A^0_{i_1}, A^0_{i_2}, \ldots, A^0_{i_\ell}$ of $\lambda_0 \cdot \mathcal{O}$ into the points $\underline{t} + A_{i_1}, \underline{t} + A_{i_2}, \ldots, \underline{t} + A_{i_\ell}$ and also, the supporting half-spaces $\overrightarrow{H}^+_{i_j}$ $(1 \leq j \leq \ell)$ of B_0 will be translated into the supporting half-spaces $\overrightarrow{H}^+_{i_j}$ $(1 \leq j \leq \ell)$ of B having all the properties mentioned in the lemma. ∎

Definition. *For any convex body* B *of* E^{d} $(d \geq 2)$ *and* $L > 0$, *let* $\mathcal{G}_B = \{(d+1)$*-gon* $\Diamond$ *in* $E^{d} \mid$ *there is a translate of* $\Diamond$ *which is covered by* $B\}$,

$\mathcal{B}_L = \{(d+1)$*-gon* $\Diamond$ *in* $E^{d} \mid \ell(\Diamond) \leq L$, *where* $\ell(\Diamond)$ *means the length of* $\Diamond\}$.

We need the following technical assertion.

Lemma 5. *For any convex body* B *of* E^{d} $(d \geq 2)$ *exists a* $(d+1)$*-gon* $\Diamond^{*} \in \mathcal{G}_B \setminus \mathcal{G}_{\text{int } B}$ *(of* E^{d}*) such that* $\ell(\Diamond^{*}) = \inf\{\ell(\Diamond) \mid \Diamond$ *is a* $(d+1)$*-gon with* $\Diamond \notin \mathcal{G}_{\text{int } B}\} = \sup\{L > 0 \mid \mathcal{B}_L \subset \mathcal{G}_b\}$.

Proof. We omit the simple proof. ∎

Lemma 6. *The* $(d+1)$*-gon* $\Diamond^{*}$ *of Lemma 5 has a translate which is a billiard* $(d+1)$*-gon of* B.

