Note di Matematica Vol. X - n. 2, 273-278 (1990)

SOME LATTICE PROPERTIES OF VIRTUALLY NORMAL SUBGROUPS

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Abstract. *The set $vn(G)$ of subgroups with only finitely many conjugates in a group G is a sublattice of the lattice of all subgroups of G. Here groups G are studied for which $vn(G)$ is decomposable, complemented and relatively complemented.*

1. INTRODUCTION

A subgroup H of a group G is called *virtually normal* if it has only a finite number of conjugates in G, that is if the normalizer $N_G(H)$ has finite index in G. It is clear that the intersection and the join of two virtually normal subgroups is likewise virtually normal, so that the set $vn(G)$ of all virtually normal subgroups of G is a sublattice of the lattice $l(G)$ of all subgroups of G. It was shown by B. H. Neumann [5] that the lattices $l(G)$ and $vn(G)$ coincide if and only if G is a central-by-finite group. It is also clear that $vn(G)$ contains the lattice $n(G)$ of all normal subgroups of the group G, and in particular $vn(G) = n(G)$ if G has no proper subgroup of finite index.

In the first part of this paper, we shall characterize groups G for which the lattice $vn(G)$ is decomposable. The same problem for the lattice $l(G)$ and $n(G)$ was solved by Suzuki [8] and Curzio [1], respectively, while Franciosi and de Giovanni [2] considered groups G for which the ordered set $sn(G)$ of all subnormal subgroups is decomposable.

In the second part of the paper some complementation problems for the lattice $vn(G)$ will be studied, and in particular we shall describe groups G for which $vn(G)$ is either complemented or relatively complemented. The structure of groups G for which the subgroup lattice $l(G)$ is complemented was investigated by various authors; for results in this direction we refer to the book [9]. The behaviour of groups whose subgroups lattice $l(G)$ is relatively complemented has been described by Zacher [12] in the finite case, and by Menegazzo [4] for arbitrary soluble groups. It should also be noted that the lattice $n(G)$ is complemented (and then also relatively complemented) if and only if G is a direct product of simple groups (see [11]), and the same characterization can be given also for groups whose ordered set of subnormal subgroups is complemented (see [10]).

Our notation is mostly standard. In particular we refer to [7] for general properties of groups and to [9] for properties concerning lattices of subgroups. Moreover:

A group G is a K-group if its subgroup lattice $l(G)$ is complemented.

A group G is an RK-group if its subgroup lattice $l(G)$ is relatively complemented.

A group G is a T-group if each subnormal subgroup of G is normal.

The FC-centre of a group G is the subgroup of all elements of G which have only a finite number of conjugates, and G is an FC-group if it coincides with its FC-centre.

If G is a group, $\pi(G)$ denotes the set of prime divisors of orders of elements of G.

2. STATEMENTS AND PROOFS

Our first theorem describes the structure of groups whose lattice of virtually normal subgroups is decomposable.

Theorem A. *Let G be a group. The lattice $vn(G)$ is decomposable if and only if G is a non-trivial direct product $G = G_1 \times G_2$ and, if K_i is a subgroup of finite index of G_i and N_i is a normal subgroup of K_i such that the FC-centre F_i/N_i of K_i/N_i is not trivial $(i = 1,2)$, then the groups F_1/N_1 and F_2/N_2 are periodic and coprime.*

Proof. Suppose that $\varphi : vn(G) \to \mathscr{L}_1 \times \mathscr{L}_2$ is a lattice isomorphism, where $\mathscr{L}_1$ and $\mathscr{L}_2$ are non-trivial lattices. Clearly $\mathscr{L}_i$ has minimum O_i and maximum I_i $(i = 1,2)$. Consider the virtually normal subgroups $G_1 = \varphi^{-1}(I_1, O_2)$ and $G_2 = \varphi^{-1}(O_1, I_2)$. Then $G_1 \wedge G_2 = 1$ and $G_1 \vee G_2 = G$. As (G_1, G_2) is a $\wedge$-distributive pair of $vn(G)$, for any element $x \in G_2$ we have

$$G_1^x = G_1^x \wedge (G_1 \vee G_2) = (G_1^x \wedge G_1) \vee (G_1^x \wedge G_2) = G_1^x \wedge G_1,$$

