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AROUND THE QUOTIENT BORNOLOGICAL SPACES (*)

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Dedicated to the memory of Professor Gottfried Köthe

My research is over quotient bornological spaces. Few mathematicians understand why. Quotient spaces have applications. Here, I shall explain how I arrived at the category and describe it. For the last 10 years, I have worked over it. I have some 200 pages of «drafts», the grammar is sloppy, several misprints remain. But the theory holds. Around Christmas, I have ended my last draft. The material will be rewritten. I hope that it will be accepted.

In 1960, I did not know that my Thèse d'Agrégation was «quotient» (Thèse d'Agrégation is in German, Halilitätsschrift).

1. THE HOLOMORPHIC FUNCTION CALCULUS AND REGULAR ELEMENTS

I constructed my first holomorphic function calculus in 1953-1954, Doctorate (1953) published [1Wa] in 1954. It was developed in locally convex, complete algebras $\mathscr{A}$ with a joint continuous multiplication.

It was not based onGelfand, [1Ge], [2Ge], [3Ge], (Mat. Sbornik, July 1941). The U.S.S.R. was invaded on the 22 June 1941. The issues I needed were not in the libraries to which I went. I had read the Math. Reviews.

At the Collège de France, 1946, Leray spoke of Garding's symbolic calculus [Ga]. If $C \subset \mathbb{R}^n$ is a convex cone, $\delta'(C)$ is the space of temperate distributions with support in C, with the convolution. $\Gamma \subset \mathbb{R}_n$ is the polar of C, $\Theta(\Gamma)$ is the algebra of holomorphic functions on $\Gamma + i\mathbb{R}_n$ with a polynomial growth at infinity. The Laplace transformation is approximately an isomorphism $\Theta(\Gamma), \cdot \to \delta'(C), H$. I tried to develop a theory similar to Garding's.

I introduced «regular elements» $a \in \mathscr{A}$, $\exists M \in \mathbb{R}_+ :\ s - a$ is invertible if $s \in C$, $|s| > M$ and $((s-a)^{-1} \,|\, |s| > M)$ is bounded in $\mathscr{A}$. Later in my Thèse d'Agrégation, I have studied non regular elements (e.g. $\frac{\partial}{\partial x_i}$).

The spectrum of $a \in \mathscr{A}$, regular, is the complement of $(z \in C | \exists (s-a)^{-1}$, the inverse is regular). The resolvent $z \searrow (z-a)^{-1}$ is holomorphic on the resolvent set, the spectrum is compact and not empty.

Shilov [Sh] has developed a holomorphic function calculus in function of $(a_1, \dots, a_n)$, $a_i \in \mathscr{A}$, a commutative semi-simple Banach algebra generated topologically by $(b_1, \dots, b_k)$, (the algebra generated by $(b_1, \dots, b_k)$ is dense in $\mathscr{A}$). He used the Cauchy-Weil formula.

(*) Proofs not corrected by the author.

When the structure space is not connected, even when $\mathscr{A}$ is not semi-simple, he has constructed an idempotent e.

With their «trick» Arens and Calderon [AC] have shown that Shilov's results hold even when $\mathscr{A}$ is not generated topologically by a finite number of elements. With the same construction as Shilov, they solved equations other than «$x^2 - x = 0$».

My doctorate was between Shilov's and Arens and Calderon's papers. Even in their situation, I prefer my results. Shilov, Arens and Calderon constructed a mapping $\Theta(sp\,(a_1, \dots, a_n)) \to \mathscr{A}$, showed that it is a homomorphism if $\mathscr{A}$ is semi-simple. Using K. Oka [OK] and H. Cartan [Ca], I proved that their mapping is a homomorphism though the algebra is not semi-simple.

At present, I use the Gunning and Rossi [GR] lemma 7, chapter I, section VI, and what I call the Lemma 7bis.

Lemma 7. *If $P_1, \dots, P_N$ are polynomials of n indeterminates, $\Delta = ((z_1, \dots, z_n) | \forall i = 1, \dots, n : |z_i| \le 1,\ \lambda k = 1, \dots, N : |P_k| \le 1)$, the mapping $F(z, y) \in \Theta(D^{n+N})$ onto $F(z, P(z)) \in \Theta(\Delta)$ is surjective.*

They did not include the lemma 7bis. The kernel of the homomorphism is generated by the polynomials $y_k - P_k(z)$. (Appendix).

The following is not exactly my Doctorate. If $a \in \mathscr{A}$ is regular, its spectral radius is $\rho(a) = \sup_v \overline{\lim}\, v(a^n)^{1/n} = \max(|s| \mid s \in sp\ a)$. If $f = \sum_n f_n z^n$ is holomorphic on a neighbourhood of $\rho(a)D$, we can let $f(a) = \sum f_n a^n$. In my Doctorate I used the rationnally convex spectrum of $a = (a_1, \dots, a_n)$. Here I shall use its polynomially convex spectrum:

$$\widehat{sp}(a_1, \dots, a_n) = (z \in C^n \mid \forall P \in C[z_1, \dots, z_n] : |P(z) \le \rho(P(a)))$$

(The joint spectrum is not yet defined).

Let U be a neighbourhood of the polynomially convex spectrum. Polynomials $P_k(z) \in C[z_1, \dots, z_n] (k = 1, \dots, N)$, real numbers $r_i > \rho(a_i)$, $(i = 1, \dots, n)$, $R_k > \rho(P_k(a))$ $(k = 1, \dots, N)$ exist such that $\Delta \subset U$ if $\Delta = ((z_1, \dots, z_n) \mid \forall i : |z_i| \le r_1,\ \forall k : |P_k(z)| \le R_k)$.

With Taylor series, if $F \in \Theta(\prod r_i D, \prod R_k D)$ we can define $F(a_1, \dots, a_n, P_1(a), \dots, P_N(a))$. If $f \in \Theta(U)$, Gunning and Rossi's lemma 7 gives $F \in \Theta(\prod r_i D, \prod R_k)$ such that $f(z) = F(z, P(z))$ near to Δ. Lemma 7 bis says that F belongs to the ideal generated by the functions $y_k - P_k(z)$ if $F \in \Theta(\prod_i r_i D, \prod_k R_k D)$, $F(z, P(z)) = 0$, the Taylor series maps F onto $0 \in \mathscr{A}$.