Proof. Let $\Diamond^{*}$ be the $(d+1)$-gon $\overline{A_1^*A_2^*} \cup \ldots \cup \overline{A_{s-1}^*A_s^*} \cup \overline{A_s^*A_1^*}$ $(2 \leq s \leq d+1, d \geq 2)$ in E^{d}. We know that $\Diamond^{*} \notin \mathcal{G}_{\text{int } B}$ so because of Lemma 4 there exists a translate of $\Diamond^{*}$ by a vector $\underline{t}$ such that for certain vertices of $\underline{t} + \Diamond^{*}$ say, for $\underline{t} + A_{i_1}^*, \underline{t} + A_{i_2}^*, \ldots, \underline{t} + A_{i_\ell}^*(\{i_1, i_2, \ldots, i_\ell\} \subset \{1, 2, \ldots, s\}, 2 \leq \ell \leq s)$ we can find a nearly bounded set of supporting half-spaces $H_{i_1}^+, H_{i_2}^+, \ldots, H_{i_\ell}^+$ of B with $\underline{t} + A_{i_1}^* \in H_{i_1}^-, \underline{t} + A_{i_2}^* \in H_{i_2}^-, \ldots, \underline{t} + A_{i_\ell}^* \in H_{i_\ell}^-$ where $H_{i_1}^- = (E^{d} \setminus H_{i_1}^+) \cup \text{bd}\, H_{i_1}^+$, $H_{i_2}^- = (E^{d} \setminus H_{i_2}^+) \cup \text{bd}\, H_{i_2}^+, \ldots, H_{i_\ell}^- = (E^{d} \setminus H_{i_\ell}^+) \cup \text{bd}\, H_{i_\ell}^+$. First of all $\{i_1, i_2, \ldots, i_\ell\} = \{1, 2, \ldots, s\}$. Namely, suppose that say, $1 \notin \{i_1, i_2, \ldots, i_\ell\}$. Then because of Lemma 4 the new $(d+1)$-gon $\Diamond_1^{*} = \overline{A_2^*A_3^*} \cup \ldots \cup \overline{A_{s-1}^*A_s^*} \cup \overline{A_s^*A_2^*}$ cannot be covered by a translate of int B i.e. $\Diamond_1^{*} \notin \mathcal{G}_{\text{int } B}$ and $\ell(\Diamond_1^{*}) < \ell(\Diamond^{*})$ which is a contradiction. Hence we have a nearly bounded set of supporting half-spaces $H_1^+, H_2^+, \ldots, H_s^+$ of B with $\underline{t} + A_1^* \in H_1^-, \underline{t} + A_2^* \in H_2^-, \ldots, \underline{t} + A_s^* \in H_s^-$ where $H_1^- = (E^{d} \setminus H_1^+) \cup \text{bd}\, H_1^+, \ldots, H_s^- = (E^{d} \setminus H_s^+) \cup \text{bd}\, H_s^+$. Now we claim that $\underline{t} + A_i^* \in \text{bd}B \cap H_i^-$ for any $i \in \{1, 2, \ldots, s\}$. In order to prove it suppose that $\underline{t} + A_{i_0}^* \notin \text{bd}B \cap H_{i_0}^*$ for a certain $i_0 \in \{1, 2, \ldots, s\}$. Since $\Diamond^{*} \in \mathcal{G}_B$ therefore there exists a non-zero vector $\underline{t}^*$ such that $\{\underline{t}^* + \underline{t} + A_1^*, \underline{t}^* + \underline{t} + A_2^*, \ldots, \underline{t}^* + \underline{t} + A_s^*\} \subset B$. Consequently $\underline{t}^* + \cap_{i=1}^{s} H_i^+$ is covered by $\cap_{i=1}^{s} H_i^+$ with $\underline{t}^* \neq \underline{0}$. So using the fact that $\{H_i^+\}_{i=1}^{s}$ is nearly bounded we get immediately that there exists a real subset I of $\{1, 2, \ldots, s\}$ such that $\{H_i^+\}_{i \in I}$ is still a nearly bounded set of closed half-spaces. But then the new $(d+1)$-gon $\Diamond^{**}$ which passes through the vertices $\{\underline{t} + A_i^*\}_{i \in I}$ of $\underline{t} + \Diamond^{*}$ according to the natural order of the set $\{1, 2, \ldots, s\}$ has the property that $\Diamond^{**} \in \mathcal{G}_B \setminus \mathcal{G}_{\text{int } B}$ and $\ell(\Diamond^{**}) < \ell(\Diamond^{*})$ which is a contradiction. Hence $\underline{t} + A_i^* \in \text{bd}\, B \cap H_i^-$ with $H_i^- = (E^{d} \setminus H_i^+) \cup \text{bd}\, H_i^+$ for any $i \in \{1, 2, \ldots, s\}$ and the set $\{H_i^+\}_{i=1}^{s}$ of certain supporting half-spaces of B is nearly bounded in E^{d}. Finally we show that $\underline{t} + \Diamond^{*}$ is a billiard $(d+1)$-gon of B indeed. In the definition of billiard $(d+1)$-gons we required three properties. Now $H_i = bd\, H_i^+ = bd\, H_i^-$ for any $i \in \{1, 2, \ldots, s\}$. It is obvious that (1) and (2) is true for $\underline{t} + 0^*$. Considering the property (3) we have to distinguish the cases $s = 2$ and $s > 2$. In the first case $\underline{t} + 0^*$ is a double covered segment with endpoints $\underline{t} + A_1^*, \underline{t} + A_2^*$. Since the set $\{H_1^+, H_2^+\}$ has to be nearly bounded in E^d we obviously have that $H_1 \parallel H_2$. So because of the definition of 0^* we get that $\overrightarrow{A_1^* A_2^*} \parallel \underline{n}_1$ and $\overrightarrow{A_2^* A_1^*} \parallel \underline{n}_2$, which means that also (3) is satisfied in this case. For $s > 2$ suppose that there exists a vertex of $\underline{t} + 0^*$ say, $\underline{t} + A_2^*$ where (3) is not true. Then it is easy to see that there exists a point $\underline{t} + A_2^{**} \in H_2$ with the property that the new $(d+1)$-gon $^{**} = \overline{A_1^* A_2^{**}} \cup \overline{A_2^{**} A_3^*} \cup \ldots \cup \overline{A_{s-1}^* A_s^*} \cup \overline{A_s^* A_1^*} \notin \mathcal{F}_{\mathrm{int}\, B}$ and $\ell(0^{**}) < \ell(0^*)$, which is a contradiction. This completes the proof of the lemma. ∎

Definition. *For any convex body B of E^d $(d \geq 2)$, let* $m_b = \inf\{\ell(0_b) \mid 0_b$ *is a billiard $(d+1)$-gon of B*$\}$.