so that $G_1^x \leq G_1$ and G_1 is normal in G. Similarly G_2 is a normal subgroup of G, so that $G = G_1 \times G_2$. Let K_i be a subgroup of finite index of G_i and let N_i be a normal subgroup of K_i such that the FC-centre F_i/N_i of K_i/N_i is not trivial $(i = 1,2)$. Put $N = N_1 \times N_2$, $K = K_1 \times K_2$, $E_1 = F_1 \times N_2$, $E_2 = F_2 \times N_1$ and $\overline{H} = H/N$ for any subgroup H/N of K/N. Clearly $\overline{F} = \overline{E}_1 \times \overline{E}_2$ is the FC-centre of $\overline{K}$. Let $\overline{H}$ be a cyclic subgroup of $\overline{F}$. As K has finite index in G, the subgroup H is virtually normal in G, and so we have

$$H = (H \wedge G_1) \vee (H \wedge G_2) = (H \wedge F_1) \vee (H \wedge F_2).$$

Therefore $\overline{H} = (\overline{H} \wedge \overline{E}_1) \vee (\overline{H} \wedge \overline{E}_2)$ for each cyclic subgroup $\overline{H}$ of $\overline{F}$. This proves that $(\overline{E}_1, \overline{E}_2)$ is a $\wedge$-distributive pair in the lattice $l(\overline{F})$ of all subgroups of $\overline{F}$, and hence $\overline{E}_1$ and $\overline{E}_2$ are periodic and coprime (see [9], p. 4). Since $\overline{E}_i \simeq F_i/N_i$ $(i = 1,2)$, this completes the first part of the proof.

Suppose now that $G = G_1 \times G_2$ has the required structure, and let H be a virtually normal subgroup of G. Then the normalizer $N = N_G(H)$ has finite index in G, so that also its core $L = N_G$ has finite index in G. The factor groups $G_1/(G_1 \wedge L)$ and $G_2/(G_2 \wedge L)$ are finite, and they have coprime orders. Therefore $(G_1L/L, G_2L/L)$ is an $\wedge$-distributive pair in the lattice $l(G/L)$ of all subgroups of the finite group G/L. Therefore

$$N = N \wedge (G_1L \vee G_2L) = (N \wedge G_1L) \vee (N \wedge G_2L) = L((N \wedge G_1) \vee (N \wedge G_2)).$$

It follows from a result of Curzio (see [1]) that $n(G) \simeq n(G_1) \times n(G_2)$, so that $L = (L \wedge G_1) \vee (L \wedge G_2) \leq (N \wedge G_1) \vee (N \wedge G_2)$. Thus $N = (N \wedge G_1) \vee (N \wedge G_2)$. As the indices $|G_1 : N \wedge G_1|$ and $|G_2 : N \wedge G_2|$ are finite, we have that $n(N \wedge G) \simeq n(N \wedge G_1) \times n(N \wedge G_2)$ (see [1]). Therefore $H = (H \wedge (N \wedge G_1)) \vee (H \wedge (N \wedge G_2)) = (H \wedge G_1) \vee (H \wedge G_2)$, and the lattice $vn(G)$ is isomorphic to $vn(G_1) \times vn(G_2)$. ∎

Lemma 1. *Let the group $G = A \times B$ be the direct product of two subgroups A and B, and let π be the natural projection of G onto B. If H is a subgroup of G such that $H \wedge A$ has a complement V in A and H^π has a complement W in B, the subgroup $K = V \times W$ is a complement of H in G.*

Proof. See [9], p. 29. ∎

Our next lemma is probably already known.

Lemma 2. *Let the group G be the direct product of a system of K-groups. Then G is a K-group.*

Proof. Suppose $G = Dr_{\alpha<\beta}K_\alpha$, where each K_α is a K-group and β is an ordinal number. Let H be a subgroup of G, and for every ordinal $\alpha \leq \beta$ put $G_\alpha = Dr_{\delta<\alpha}K_\delta$ and $H_\alpha = H \wedge G_\alpha$. Assume that $\alpha < \beta$ is an ordinal such that for each $\delta \leq \alpha$ there exists a complement V_δ of H_δ in G_δ such that $V_\mu \leq V_\delta$ if $\mu \leq \delta$. Let π be the natural projection of $G_{\alpha+1} = G_\alpha \times K_\alpha$ onto K_α, and let W be a complement of $H^\pi_{\alpha+1}$ in the K-group K_α. Then the subgroup $V_{\alpha+1} = V_\alpha \times W$ is a complement of $H_{\alpha+1}$ in $G_{\alpha+1}$ by Lemma 1. As the situation is clear for limit ordinals, it follows by induction on α that $H = H_\beta$ has a complement in $G = G_\beta$. Therefore G is a K-group. ∎