We have a homomorphisms $\Theta(\Delta) \to \mathscr{A}$ mapping the functions z_i onto a_i and the constant function 1 onto the identity of $\mathscr{A}$. $\Theta(\widehat{sp}(a))$ is a union of algebras $\Theta(\Delta)$, we have a homomorphism $\Theta(\widehat{sp}(a)) \to \mathscr{A}$ mapping again z_i onto a_i and 1 onto 1 (the polynomials are dense in $\Theta(\widehat{sp}(a_1, \dots, a_n))$.

The Arens and Calderon trick comes now. Let $(b_1, \dots, b_m)$ be further regular elements of $\mathscr{A}$. Consider $\widehat{sp}(a_1, \dots, a_n, b_1, \dots, b_m) \subset C^{n+m}$, its projection $sp_b(a) \subset C^n$, and $\cap_b sp_b(a_1, \dots, a_n)$. Let f be holomorphic on a neighbourhood U of $\cap_b sp_b(a) \exists (b'_1, \dots, b'_m)$, regular elements of $\mathscr{A}$ such that the projection maps $\widehat{sp}(a, b')$ into U. The composition of the mapping $\Theta(U) \to \Theta(\widehat{sp}(a, b'))$ with the holomorphic function calculus $\Theta(\widehat{sp}(a, b') \to \mathscr{A}$ is a homomorphism $\Theta sp_{b'}(a)) \to \mathscr{A}$ mapping z_i (the variable) onto a_i and 1 onto 1.

Let $b_1", \dots, b"_{m"}$ be other elements of $\mathscr{A}$ such that the projection $C^{n+m"} \to C^n$ maps $\widehat{sp}(a, b")$ into U. We have another homomorphism $\Theta(U) \to \Theta(\widehat{sp}(a, b")) \to \mathscr{A}$. They factor through the homomorphism $\Theta(U) \to \Theta(\widehat{sp}(a, b', b"))$. They are equal. Using again a union, now $\Theta(\cap_b sp_b(a)) = \cup \Theta(U)$, we obtain a holomorphic function calculus, the set $\cap_b sp_b(a)$ is not empty.

An equivalent definition of the joint spectrum can be given. Let a_i be regular. Its joint spectrum $sp(a_1, \dots, a_n)$ is the complement of the resolvent set $\{s \in C^n | \exists (c_i, \dots, c_n) \in \mathscr{A}$, regular $1 = \sum_i (s_i - a_i) \cdot c_i\}$.

$\cap_b sp_b(a)$ is the complement of the set of the resolvent set. Assume that $\exists (c_i, \dots, c_n) \in \mathscr{A}$, regular such that $1 = \sum_i (s_i - a_i) \cdot c_i$. The set $\widehat{sp}(a, c)$ is contained in $((z, z') \in C^n | 1 = \sum_i (z_i - s_i) \cdot z'_i)$, $s \notin sp_c(a)$.

Let $b \in \mathscr{A}^m$, $s \notin sp_b(a)$, $\forall t \in C^m$, $(s, t) \notin \widehat{sp}(a, b)$. The polynomially convex joint spectrum is polynomially convex. $\forall t \in C^m : (s, t) \notin \widehat{sp}(a, b)$, $\exists P \in C[z, y]$, $P(s, t) = 0$, $\forall (z, z') \notin \widehat{sp}(a, b) : P(z, z') \neq 0$. The holomorphic function calculus gives $(P(a, a'))^{-1}$, and therefore regular elements $c_1, \dots, c_n$ of $\mathscr{A}$ such that $1 = \sum_i (s_i - a_i) \cdot c_i$.

The second definition is essentially Gelfand's.

Köthe has reviewed my papers in the Zentralblatt, his reviews where taken in the Mathematical Reviews.

2. BORNOLOGICAL ALGEBRAS

In 1956, I spoke to H. Cartan of my Doctorate, did not know that he should speak at the Bourbaki Seminar and did not go there.

I met him again after his talk. Bourbaki appreciated my machinery but it could be generalised. I considered complete algebras, could have used quasi-complete algebras.

More important, I used algebras with a joint continuous multiplication. The Banach-Steinhaus theorem shows that $B_1 \cdot B_2$ is bounded set in $\mathscr{A}$ if B_1 and B_2 are bounded in $\mathscr{A}$, quasi-complete algebra with a separately continuous multiplication.

Bourbaki wished the multiplication to be also quasi-continuous, if (x_i) and (y_i) are generalised sequences, $x_i \to x$, $y_i \to y$ and either (x_i) or (y_i) is bounded, then $x_i \cdot y_i \to x \cdot y$.

Cartan did not tell me, but Bourbaki knew that $\mathscr{L}_s(E)$, the continuous operators $E \to E$ with the simple (or strong) topology has a quasi-continuous multiplication if E is barreled;

$\mathscr{L}_s(E)$ is quasi-complete if E is also quasi-complete. Bourbaki could apply my machinery to closed commutative subalgebras of $\mathscr{L}_s(E)$, E barreled, quasi-complete.

I had used Cauchy sequences in the bornological sense: $\exists B$ bounded $\exists \varepsilon_n \to 0$ such that $x_n - x_{n'} \in \varepsilon_{n_0} B$ if $\forall n, n'1 > n_0$. Let E be a vector space, $B \subset E$ is completant if it is absolutely convex and its Minkowski functional (or gauge) is a Banach norm on the vector space E_B absorbed by B. Definition of a bornology is obvious. A boundedness is completant if $\forall B$ bounded in $E \exists B'$ bounded completant such that $B \subset B'$. A b-algebra is an algebra $\mathscr{A}$ with a completant boundedness and $B_1 \cdot B_2$ is bounded in $\mathscr{A}$ if B_1 and B_2 are bounded in $\mathscr{A}$.