Lemma 7. $m_b = \ell(0^*)$ *where* 0^* *is the $(d+1)$-gon of Lemma 5.*

Proof. Because of Lemma 2 we know that $m_b \geq \ell(0^*)$. On the other hand Lemma 6 shows that $m_b \leq \ell(0^*)$. Consequently $m_b = \ell(0^*)$. ∎

Now we are in the position to prove the following.

Theorem 1. *Any closed curve of length one or less of E^d $(d \geq 2)$ can be covered by a translate of a given convex body $B \subset E^d$ if and only if* $m_b = \inf\{\ell(0_b) \mid 0_b$ *is a billiard $(d+1)$-gon of B*$\} \geq 1$.

Proof. We have proved (see Lemma 7) that $m_b = \ell(0^*)$ where the $(d+1)$-gon 0^* of E^d has the property (see Lemma 5) that $0^* \mathcal{F}_B \setminus \mathcal{F}_{\mathrm{int}\, B}$ and $\ell(0^*) = \inf\{\ell(0) \mid 0$ is a $(d+1)$-gon with $0 \notin \mathcal{F}_{\mathrm{int}\, B}\} = \sup\{L > 0 \mid \mathcal{L}_L \subset \mathcal{F}_B\}$. So the result follows using Lemma 3. ∎

Definition. *Let* $s_b(d) = \inf\{\ell(\Delta) \mid \Delta$ *is a real simplicical billiard path (of dimension d) of a d-simplex of minimal width 1*$\}$ *where $\ell(\Delta)$ means the length of the billiard path Δ $(d \geq 2)$.*

Theorem 2. *For any integer $d \geq 2$ we have* $w(d) = \frac{1}{s_b^*(d)}$ *where* $s_b^*(d) = \min\{s_b(2), s_b(3), \ldots, s_b(d)\}$.

Proof. Obviously, it is enough to prove that $s_b^*(d) = L^*(d) = \sup\{\ell^* > 0 \mid$ any convex body of E^d of minimal width 1 is a translation cover for the collection of closed curves of length $\leq \ell^*$ in $E^d\}$. From Theorem 1 we get that $s_b^*(d) \geq L^*(d)$. On the other hand we prove that $s_b^*(d) \leq L^*(d)$ i.e. any convex body B of E^d $(d \geq 2)$ of minimal width 1 is a translation cover for the collection of closed curves of length $\leq s_b^*(d)$. Prove it by induction on d. For

$d = 2$ $s_b^*(2) = \sqrt{3} < 2$ so it is easy to show that any convex domain of E^2 of minimal width 1 is a translation cover for the collection of closed curves of length $\leq \sqrt{3}$ (use Theorem 1 again). Now suppose that we have proved: $s_b^*(d') \leq L^*(d')$ for any $2 \leq d' < d$. We show that from these follows the inequality $s_b^*(d) \leq L^*(d)$. Consider an arbitrary convex body B of E^d of minimal width 1. Because of Theorem 1 B is a translation cover for the collection of closed curves of length $\leq s_b^*(d)$ if and only if $m_b = \inf\{\ell(0_b) | 0_b$ is a billiard $(d+1)$-gon of $B\} \geq s_b^*(d)$. So we have to prove that $m_b \geq s_b^*(d)$. Therefore let 0_b be an arbitrary billiard $(d+1)$-gon of B. If $\dim(\mathrm{aff}(0_b)) < d$, then by induction we get that $\ell(0_b) \geq s_b^*(d-1) \geq s_b^*(d)$ in which case we are done. So we are left with the case where $\dim(\mathrm{aff}(0_b)) = d$. This means that 0_b is a simplicial billiard path (of dimension d) of the convex body $B \subset E^d$. But then 0_b is a real simplicial billiard path of the d-simplex of minimal width ≥ 1 determined by the supporting hyperplanes of B passing through the vertices of 0_b consequently, $\ell(0_b) \geq s_b(d) \geq s_b^*(d)$ indeed. ∎

Because of Theorem 2 our first problem is equivalent to the following special

Problem 2. *Determine* $s_b(d)$ *for all* $d \geq 3$.

Finally we mention the following

Proposition 1. *For any integer* $d \geq 3$ *we have the inequalities:* $s_b(d) \geq \frac{d+1}{d}, s_b^*(d) \geq \frac{d+1}{d}, w(d) \leq \frac{d}{d+1}$.

Proof. The proof is based on the fact (see [2]) that any convex body of E^d $(d \geq 2)$ of minimal width 1 is a translation cover for the collection of arcs (i.e. of not necessarily closed curves) of length ≤ 1 in E^d. We leave the further details of the proof to the reader since they are simple. ∎

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