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ABEL'S FORMULA AND β-DUALITY IN SEQUENCE SPACES

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

The real Banach space m (resp. c) can be identified with the set of formal, real infinite series $x \sim \sum_{j=1}^{\infty} x_j$ such that x has bounded partial sums, $S_n(x) = \sum_{j=1}^{n} x_j$, $n \geq 1$ (resp. x is convergent). Let $\| x \| = \sup_n |S_n(x)|$, $x \in M \supseteq c$.

Recall that the $\beta - dual$ ([K], p. 453) of a linear space E of formal series $x \sim \sum x_j$ is the set of sequences $y = (y_1, y_2, \ldots)$ such that (x, y) exists for all $x \in E$, where

$$(1.1) \qquad (x,y) = \sum_{n=1}^{\infty} x_n y_n \quad (x \in E)$$

If $E = m$, the principle of uniform boundedness and Abel's formula

$$(1.2) \qquad \sum_{n=1}^{N} x_n y_n = \sum_{n=1}^{N-1} S_n(x) \left(y_n - y_{n+1}\right) + S_N(x) y_N$$

imply that each $y \in \beta$-dual (m) satisfies (a) $\sum_{n=1}^{\infty} |y_n - y_{n+1}| < \infty$ and (b) $\psi_0(y) \stackrel{\text{def}}{=}$ $\stackrel{\text{def}}{=} \lim_n y_n = 0$. In the case of $c \subseteq m$, the requirement (b) is replaced by (b') $\psi_0(y) =$ $= \lim_{n\to\infty} y_n$ *exists.* It follows that the map $y(\cdot)$ which assigns to $(u,t) \in l^1 \times \mathbf{R}$ the sequence $y_n(u,t) = t + \sum_{k=n}^{\infty} u_k$ is an isomorphism onto the β-dual of c with the image of $l^1 \times \{0\}$ being the β-dual of m.

In the present paper we shall define Banach sequence (series) spaces $c(\alpha) \subseteq m(\alpha)$, $0 < \alpha \leq 1$, and we shall observe that the β-dual, $l(\alpha)$, of $c(\alpha)$ stands in relation to $c(\alpha)$ and $m(\alpha)$ as $l(1) \simeq l^1 \times \mathbf{R}$ stands to $c = c(1)$ and $m = m(1)$. This relation will be exhibited through a characterization of $l(\alpha)$ in terms of a «generalized Abel's formula», (1.14) below.

In preparation of the definition of $c(\alpha)$ and $m(\alpha)$ let $\sum$ be the group of permutations of $\mathbf{N} = \{1, 2, \ldots\}$ with finite supports. $\sum$ acts upon m, preserving c, by the rule

$$(1.3) \qquad T_\sigma x \sim \sum_{j=1}^{\infty} x_{\sigma^{-1} j} \left(x \sim \sum_{j=1}^{\infty} x_j \in m \right)$$

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$\| \sigma \|$ denotes the operator norm of T_σ. Define $\| \cdot \|_\alpha$ on m by

$$\| x \|_\alpha = \sup_{\sigma \in \sum} \frac{\| T_\sigma x \|}{\| \sigma \|^\alpha}. \tag{1.4}$$

Now define $m(\alpha)$ and $c(\alpha) \subseteq m(\alpha)$ by

$$\begin{aligned} m(\alpha) &= \{x \in m \mid \| x \|_\alpha < \infty\} \\ c(\alpha) &= \left\{x \in m(\alpha) \mid \lim_{\|\sigma\| \to \infty} \frac{\| T_\sigma x \|}{\| \sigma \|^\alpha} = 0\right\}. \end{aligned} \tag{1.5}$$

If $\alpha = 1$, then $m(\alpha) = m$ and $\| \cdot \|_1$ reduces to $\| \cdot \|$. (In this paper the l^p-norms $\| \cdot \|_p$ will be avoided, and the notation (1.4) for $\| \cdot \|_\alpha$ should not cause confusion). It will be seen that $c(1) = c$. If $\alpha = 0$, then $m(\alpha) = m(0) = l^1$ is the space of absolutely summable sequences while $c(0) = \{0\}$. If $\epsilon < \alpha$, then clearly $m(\alpha - \epsilon) \subseteq c(\alpha)$. In particular, it will follow that $m(\alpha) \subseteq c$, $\alpha < 1$.