We can define regular elements of a b-algebra. If $(a_1, \ldots, a_n)$ are regular, we can define its joint spectrum, the above construction is a homomorphism $\Theta(sp(a_1, \ldots, a_n)) \to \mathscr{A}$.

In 1956, I had not observed that one had not proved not that the homomorphism maps bounded subset of $\Theta(sp(a))$ onto a bounded subset of $\mathscr{A}$.

It does. The easiest is to use a result of Grothendieck [Gr] rewritten in a bornological language by H. Buchwalter [Bu]: Let $E, \mathscr{B}_E$ a b-space with a countable basis of boundedness; $B \subset E$ completant, compatible with all bounded completant subset of E is bounded in E.

[Two completant sets are compatible if $B \cap B'$ is completant, i.e. if $B + B'$ does not contain a non zero vector subspace of E.

[For Grothendieck, if B is a Banach ball in a vector space and (B_n) an increasing sequence of Banach balls B_n compatible with B such that $B = \cup_{n_0} B_{n_0}$ then B is absorbed by some B_{n_0}].

Let $(\underline{E}, \mathscr{B}_E)$ be a b-space with a countable basis of boundedness, $(\underline{F}, \mathscr{B}_F)$ another b– -space. A bounded mapping $u : E \to F$ such that $u(\underline{E}) = \underline{F}$ is then surjective in the bornological sense, i.e. $\forall C \in \mathscr{B}_F \exists B \mathscr{B}_E : u(B) = C$.

The boundedness of $\Theta(X_1 \times \ldots \times X_n \times Y_1 \times \ldots Y_N)$ has a countable basis. The mapping $F(z,y) \searrow F(z, P(z))$ is surjective $\underline{\Theta}(X_1 \times \ldots \times X_n \times Y_1 \times \ldots Y_N) \searrow \underline{\Theta}(\Delta)$, is therefore bornologically surjective. B bounded in $\Theta(\Delta)$ can be lifted to B' bounded in $\Theta(X_1 \times \ldots \times X_n \times Y_1 \times \ldots Y_N)$. The Taylor series maps B' into a bounded subset of $\mathscr{A}$. The h.f.c. maps B onto a bounded subset of $\mathscr{A}$.

My results can be applied to regular elements of a commutative unital b-algebra.

[A topologically vector space is complete enough if B is completant if it is closed, bounded, and absolutely convex].

All unital locally convex complete enough algebras $\mathscr{A}$ with a separately continuous multiplication become b-algebras: $\mathscr{A}_b$ has the same sets, with its von Neumann boundedness. (B is bounded in $\mathscr{A}_b$ if it is absorbed by all neighbourhood of zero in $\mathscr{A}$).

$\mathscr{L}(E)$, the algebra of continuous linear operators $E \to E$ with the equicontinuous boundedness is a b-algebra if E is a complete enough locally convex space.

If E is a b-space, $\mathscr{L}(E)$ the bounded mappings $E \to E$ with the equibounded boundedness is a b-algebra.

We can apply my results to many algebras.

3. ANOTHER APPLICATION OF BOUNDEDNESSES

Many functional Analysists consider a Banach space E and an operator $a : D(a) \to E$ whose resolvent set is not empty. We assume that $0 \notin sp\, a$, $E_{-1} = D(a)$, with the graph norm $\| x \|_{-1} = \max(\| x \|, \| ax \|)$, $E_0 = E$. Then $a = a_0$ is an isomorphism $a_0 : E_{-1} \to E_0$. We let $E_1 \simeq (E \oplus E)/gr\, a;\ gr\, a = ((x, ax) | x \in E_{-1})$. We embed $E_0 \subset E_1$, identify $x \in E$ with its class of equivalence in E_1. The operator $a_0 : E_{-1} \to E_0$ can be extended to $a_1 : E_0 \to E_1$, $sp(a_1) = sp(a_0)$.

By induction, we construct $E_{k+1} \simeq (E_k \oplus E_k)/gr\, a_k$, $a_{k+1} : E_k \to E_{k+1}$, imbed $E_k \subset E_{k+1}$; $E_\infty \simeq \cup_k E_k$, as a b-space and a linear bounded mapping $a_\infty : E_\infty \to E_\infty$. A large part of Functional Analysis about a is about $a_\infty : E_\infty \to E_\infty$, (resolvent set not empty). The operator a_∞ extends a.

4. A SINGLE NON REGULAR OPERATOR

The following is part of the preparation of my Thèse d'Agrégation. It is not in it, since more general results are there.

We let $w_0(s) = (1+|s|^2)^{-1/2}$. If S is open in C, $w_s(s) = \min(w_0(s), |z-s|\, |z \notin S)$. If w is Lipschitz $C^n \to R_+$ and $|z|w(z)$ is bounded, we let $S_w = (s \in C | w(s) > 0)$, $\Theta(w) = (f \in \Theta(S_w) | \exists N \in N,\ M \in R_+ : |w(s)^N f(s)| \leq M)$ with the obvious boundedness. It is a nuclear b-algebra.

$S \subset C$ is spectral for $a \in \mathscr{A}$ if $\lambda z \notin S$, $\exists (z-a)^{-1}$ and $(w_0(z)^N (z - a^{-1} | z \notin S)$ is bounded in $\mathscr{A}$. The resolvent is bounded on the complement of S, tends to zero for $z \to \infty$, $z \notin S$, it is Lipschitz on $C \setminus S$, can be extended to $\overline{C \setminus S}$, is holomorphic on the interior of $\overline{C \setminus S}$. The restriction of the resolvent to the interior of $C \setminus S$ (or of $\overline{C \setminus S}$) is holomorphic, $\mathscr{A}$-valued. Liouville shows that the spectral sets is a filter on C. It has an open basis.

From obvious identities, $\forall r \in N \exists B_r$ bounded completant such that the restriction of the resolvent to $\overline{C \setminus S}$ belongs to $\mathscr{C}^r(\overline{C \setminus S}, \mathscr{A}_{B_r})$.