Lemma 3. *Let the group $G = A \times B$ be the direct product of two subgroups A and B. If the lattices $vn(A)$ and $vn(B)$ are complemented, then also $vn(G)$ is complemented.*

Proof. Let H be a virtually normal subgroup of G. Then $H \wedge A$ is a virtually normal subgroup of A, and hence there exists a virtually normal complement V of $H \wedge A$ in A. If $\pi : G \to B$ is the natural projection, the image H^π is a virtually normal subgroup of B, so that there exists a virtually normal complement W of H^π in B. Thus $K = V \times W$ is a complement of H in G by Lemma 1, and it is obviously a virtually normal subgroup of G. ∎

Theorem B. *Let G be a group. The lattice $vn(G)$ is complemented if and only if $G = H \times E \times C$, where H is a direct product of infinite simple groups, E is a finite K-group and C is a K-group which is a direct product of finite simple groups.*

Proof. Suppose that $vn(G)$ is a complemented lattice, and let F be the FC-centre of G. If M is a subgroup of finite index of G, there exists a virtually normal subgroup L of G

such that $M \vee L = G$ and $M \wedge L = 1$. Clearly L is finite, and hence it is contained in F, so that $G = MF$. Therefore the factor group G/F has no proper subgroups of finite index. Let H be a virtually normal complement of F in G. Then $H \simeq G/F$ has no proper subgroups of finite index, and hence $[F, H] = 1$. Thus $G = H \times F$, and the lattice $vn(H)$ is complemented. As every virtually normal subgroup of H is normal, it follows that H is a direct product of infinite simple groups (see [11]). Therefore we may assume that G is an FC-group. Let $A = Z(G)$ be the centre of G. Then A has a complement B in G, which is obviously normal. From $G = A \times B$ it follows that the abelian group A is periodic, and its primary components have prime exponent. As $Z(B) = 1$ and $vn(B)$ is complemented, we may suppose that G is an FC-group with trivial centre. Let U be an abelian normal subgroup of G, and let V be a virtually normal subgroup of G such that $UV = G$ and $U \wedge V = 1$. The normalizer $N = N_G(V)$ has finite index in G, and hence $G = \langle N, X\rangle$ for some finite subset X. Clearly $Z(N) \wedge C_G(X) \leq Z(G) = 1$, and so $Z(N)$ is finite, since G is an FC-group. Moreover $N = UV \wedge N = V \times (U \wedge N)$, so that $U \wedge N \leq Z(N)$ is also finite, and U is finite. This proves that G has no infinite abelian normal subgroups. Let J be a virtually normal complement of the socle S of G. If $J^{x_1}, \dots, J^{x_n}$ are the conjugates of J in G, the core J_G of J in G contains the subgroup $J \wedge (\cap_{i=1}^{n} C_G(x_i))$, so that J/J_G is finite. But J_G contains no minimal normal subgroups of the periodic FC-group G; hence $J_G = 1$ and J is finite. Thus also the normal closure J^G of J is a finite group. The socle S is a direct product $S = S_0 \times S_1$, where S_0 is abelian (and hence finite) and S_1 is the direct product of all non-abelian minimal normal subgroups of G. Since a minimal normal subgroup of an FC-group is finite, S_1 is a direct product of finite non-abelian simple groups. The normal subgroup $E = J^G S_0$ is finite and $S_1 = (E \wedge S_1) \times S_2$, where S_2 is a normal subgroup of G (see [7], Part 1, p. 179). Then $G = JS = ES = ES_1 = E \times S_2$, where E is a finite K-group and S_2 is a K-group by Lemma 2.