$l(\alpha)$, $0 < \alpha \leq 1$, denotes the β-dual of $c(\alpha)$. We shall find that $\psi_0(y) = \lim_n y_n$ exists for each $y \in l(\alpha)$, and we define $l_0(\alpha) = l(\alpha) \cap \psi_0^{-1} 0$. In the statement below * and ^ stand for Banach dual and β-dual, respectively.

Theorem 1.6. *Let $\alpha \in (0, 1]$, and let $l(\alpha) = c(\alpha)\hat{}$. Then*

$$\begin{aligned} & c(\alpha)^* = l(\alpha) \\ & l(\alpha)\hat{} = m(\alpha) \quad (\alpha < 1) \\ & l_0(\alpha)\hat{} = l_0(\alpha)^* = m(\alpha) \\ & l(\alpha)^* = m(\alpha) \oplus \mathbf{R}\psi_0. \end{aligned} \tag{1.7}$$

When $\alpha = 1$, only the second line of (1.7) requires modification (replace $m(1)$ by $c(1)$). As indicated earlier, Theorem 1.6 hinges on a generalized Abel formula which we shall now describe.

Define $\mathcal{F}_0$ to be the set of finite subsets of $\mathbf{N}$, and let $\mathcal{F} = \{F \subseteq \mathbf{N} \mid F \in \mathcal{F}_0$ or $F^c \in \mathcal{F}_0\}$. Define $r(\cdot)$ on $\mathcal{F}$ by $r(F)$ = «connectivity» of F, the number of maximal segments $[a, b) \subseteq F$, $1 \leq a < b \leq \infty$. Note that $\mathcal{F} = \{F \subseteq \mathbf{N} \mid r(F) < \infty\}$ and $|r(F) - r(F^c)| = 1$, $F \in \mathcal{F}$. We shall deal with classes of functions $z(\cdot)$ on $\mathcal{F}$, and in all cases it will be true that $z(\emptyset) = 0$.

If $0 \leq \alpha \leq 1$, define $\| z \|_\alpha^*$, z a real function on $\mathcal{F}$, by

$$\| z \|_\alpha^* = \sum_{F \in \mathcal{F}} |z(F)| r(F)^\alpha . \tag{1.8}$$

Also, define $Z(\alpha)$ by

$$Z(\alpha) = \{ z | \; \| z \|_\alpha^* < \infty \} . \tag{1.9}$$

Each $z \in Z(\alpha)$ is absolutely summable, and therefore the operator $R : Z(\alpha) \to$ sequences, defined by $y = Rz$ with

$$(Rz)_j = \sum_{F \in \mathcal{F}} \chi_F(j) z(F) \tag{1.10}$$

is well-defined. As $\lim_{j \to \infty} \chi_F(j)$ exists for all $F \in \mathcal{F}$, it is true that $\lim_{j \to \infty} (Rz)_j = \sum_{F \in \mathcal{F}_0^c} z(F)$ exists.