A holomorphic function calculus can be constructed if a is regular, S is open spectral, with a smooth border of finite connectivity, i.e. $S = S_\infty \setminus \cup_1^N T_k$ S_∞ open simply connected $\forall k : T_k$ is closed and connected, $T_k \cap T_{k'} = \Theta$ if $k \neq k'$.

Let $f \in \Theta(w_s)$. For all $i \in N$, $\forall r_i(z)$ rational on C whose poles are off S, all the primitives of $f - r_i$, order $j, j \leq i$ are functions on S. Consider $D^i f(z) \in \Theta(S_w)$, a rational function $R_i(z)$ exists such that $D^i f - R_i$ has a primitive. Let F_{i-1} be one of these primitives. A function R_{i-1} exists such that $F_{i-1} - R_{i-1}$ has a primitive F_{i-2}. The rational function $r_i(z) = \sum_{j=1}^{i} D^j R_{i-j}$ is such that all primitives of $f - r_i$ are functions of $\overline{C \setminus S}$.

If $N \geq 2$ and $w_s(z)^N \cdot f(z)$ is bounded then $w_s(z)^{N-1} Pf(s)$ is bounded. If $w(z)f(z)$ is bounded then $\log w_s(z) Pf(z)$ is bounded. (Pf is any primitive of f if f has a primitive).

If $f \in \Theta(w_s)$, $\exists N' \in N$, $\exists F \in \mathscr{E}(\overline{S}, \mathscr{A})$, $F_{|S} \in \Theta(S)$, $\exists r$ rational on C whose poles are off S such that $f = (\frac{d^{N'}}{dz^{N'}} F) + r$. The Cauchy formula gives the derivative of order N' of F,

$$f(z) - r(z) = \frac{d^{N'}}{dz^{N'}} F(z) = \frac{1}{2\pi i} \int \frac{F(s)}{(s-z)^{N'+1}} ds.$$

We can let

$$f(a) = r(a) + \frac{N'!}{2\pi i} \int f(s)(s-a)^{-N'-1} ds$$

show that it is a linear bounded mapping, $\Theta(S_w) \to \mathscr{A}$, maps z onto a, the constant 1 onto 1, and is an algebraic homomorphism.

If S is open, not bounded, (a is not regular), if $f \in \Theta(S_w)$, we can consider $z^{-N''} f(z)$, N'' large enough. Successive primitives of $z^{-N''} f$, or of $(z^{-N''} f - r)$ give linear bounded mapping $\Theta(S_w) \to \mathscr{A}$. We must again show that it is a homomorphism, maps z onto a, 1 onto 1.

The Stokes formula is another construction of $f(a)$, $f \in \Theta(w_s)$, S spectral for a. For all $r \in N$, the resolvent function $z \searrow (z-a)^{-1}$ belongs to $\mathscr{E}^r(C \setminus S, \mathscr{A}_{B_r})$, B_r bounded completant; and $g(z) = (z-a)^{-1} = \sum_{k=1}^{r} z^{-k-1} a^k + z^{-r-1} a^{r+1} (z-a)^{-1}$ on $C \setminus S$. H. Whitney [Wh] has extended $g \in \mathscr{E}^r(\overline{C \setminus S})$ to $g_1 \in \mathscr{E}^r(C_\infty)$, the extension is such that $g_1(z) = \sum_{k=1}^{r} z^{-k-1} a^k + z^{-r-1} a^{r+1} h(z)$ with $h \in \mathscr{E}^r(C_\infty, \mathscr{A})$ ($C_\infty = CU(\infty)$).

If f is holomorphic on S and can be extended to a continuous function with a rectifiable border,

$$f(z) = \frac{1}{2\pi i} \int f(s)(s-z)^{-1} ds.$$

We replace z by a, $(s-z)^{-1}$ by $g(z)$, extend g to $g_1(z)$, and obtain

$$f(a) = \frac{1}{2\pi i} \int f(s) \overline{\partial} g_1(s, d\overline{s}) ds.$$

It is again bounded, $\Theta(w_s) \to \mathscr{A}$, maps z onto a and 1 onto 1.

It is a homomorphism: $\Theta(w_s \times w_s)$ is the b-algebra of holomorphic functions on $S \times S$ such that $\exists N : w_s(z)^N w_s(y)^N f(z,y)$ is bounded. The function $f(z) - f(y)$ belongs to the ideal generated by $z - y$ in $\Theta(w_s \times w_s)$: the quotient $(f(z) - f(y))/(z - y)$ has an eliminable singularity when $z = y$. We must check that it «does not grow too fast» when z and y grow to the border of S. If z and y are relatively far from each other, i.e. $|z - y| > \varepsilon w_s(z)$, then $1/|z - y| < M/w_s(z)$, $(f(z) - f(y))/(z - y)$ does not grow too fast.

if z and y are near each to the other, $|z - y| < \varepsilon' w_s(z)$, the Cauchy formula around a circle j_z of radius $\varepsilon' w(z)$ around z gives

$$f(z) - f(y) = \frac{z - y}{(2\pi i)^2} \int_{iz} \frac{f(s)}{(s - z)(s - y)} \mathrm{d}s.$$

The result is proved.

The functions $f(z)g(z)$ and $f(z)g(y)$ are congruent modulo the ideal generated by $z - y$; $(z - a)\overline{\partial} g_s$ and $(y - a)\overline{\partial} g_s$ are $\overline{\partial}$-coborders of functions of class $\mathscr{C}^r$ which vanish off S, $z - y = (a - y)(a - z)$. The h.f.c. constructed maps $f(z)g(y)$ onto $(f \cdot g)(a)$, its direct product maps $f(z)g(y)$ onto $f(a)g(a)$ hence $(f \cdot g)(a) = f(a)g(a)$.

5. SYSTEMS OF n OPERATORS

To construct the h.f.c. in $(a_1, \ldots, a_n) \in \mathscr{A}^n$, we must define «spectral sets» of $(a_1, \ldots, a_n)$. A set $S \subset C^n$ is spectral if $\exists u_i$ with polynomial growth such that $1 = \sum_i (z_i - a_i) u_i(s)$ off S. We must prove that the spectral sets is a filter, that the empty set is not spectral. I use a lemma, called Fundamental Lemma in my Thèse d'Agrégation [2Wa], and a Preliminary Lemma at a talk at the Operator Theory x meeting [7Wa].