Conversely, let $G = H \times E \times C$, where H is a direct product of infinite simple groups, E is a finite K-group and C is a K-group which is a direct product of finite simple groups. As H has no proper subgroups of finite index, it follows from Theorem A that the lattices $vn(G)$ and $vn(H) \times vn(E \times C)$ are isomorphic. The lattice $vn(H) = n(H)$ is complemented, and hence Lemma 3 allows us to assume that $G = E \times C$. Write $C = C_0 \times C_1$, where C_0 is abelian and C_1 is a direct product of finite non-abelian simple groups. Clearly $E \times C_0$ is a centre-by-finite K-group, and hence by Lemma 3 it is enough to prove that $vn(C_1)$ is a complemented lattice. Therefore we may suppose that $G = Dr_{i\in I} G_i$ is a direct product of finite non-abelian simple groups. Let V be a virtually normal subgroup of G. Then the normalizer $N = N_G(V)$ has finite index in G, so that $G = \langle N, X\rangle$ for some finite subset X of G. Thus the subgroup $V \wedge C_G(X^G)$ is normal in G and has finite index in V. Hence $V = V_0 \times (Dr_{i\in J} G_i)$, where V_0 is finite and J is contained in I. Clearly $V_0 \leq Dr_{i\in Y} G_i$ for some finite subset Y of $I \setminus J$. Let W_0 be a complement of V_0 in the finite K-group

$Dr_{i\in Y}G_i$. Then $W = W_0 \times (Dr_{i\in I\setminus(J\cup Y)}G_i)$ is a virtually normal complement of V in G. The theorem is proved. ∎

If G is a group, a sublattice $\mathscr{L}$ of $l(G)$ containing 1 and G is said to be *permutably complemented* if for each $H \in \mathscr{L}$ there exists $K \in \mathscr{L}$ such that $HK = G$ and $H \wedge K = 1$. P. Hall [3] proved that a finite group G is a Hall K-group (i.e. the lattice $l(G)$ of all subgroups of G is permutably complemented) if and only if G is a supersoluble K-group. Using this result, Theorem B has the following consequence:

Corollary. *Let G be a group. The lattice $vn(G)$ is permutably complemented if and only if $G = H \times E \times C$, where H is a direct product of infinite simple groups, E is a finite Hall K-group and C is a periodic abelian group whose primary components have prime exponent.*

Remark. From Theorem B it follows that if G is a soluble group such that $vn(G)$ is complemented, then G is central-by-finite. In particular there exist soluble Hall K-group G for which $vn(G)$ is not complemented, as the following example shows.

Let H be an infinite elementary abelian 3-group, and let x be the inverting automorphism on H. Then the semidirect product $H \rtimes \langle x\rangle$ is a Hall K-group, but it is not central-by-finite.

Theorem C. *Let G be a group. The lattice $vn(G)$ is relatively complemented if and only if $G = H\times E\times C$, where H is a direct product of infinite simple groups, C is a periodic abelian group whose primary components have prime exponent and E is a finite soluble T-group whose Sylow subgroups are elementary abelian and such that the set of primes $\pi(E') \cap \pi(EC/E')$ is empty.*

Proof. Suppose that $vn(G)$ is relatively complemented. It follows from Theorem B that $G = H \times E \times C$, where H is a direct product of infinite simple groups, E is a finite K-group and C is a K-group which is a direct product of finite simple groups. It is well known that a finite RK-group is soluble, so that C is abelian and its primary components have prime exponent. Clearly the lattice $l(E \times C) = vn(E \times C) \simeq vn(G/H)$ is relatively complemented, and in particular E is soluble. From the structure of soluble RK-groups it follows that $E \times C$ is a T-group whose Sylow subgroups are elementary abelian (see [4], Th. 1.2). Moreover the set of primes $\pi(E') \cap \pi(EC/E')$ is empty (see [6]).

Conversely, it follows from Theorem A that $vn(G) \simeq vn(H) \times vn(E \times C) = n(H)\times l(E \times C)$, and hence it is enough to prove that $E\times C$ is an RK-group. As E is a T-group, every subnormal subgroup of E' is normal in $E \times C$, and then $E \times C$ is a T-group (see [6], Lemma 5.5.2). Moreover E' is a Hall subgroup of the finite soluble group E, so that each Sylow $\pi(E/E')$-subgroup of EC is a complement of E' in E. Therefore every Sylow $\pi(EC/E')$-subgroup of EC is a complement of E' in $E \times C$. Application of [4], Th. 1.2, gives that ExC is an RK-group. ∎

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Received April 9, 1990.

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