If $x \in c(1)$, then for each $F \in \mathcal{F}$ the number $x(F)$ is well-defined, where

$$x(F) = \sum_{j=1}^{\infty} \chi_F(j) x_j . \tag{1.11}$$

(This uses the fact $x \sim \sum_{j=1}^{\infty} x_j \in c(1)$ is a convergent series and $r(F) < \infty$, $F \in \mathcal{F}$.) We shall prove

Theorem 1.12. *If $0 < \alpha \leq 1$, then $l(\alpha)$ is the R-image of $Z(\alpha)$, i.e.*

$$l(\alpha) = RZ(\alpha) . \tag{1.13}$$

The duality between $l(\alpha)$ and $m(\alpha)$ (or $c(1)$ when $\alpha = 1$) is given by

$$(x, y) = \sum_{F \in \mathcal{F}} x(F) z(F) \quad (x \in m(\alpha),\ z \in Z(\alpha),\ y = Rz) . \tag{1.14}$$

Example 1.15. Let $y = (y_1, y_2, \ldots)$ be such that $\sum_{n=1}^{\infty} |y_n - y_{n+1}| < \infty$, and define $\psi_0(y) = \lim_{n \to \infty} y_n$. Define $z \in Z(1)$ by $z(\mathbf{N}) = \psi_0(y)$, $z([1, n]) = y_n - y_{n+1}$ and $z(F) = 0$ for all other $F \in \mathcal{F}$. For each n we have

$$(Rz)_n = \psi_0(y) + \left(y_n - \lim_{N \to \infty} y_N \right) = y_n$$

and therefore $Rz = y$. If $x \in c$, then Abel's formula says

$$(1.15) \qquad (x,y) = \sum_{n=1}^{\infty} x([1,n])z([1,n]) + x(\mathbf{N})z(\mathbf{N}).$$

By (1.14) it is true for *any* $z \in Z(1)$ such that $Rz = y$ that $(x,y) = \sum_{F\in\mathcal{F}} x(F)z(F)$ ((1.14)). We remark that (1.15) makes sense for $x \in m$ as soon as $z(\mathbf{N}) = \psi_0(y) = 0$. This remains true for all z such that $Rz = y$, $\psi_0(y) = 0$, with (1.14) in place of (1.15).

2. COMPUTATIONS WITH $\| \sigma \|$

If $x \in m$, define $Sx = \{S_n(x) | n \geq 1\}$ to be the sequence of partial sums. If $\sigma \in \sum$, the operator $ST_\sigma S^{-1}$ is an operator on the space of bounded *sequences* which, since $x \to Sx$ is an isometry, has the same operator norm ($=\| \sigma \|$) as T_σ. If a bounded sequence is taken to be a column vector, the operators S, T_σ and S^{-1} can be expressed in infinite matrix form. S (resp. S^{-1}) is lower triangular with $1's$ on and below the diagonal (resp. $1's$ on the diagonal, $-1's$ just below the diagonal and $0's$ elsewhere). T_σ is represented by $(T_\sigma)_{ij} = \delta_{j\sigma^{-1}i}$. It is clear $ST_\sigma S^{-1}$ is then a matrix whose entries are $0, \pm 1$. Therefore, $\| \sigma \|$ is the maximum number of nonzero entries (= maximum l^1-norm) of any row of $ST_\sigma S^{-1}$.

If $i, j \in \mathbf{N}$, the entry $(ST_\sigma S^{-1})_{ij}$ is 1 if $\sigma j \leq i$ and $\sigma(j+1) > i, -1$ if $\sigma j > i$ and $\sigma(j+1) > i$ and 0 otherwise. Define $\nu(\sigma)$ by

$$(2.1) \qquad \nu(\sigma) = \operatorname*{Max}_i \ \operatorname{Card} \left\{ j | \chi_{[1,i]}(\sigma j) \neq \chi_{[1,i]}(\sigma(j+1)) \right\}.$$