Let $\mathscr{A}$ be a unital b-algebra, α a two-sided b-ideal, (u_k) a sequence of elements of $\mathscr{A}$, (v_k) a sequence of elements of α such that $\forall k : u_k + v_k = 1$, $\exists M \in R_+$, $u_k = 0_{\mathscr{A}}(M^k)$, $\exists r \in R_+$, $0 < r < 1$, $v_k = 0_\alpha(r^k)$. Then $\mathscr{A} = \alpha$.

In its proof, we use the fact that $\sum_k (Mr^N)^k < \infty$ if $Mr^N < 1$.

The lemma was again used to show that if $\exists u_i(z), y(z)$, $1 = \sum_i (z_i - s_i) + y(z)\ \ u_i(z)$, $y(z)$ of polynomial growth, $y(z)$ vanishes off S, $\exists u_i'(z)$, $\exists y'(z)$, of class $\mathscr{C}^r$ of $C^n \setminus S$, of polynomial growth whose derivatives have a polynomial growth, y' vanishes off S and $1 = \sum_i (z_i - a_i) u_i'(z) + y'(z)$.

The ideal was introduced here. It stayed there. For geometric reasons, I introduced $w : C^n \to R_+$ spectral if $1 = \sum_i (z_i - a_i) u_i(z) + v(z) + w(z) y_0(z)$, $u_i(z)$ and $y_0(z)$ of polynomial growth $C^n \to \mathscr{A}$, $v(z)$ of polynomial growth $C^n \to \alpha$.

$\forall w$ spectral for $(a_1, \ldots, a_n)$ mod α, $\exists w'$ Lipschitz, even of class $\mathscr{C}^r$, such that all its derivatives of order k, $|k| \leq r$ are bounded, w' spectral for $(a_1, \ldots, a_n)$ modulo α, such that $w' \leq w$. And $\exists u_i \in \mathscr{C}^r(C^n, \mathscr{A})$, $\exists v \in \mathscr{C}^r(C^n, \alpha)$ $\exists y_0 \in \mathscr{C}^r(C^n, \mathscr{A})$ such that $1 = \sum_i (z_i - a_i) u_i(z) + v(z) + w'(z) y_0(z)$. The derivatives of u_i and y_0 are bounded $\mathscr{A}$-valued, those of v are bounded α-valued.

If $w : C^n \to R_+$ is a Lipschitz spectral function, $f \in \Theta(w)$, I constructed $f(a)$ if $f \in \Theta(w)$. This was not an element of $\mathscr{A}$ but one of $\mathscr{A}/\alpha$.

Why introduce spectral functions?

$u \in \mathscr{C}^{\infty}(R_+, \mathscr{A})$ is a semi-group of class $\mathscr{C}^{\infty}$, $R_+ \to \mathscr{A}$, if $\forall s,t \in R_+$: $u(s+t) = u(s) \cdot u(t)$. Its infitesimal generator is $a = \frac{du}{dx}(0)$. The resolvent set of a can be empty, but a has an asymptotic resolvent, i.e. $1 = (z-a)u(z) + w(z)y_0(z)$, $w(z) = \min(w_0(z), exp(-Re\ z))$. And we can construct $exp(sa) = u(s)$ when $s > 0$.

6. THE BANACH SEMINAR

Before 1960, two French mathematicians (Ch. Houzel and Overaert) have thought of spaces with a structure defined by bounded sets. They were happy that I had used boundedness and results.

In 1962-63, Ch. Houzel has organized a seminar, the «Séminaire Banach» (with the collaboration of J.P. Ferrier, H. Jacquet, L. Gruson and G. Schiffman).

I would have liked to collaborate. I have gone every week to Paris between 1963 and 1967, the second semester. Their results have been written in 1967, and published in 1972 [Ho]. My Thèse d'Agrégation cannot be formulated in their language.

I considered a b-algebra $\mathscr{A} = (\overline{\mathscr{A}}, \mathscr{B}_{\mathscr{A}})$ and a b-ideal $\alpha = (\alpha, \mathscr{B}_{\alpha})$ of $\mathscr{A}$, i.e. $\underline{\alpha}$ is a two-sided ideal of $\underline{\mathscr{A}}$, $\mathscr{B}_{\alpha} \subset \mathscr{B}_{\mathscr{A}}$, and both $B \cdot C$ and $C \cdot B \in \mathscr{B}_{\alpha}$ if $B \in \mathscr{B}_{\mathscr{A}}$, $C \in \mathscr{B}_{\alpha}$.

The Banach Seminar did not study b-subspaces of b-spaces. Let $E = (\underline{E}, \mathscr{B}_E)$ and $F = (\underline{F}, \mathscr{B}_F)$ be b-spaces. Then F is a b-subspaces of E if $\underline{F}$ is a vector subspace of $\underline{E}$, and $\mathscr{B}_F \subset \mathscr{B}_E$.

[In the two paragraphs above, I use my present notations. A b-space is $E = (\underline{E}, \mathscr{B}_E)$, $\underline{E}$ is a vector space, $\mathscr{B}_E$ is a completant boundedness on $\underline{E}$].

7. QUOTIENT PROBLEMS

I did not have any application of the b-ideal. The computations went through modulo the ideal. I read Godement [Go] in 1962, learned what abelian categories are. No abelian category is applicable to Functional Analysis. A month or two later, I had the abelian category of quotient bornological spaces.

For three months,I tried to develop Functional Analysis in it, and could not do it, I did not have any application either. My Thèse d'Agrégation was «quotient» but that is not an application.

A few years later I gave my 1962 notes to G. Noël. In his Doctorate, 1969, [1No], he gave another definition of the category, wrote the tensor product of q-spaces [2No]. The functor is right-exact, as the tensor product of modules. He showed that all b_0-spaces can be embedded in q-spaces. (A b-space is a union of Banach spaces, a b_0-space is a union of normed space, a q-space is the formal quotient $E|F$ of a b-space by a b-subspace).

I gave another problem put in my Thèse d'Agrégation to I. Cnop, who also presented his Doctorate under my direction. If w Lipschitz, $C^n \to R_+$, -log$w$ p.s.h, is w spectral for $(z_1, \ldots, z_n)$ in the algebra $\Theta(w)$. Using a theorem of L. Hörmander [Hö], he said that it was true [Cn].

I went to Nancy, spoke with J.P. Ferrier, told him that the function $f(z) - f(y)$ belongs to the ideal generated by the functions $z_i - y_i$ if w is Lipschitz, -log$w$ p.s.h. He said that the difference belongs to the closure of the generated ideal. My Thèse d'Agrégation was «quotient». The difference belongs to the ideal. Ferrier needed the fact [Fe].

This is my first «quotient» application.

In 1977, at Warsaw, [3Wa], I spoke of quotient Banach spaces. Thanks to the closed graph theorem, we can identify a quotient Banach space with the vector space E/F, object of the category $qBan$, I spoke of a sketch of proof given by J.L. Taylor [Ta], which he could not use. His spectrum has the projection property. One should consider a vector space, it is not a Banach space. It is quotient Banach space, Taylor's sketch is valid. He gave another proof.

F.H. Vasilescu attended my talk. He is interested in Fréchet spaces, studies now quotient Fréchet spaces. He [1Va] and I [6Wa] have independently defined the category of quotient Fréchet spaces. Our categories are naturally isomorphic.

In 1977, I obtained another application. At the Banach Institute, Y. Domar spoke of ideals of the disc algebra $\mathscr{A}(D)$ such that Hull $\alpha \subset D^0$. Primary ideals whose hull belongs to the open disc are closed. (Primary ideals have a unique element in their hull).

The next day I said that ideals of $\mathscr{A}(D)$ whose hull is disjoint from the circle are closed. The essential of my answer is that if α is an ideal of a unital Banach algebra whose hull $X = X_0 \cup X_1$ is not connected, $\exists e \in \mathscr{A}$ such that $e^2 - e \in \alpha$, $e(m) = i$ if $m \in X_i$. [4Wa], [5Wa]. This is Shilov's theorem if α is closed.

8. OTHER APPLICATION

In 1967, [HH] Helton and Howe have constructed a functional calculus in function of a, b «nearly commuting» self-adjoint operators on a Hilbert space; a and b commute nearly if their commurator $[a, b]$ is of finite trace.

The essential spectrum of (a, b) is

$$sp_e(a,b) = \big((s,t) \in R^2 \,|\, (a+ib) - (s+it) \quad \text{is invertible in} \quad \mathscr{L}(H) \bmod \mathscr{K}(H)\big)$$

$\mathscr{K}(H)$ is the ideal of compact operators $H \to H$. They constructed $f(a, b) \in \mathscr{L}(H)/\tau_1(H)$ if $f \in \sigma\xi([sp_e(a,b)])$, a germ of class $\mathscr{E}^\infty$ on a neighbourhood of the essential spectrum. (If $E|F$ is a q-space, $\sigma(E|F) \simeq \underline{E}/\underline{F}$, $\xi([sp_e(a,b)])$ is a quotient bornological space).

M. Berkani's Doctorate [Be] comes from Helton and Howe's functional calculus.

The continuous germs, the germs of class $\mathscr{E}^{\infty}$, the germs of distributions near to a compact subset are not vector space even with a separated convergence structure. They are quotient bornological spaces.

Vasilescu [1Va] has observed that the hyperfunctions is a quotient Fréchet space. It is also a quotient bornological space. (Hyperfunctions come from Sato's papers 1958, 1959, [1Sa], [2Sa]; before my Thèse d'Agrégation).

The singularities of distributions on U, i.e. $\mathscr{D}'(U)|\xi(U)$, the distributions on U modulo the functions of class $\mathscr{E}^{\infty}$ is a q-space.

In Oberwolfach, 1981, J.F. Colombeau presented his New Generalised Functions, an algebra $\mathscr{G}(U)$ containing $\mathscr{D}'(U)$, $\mathscr{G}(U) \simeq \xi_M(\mathscr{D}(U)|\mathscr{N}(U)$, $\mathscr{N}(U)$ a b-ideal of $\xi_M(\mathscr{D}(U))$. He could not prove that $\mathscr{N}(U)$ is closed in $\xi_M(U)$, but $\mathscr{G}(U)$ is a q-algebra. He encouraged me creating the category of quotient bornological spaces and algebras.

The space of pseudo-differential operators modulo the regularising operators is a quotient space. I know that microlocalisation is a quotient problem, but cannot microlocalise.

9. THE CATEGORY AS IT IS

I must recall my notations. $E = (\underline{E}, \mathscr{B}_E)$ is a b-space: $\underline{E}$ is a vector space, $\mathscr{B}_E$ is a completant boundedness over $\underline{E}$. A b-subspace of E is $F = (\underline{F}, \mathscr{B}_F)$, $\underline{F}$ a vector subspace of $\underline{E}$, $\mathscr{B}_F$ is a completant boundedness over $\underline{F}$, $\mathscr{B}_F \subset \mathscr{B}_E$.

An object of the category q is a couple (E, F), E is a b-space, F is a b-subspace of E. We write $E|F = (E, F)$. q is a category, its objects are formal quotients: $E|F \simeq 0$ if $\underline{E} = \underline{F}$, $\mathscr{B}_E = \mathscr{B}_F$; $E|F \not\simeq 0$ though $\underline{E} = \underline{F}$ if $\mathscr{B}_F$ is stronger than $\mathscr{B}_E$ (i.e. $\mathscr{B}_F \subset \mathscr{B}_E$).

Vasilescu considers quotient Fréchet spaces. If E and F are Fréchet, and $\underline{E} = \underline{F}$ then $\mathscr{T}_E = \mathscr{T}_F$ (closed graph, open mapping theorem). One can say that $E|F$ = «$\underline{E}/\underline{F}$, object of the category qFré».