We have

Proposition 2.2. *The operator norm* $\| \sigma \| = \| T_\sigma \|$ *satisfies*

$$(2.3) \qquad \| \sigma \| = \nu(\sigma).$$

Let σ be extended from $\mathbf{N}$ to a PL-function on $[1, \infty)$, and let $\mu(\sigma)$ be the maximum number of intersections of the graph $G(\sigma)$ with horizontal lines. If i is chosen to maximize (2.1), then $\nu(\sigma)$ is precisely the number of times $G(\sigma)$ intersects the horizontal $y = i + \epsilon$, $0 < \epsilon < 1$. Therefore, $\mu(\sigma) \geq \nu(\sigma)$. On the other hand, if $\mu(\sigma)$ is the number of intersections of $G(\sigma)$ with a line of height $i + \epsilon$, $0 < \epsilon < 1$, each point of intersection accounts for a j such that $\chi_{[1,i]}(\sigma j) \neq \chi_{[1,i]}(\sigma(j+1))$. Therefore,

Proposition 2.4. *With notations as above*

$$(2.5) \qquad \nu(\sigma) = \| \sigma \| = \mu(\sigma).$$

Let $r(\cdot)$ be the «connectivity» function on $\mathcal{F}$ as in section 1. Define $\rho(\sigma)$, $\sigma \in \sum$, by

$$(2.6) \qquad \rho(\sigma) = \operatorname{Max}_i r\left(\sigma^{-1}[1,i]\right).$$

Fix i to maximize (2.6) and let $r^{-1}([1,i]) = \bigcup_{k=1}^{\rho}[a_k, b_k)$, $a_1 < b_1 < a_2 < \ldots < b_\rho$. The graph $G(\sigma)$ crosses the horizontal $y = i + \epsilon$, $0 < \epsilon < 1$, twice on each complementary interval $[b_k, a_{k+1})$, $k < \rho$, once on $[b_\rho, \infty)$ and, when $a_1 > 1$, once on $[1, a_1]$. It follows that $\mu(\sigma) \geq 2\rho(\sigma) - 1$ in all cases. Conversely, if $G(\sigma)$ crosses $y = i + \epsilon$ $\mu(\sigma)$ times, it must be that $r(\sigma^{-1}[1,i]) = [\frac{\mu(\sigma)-1}{2}]$. Therefore,

$$(2.7) \qquad 2\rho(\sigma) - 1 \leq \| \sigma \| \leq 2\rho(\sigma) + 1.$$

In particular, $\rho(\sigma)$ and $\| \sigma \| = \mu(\sigma) = \nu(\sigma)$ are of the same magnitude.

Lemma 2.8. *Let $x \in m$, and suppose*

$$(2.9) \qquad \lim_{\|\sigma\| \to \infty} \frac{\| T_\sigma \|}{\| \sigma \|} = 0.$$

Then $x \sim \sum_{j=1}^{\infty} x_j$ is a convergent series.

Proof. To say x is not convergent is to say that there exists $\epsilon > 0$ and, replacing x by $-x$ and relettering, if necessary, $a_1 < b_1 < a_2 < b_2 < \ldots$ such that $\epsilon < x([a_k, b_k)) = \sum_{j=a_k}^{b_k-1} x_j$. For each N define $i_N = \sum_{k=1}^{N}(b_k - a_k)$ and let $\sigma_N \in \sum$ be constructed so that (1) $\sigma_N(\bigcup_{k=1}^{N}[a_k, b_k)) = [1, i_N]$, (2) σ_N is supported on $[1, i_N] \cup \bigcup_{k=1}^{N}[a_k, b_k)$ and (3) σ_N is monotone on $\bigcup_{k=1}^{N}[a_k, b_k)$ and on its complement in $[1, i_N]$. $G(\sigma_N)$ has at most $2N$ (maximal) intervals of negative slope, and therefore $\| \sigma_N \| = 0(N)$. As $\| T_\sigma x \| \geq x(\sigma^{-1}[1, i_N]) > N\epsilon$, (2.9) cannot be true. This is a contradiction, and we conclude x is convergent.