I have three isomorphic definitions of the category q. (I do not keep my 1962 definition).

a) Begin my defining $\widetilde{q}(E|F, E'|F')$. A strict morphism $u : E|F \to E'|F'$ is induced by $u_1 : \underline{E} \to \underline{E}'$, linear bounded mappins $E \to E'$ whose restriction to $\underline{F}$ is bounded $F \to F'$; u_1 induces the zero strict morphis iff it is linear bounded $E \to F'$. The strict morphisms are a category.

I also need «pseudo-isomorphisms» $s : E|F \to E'|F'$, strict morphism induced by $s_1 : E \to E'$, «bornologically surjective» $(\forall B' \in \mathscr{B}_E, \exists B \in \mathscr{B}_E : u(B) = B')$ such that $s_1^{-1} F' = F$ in the bornological sense (B is bounded in F if B is bounded in E and $s_1(B)$ is bounded in F').

All pseudo-isomorphisms are isomorphisms of q, a strict functor $\widetilde{\Phi} : \widetilde{q} \to Cat$ (Cat a category) can be extended to a functor $\Phi : q \to Cat$ if and only $\Phi(s)$ is an isomorphism of

Cat for all pseudo-isomorphisms s of $\tilde{q}$.

b) The second definition is due to Vasilescu. He considers the category qFré of quotient Fréchet spaces. A similar definition of the category q can be given.

A morphism $u : E|F \to E'|F'$ has a «lifted graph», a b-subspace of E, E' such that $G(u) \cap (E \times F') = F \times F'$ in the bornological sense, the restriction of the projection $E \times E' \to EG(u)$ is bornologically surjective.

Vasilescu considers quotient Fréchet spaces. A Fréchet subspace $F = (\underline{F}, \mathscr{T}_F)$ of a Fréchet space $E = (\underline{E}, \mathscr{T}_E)$ is a vector space $\underline{F}$ of $\underline{E}$, with a Fréchet topology $\mathscr{T}_F$ sucht that inclusion $F \to E$ is continuous. $F_1 \cap F_2$ is a Fréchet subspace if F_1 and F_2 are two Fréchet subspaces of E, its topology is the weakest stronger than those induced by those of F_1 and of F_2. And $F_1 + F_2$ is isomorphic to the quotient of $F_1 \oplus F_2$ by $((x, -x) | x \in \underline{F}_1 \cap \underline{F}_2)$. It is also a Fréchet subspace of E. The set of Fréchet subspaces of E is a lattice with $\cap$ as inter and $+$ as joint.

In a similar way, a lattice of b-subspaces of a b-space E can be defined. Its elements are the b-subspaces of E. The inter is $\cap$, $\underline{F}_1 \cap \underline{F}_2 = \underline{F_1 \cap F_2}$, $\mathscr{B}_{F_1 \cap F_2} = \mathscr{B}_{F_1} \cap \mathscr{B}_{F_2}$. And $\underline{F_1 + F_2} = \underline{F}_1 + \underline{F}_2$, B is bounded in $F_1 + F_2$ iff $\exists B_1 \in \mathscr{B}_{F_1}$, $\exists B_2 \in \mathscr{B}_{F_2}$ such that $B \subset B_1 + B_2$. We have a lattice of b-subspaces of E.

c) The third definition is due to G. Noël [1No]. If E is a b-space and X a set, we let $\underline{\beta}(X, E)$ the mappings $u : X \to E$ such that $u(X)$ is bounded in E. If $E|F$ is a q-space, $\sigma\beta(X, E|F) \simeq \underline{\beta}(X, E)/\underline{\beta}(X, F)$. If X and Y are two sets, we call $\ell^1(X, Y)$ the bounded linear mappings $u : \ell^1(X) \to \ell^1(Y)$. With the obvious composition ℓ^1 is a category. We make $\sigma\beta(X, E|F)$ a functor $(\ell^1)^{op} \times q \to q$. The category q is isomorphic to the category of morphism of functors $\ell^1 \to EV$, $E|F \searrow \sigma\beta(\cdot, E|F)$ (EV is the category of vector spaces and linear mappings).

b is a full subcategory of q: $b(E, F) \simeq q(EIO, FIO)$ if E, F are b-spaces. In b a complex $(u, v) : E \to F \to G$ is exact if $\forall B \in \mathscr{B}_F$, $v(B) = 0 \exists B_1 \in \mathscr{B}_E$ such that $u(B_1) = B$. It is exact in q iff it is exact in b. (A b-space E is identified to the q-space EIO).

If Cat is an abelian category, a functor $\Psi_1 : b \to Cat$ is exact iff $(\Psi_1(u), \Psi_1(v))$ is exact in Cat when (u, v) is exact in b. It can be extended in a unique way to an exacat functor $\Psi : q \to Cat$.

A right-exact functor $\Psi_1 : b \to Cat$ has a unique right exact extension $\Psi : q \to Cat$. No similar theorem is proved about left exact functors $b \to Cat$. I know left exact functors $\Psi_1 : b \to Cat$ with several not isomorphic left exact extensions Ψ, $\Psi' : q \to Cat$. I am convinced that a left exact functor $\Psi_1 : b \to Cat$ exists without extension $\Psi : q \to Cat$.

G. Noël has defined the tensor product $E|F \otimes_q E'|F'$ of q-spaces. It is right exact, often difficult to compute. Two Banach spaces E and F exist such that $E \otimes_q F$ is not a Banach space. Let E be a reflexive dimension of infinite dimension. The unit ball X of its dual space

E' with the topology induced by $\sigma(E', E)$ is a compact space. $\mathscr{C}(X)/E = F$ is a Banach space and $E \otimes_q F$ is a genuine Banach space.

He has proved that $E \otimes_q F \simeq E \widehat{\oplus} F$ if E is a Banach $\mathscr{L}_1$-space [5No]. If $\mathscr{N}$ is a nuclear b-space, $\mathscr{N} \otimes_q E \simeq \mathscr{N} \widehat{\oplus} E$.

The tensor product is right exact in the category q. A right exact functor respects direct limits. Bonet [Bo] has shown that the projective tensor product does not respect direct limits. I consider this a reason for studying the tensor product in the category q.