Proposition 2.10. *If $\alpha = 1$, then $c(\alpha) = c(1)$ is the space c of convergent series.*

Proof. Lemma 2.8 and the definition (1.3) imply $c(1) \subseteq c$. Conversely, let $x \sim \sum_{j=1}^{\infty} x_j$ be a convergent series. Given $\epsilon > 0$ choose i so that $|x([a,b))| < \epsilon$ if $[a,b) \subseteq [i, \infty)$. If

$\sigma \in \sum$ is such that $\| T_\sigma x \| > \| x \|$, there exists i_0 such that $\| T_\sigma x \| = |x(\sigma^{-1}[1, i_0])|$. As $|x(I)| < \epsilon$ for any component I of $\sigma^{-1}[1, i_0]$ which is contained in $[i, \infty)$, we have

$$\| T_\sigma x \| \leq \sum_{j=1}^{i} |x_j| + r\left(\sigma^{-1}\left[1, i_0\right]\right)\epsilon.$$

Now divide by $\| \sigma \|$ and let $\| \sigma \| \to \infty$. By (2.7) the lim sup in (2.9) is at most ϵ. Letting $\epsilon \to 0$, the proposition obtains.

In what follows we use $e_i \sim \sum_{j=1}^{\infty} \delta_{ij}$ to denote the «unit vectors» in m.

Proposition 2.11. *The set $\{e_i | i \in \mathbb{N}\}$ is a Schauder basis for $c(\alpha), 0 < \alpha \leq 1$. More precisely, if $x \sim \sum_{j=1}^{\infty} x_j \in c(\alpha)$, then $x = \sum_{j=1}^{\infty} x_j e_j$ in $\| \cdot \|_\alpha$.*

Proof. Fix $0 < \alpha < 1$ and $x \in c(\alpha)$. Let $\epsilon_0(t)$, $t > 0$ be a function such that

$$\begin{aligned} &\| T_\sigma x \| \leq \epsilon_0(\| \sigma \|) \, \| \sigma \|^\alpha \quad \left(\sigma \in \sum\right) \\ &\lim_{t \to \infty} \epsilon_0(t) = 0. \end{aligned} \tag{2.12}$$

Given $\epsilon > 0$, use the fact x is a convergent series (Lemma 2.8) to find n such that $|x(I)| < \epsilon$ for any segment $I \subseteq [n, \infty)$. Let $\widehat{x} = x - \sum_{j=1}^{n-1} x_j e_j \sim \sum_{j=n}^{\infty} x_j$. We shall prove that

$$\| \widehat{x} \|_\alpha \leq \epsilon^\alpha + 2\epsilon_0\left(\epsilon^{\alpha-1}\right). \tag{2.13}$$

Since $\alpha < 1$, (2.13) and (2.12) yield the desired result. (For $\alpha = 1$ the statement is obvious and standard.) To establish (2.13) let $\sigma \in \sum$ be such that $\| T_\sigma \widehat{x} \| > \| \widehat{x} \|$ (since $\| \widehat{x} \| < \epsilon$!), and choose N so that $\| T_\sigma \widehat{x} \| = |\widehat{x}(\sigma^{-1}[1, N])|$. Decompose $\sigma^{-1}[1, N]$ into $r = r(\sigma^{-1}[1, N])$ maximal segments $I_1, \ldots, I_r$, from left to right, and let $s \leq r$ be the least s, if any, such that $I_s \cap [n, \infty) \neq \emptyset$. Replace I_s by $I_s \cap [n, \infty)$ and reletter so that now $\widehat{x}(I_j) = x(I_j)$, $s \leq j \leq r$ and

$$\| T_\sigma \widehat{x} \| = \left| \sum_{j=s}^{r} x\left(I_j\right) \right| \leq (r - s + 1)_\epsilon. \tag{2.14}$$

Construct $\tau \in \sum$ as in (1)-(3) of the proof of Lemma 2.8 so that

$$\| T_\sigma \widehat{x} \| = |T_\tau x([1, A])| = \left|x\left(\tau^{-1}[1, A]\right)\right| \tag{2.15}$$

where $\tau(\bigcup_{j=s}^{r} I_j) = [1, A]$. By (2.12) and (2.14) we have

$$\| T_\sigma \widehat{x} \| \leq \text{ Min } \left((r - s + 1)\epsilon,\ \epsilon_0(\| \tau \|) \| \tau \|^\alpha\right) .$$