Definition of a sheaf is more difficult. A function f is continuous, is of class $\mathscr{C}^r$, is of class $\mathscr{C}^\infty$ if it has the property locally. The reader should know what sheaves are.

I can define $\mathscr{C}(X, E)$, X metrisable locally compact space. The functor $\mathscr{C}(X, \cdot) : b \to b$ is exact, can be extended to a functor $q \to q$. If Ω is a σ-finite measure space, we can define $L^p(\Omega, \cdot)$, it is an exact functor $b \to b$. It can be extended to an exact functor $q \to q$.

Both functors behave O.K. with the tensor product by a nuclear b-space:

$$\mathscr{N} \otimes_q \mathscr{C}(X, \cdot) \simeq \mathscr{C}(X, \mathscr{N} \otimes_q \cdot); \mathscr{N} \otimes_q L^p(\Omega, \cdot) \simeq L^p(\cdot, \mathscr{N} \otimes_q).$$

We must speak of sheaves. We know spaces of functions. If X is a topological space, for each U open in X, we can define $\mathscr{F}(U)$. If $V \subset U$, we have a restriction $\mathscr{F}(U) \to \mathscr{F}(V)$. If $W \subset V \subset U$, the restriction $\mathscr{F}(U) \to \mathscr{F}(W)$ is the composition of the restrictions $\mathscr{F}(U) \to \mathscr{F}(V) \to \mathscr{F}(W)$. This is a presheaf.

If $X = (\underline{X}, d_X)$ is metrisable, locally compact, $\mathscr{C}(\cdot, E|F) \simeq \mathscr{C}(\cdot, E)\mathscr{C}(\cdot, F)$ is a sheaf.

In his Doctorate, using the results of Frampton and Tromba [FT], B. Aqzzouz [Aq] has defined $\mathscr{C}(\cdot, E|F) \simeq \mathscr{C}(\cdot, E)\mathscr{C}(\cdot, F)$, when $r \in R_+/N$.

Also in his Docotrate, M. Berkani [Be] has defined $\mathscr{C}(\cdot, E|F)$ when $r \in N$, the manifold we study is of dimension 1. He used J.L. Taylor's construction, could have defined $\mathscr{C}^{r_1 \cdots r_n}$ or $\mathscr{B}^{r_1 \cdots r_n}$ if he had needed them.

Both Aqzzouz's and Berkani's presheaves are sheaves.

I define the sheaf of holomorphic functions. I regret that holomorphic functions do not have the unique continuation property.

The choice of sheaves and their construction have asked much time. No book can be written so long they are not chosen.

APPENDIX. PROOF OF LEMMA 7BIS

I have used the Gunning and Rossi lemma 7, chapter I, section VI and what I call the lemma 7bis. I shall prove the lemma 7bis, using the lemma 7.

Let $P_1, \dots, P_N$ be polynomials on n indeterminates, $D^{n+N} \subset C^{n+N}$, $\Delta = (z \in C^n | \forall i : |z_i| \leq 1, \forall k : |P_k(z)| \leq 1)$. Let F be holomorphic on a neighbourhood of D^{n+N}. Then $\varphi(F)(z) = F(z, P(z))$ is holomorphic on a neighbourhood of Δ. The mapping φ is surjective: $\Theta(D^{n+N}) \to \Theta(\Delta)$.

Lemma 7bis. *The kernel of the homomorphism* $\varphi : \Theta(D^{n+N}) \to \Theta(\Delta)$ *of algebras is generated by the polynomials* $y_k - P_k(z)$.

Let $F \in \Theta(D^{n+N}$, $F(z,P(z)) = 0$. Consider next $\varphi_N(F)(z_1,\dots,z_n,P_1(z),\dots,P_{N-1},y_N)$. It is holomorphic on a neighbourhood of $\Delta_{N-1} = ((z_1,\dots,z_n,y_1)||z_i| \leq 1$, $|y_1| \leq 1$, $\forall k = 1,\dots,N-1 : |P_k(z)| \leq 1)$.

The function $\varphi_N(F)(z,y_N)$ is such that $\varphi_N(F)(z,P_N(z)) = 0$. Therefore $g_N(z,y_N) = (\varphi_N(F)(z,y_N))/(y_N - P_N(z))$ is holomorphic on a neighbourhood of Δ_{N-1}. A function $G_N(z,y) \in \Theta(D^{n+N})$ exists such that $g_N(z,y_N) = G_N(z,P_1,\dots,P_{N-1},y_N)$, and $F(z,P_1(z),\dots,P_{N-1},y_N) = (y_N - P_N(z))G_N(z,P_1(z),\dots,P_{N-1}(z),y_N)$.

We have found functions $G_{k+1},\dots,G_N \in \Theta(D^{n+N})$ such that $F(z,P_1(z),\dots,P_k,y_{k+1},\dots,y_N) = \sum_{k+1}^N (y_r - P_r(z))G_r(z,P_1(z),\dots,P_k(z),y_{k+1},\dots,y_N)$ on Δ_{k+1}. We consider $\Delta_k = ((z_1,\dots,z_n,y_k,\dots,y_N)|\forall r = 1,\dots,k : |P_r(z)| \leq 1)$. The function $F(z,P_1(z),\dots,P_k(z),y_{k+1},\dots,y_N) - \sum_{k+1}^N (y_r - P_r(z))G_r(z,P_1(z),\dots,P_k(z),y_{k+1},\dots,y_N)$ vanishes when we replace y_k by $P_k(z)$. We let

$$g_k(z,y_{k+1},\dots,y_N) = \\ = \frac{F(z,P_1(z),\dots,P_{k-1}(z),y_k,\dots,y_N)}{y_k - P_k(z)} \cdot \\ \cdot \frac{-\sum_{k+1}^N (y_r - P_r(z))G_r(z,P_1(z),\dots,P_k(z),y_{k+1},\dots,y_N)}{y_k - P_k(z)}.$$

It is holomorphic on a neighbourhood of Δ_{k+1}, can be lifted to $G_k \in \Theta(D^{n+N})$. The lemma 7bis is proved by induction.

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