By construction $\| \tau \| \geq r - s + 1$ and $\| \sigma \| \geq r \geq \frac{1}{2} \| \tau \|$. If $(r - s + 1)\epsilon > \epsilon^\alpha$, then $\frac{\|T_\sigma \widehat{x}\|}{\|\sigma\|^\alpha} \leq 2\epsilon_0(\epsilon^{\alpha-1})$, while if $(r - s + 1)\epsilon \leq \epsilon^\alpha$, $\| T_\sigma \widehat{x} \| \leq \epsilon^\alpha$. Now (2.13) is true, and the proposition is proved.

Corollary 2.16. *Let $l(\alpha)$ be the β-dual of $c(\alpha)$. Then $l(\alpha)$ is also the Banach dual of $c(\alpha)$.*

Proof. Clear.

3. EQUIVALENT NORMS ON $m(\alpha)$

Associate to each $\pi = (\pi_1, \pi_2, \ldots) \in \mathbf{N}^{\mathbf{N}}$ the natural partition of $\mathbf{N}$ into segments $I_j = I_j(\pi)$, $j \geq 1$, ordered from left to right with $|I_j(\pi)| = \pi_j$. π determines a contraction $Q_\pi : m \to m$ by

$$Q_\pi x \sim \sum_{j=1}^{\infty} x\left(I_j(\pi)\right) \qquad (x \in m)$$

(Since $I_j(\pi)$ is finite, $x(I_j(\pi))$ makes sense.) The partial sums of $Q_\pi x$ being a subsequence of the partial sums of x we have $\| Q_\pi x \| \leq \| x \|$ for all π, x.

$\sum$ acts upon $\pi \in \mathbf{N}^{\mathbf{N}}$ by $(\sigma\pi)_j = \pi_{\sigma-1j}$. Given $\sigma \in \sum$ and $\pi \in \mathbf{N}^{\mathbf{N}}$, define $\tau = \tau(\sigma, \pi) \in \sum$ by requiring (a) $\tau I_{\sigma-1j}(\pi) = I_j(\sigma\pi)$ and (b) τ is monotone on $I_k(\pi)$ for each k. Since $|I_j(\sigma, \pi)| = \pi_{\sigma-1j} = |I_{\sigma-1j}(\pi)$ (a) makes sense. Observe the relation

$$T_\sigma Q_\pi = Q_{\sigma\pi} T_\tau \qquad (\tau = \tau(\sigma, \pi)). \tag{3.1}$$

Indeed, $(T_\sigma Q_\pi x)_j = (Q_\pi x)_{\sigma^{-1}j} = x(I_{\sigma^{-1}j}(\pi)) = x(\tau^{-1} I_j(\sigma\pi)) = (T_\tau x)(I_j(\sigma\pi)) = (Q_{\sigma\pi} T_\tau x)_j$.

Lemma 3.2. *If $\sigma \in \sum$ and $\pi \in \mathbf{N}^{\mathbf{N}}$, and if $\tau = \tau(\sigma\pi)$ is as above, then*

$$\| \sigma \| \leq \| \tau \| \leq \| \sigma \| + 2. \tag{3.3}$$

Proof. Define $k(i)$, $i \geq 1$, to be the number of j such that $I_j(\sigma\pi) \subseteq [1, i]$. Let $I_{k+1}^0 = I_{k+1}(\sigma\pi) \cap [1, i]$, and define $I_{\sigma^{-1}(k+1)}^0 = \tau^{-1} I_{k+1}^0$. We have $\tau^{-1}[1, i] = I_{\sigma^{-1}(k+1)}^0 \cup$

$\cup \bigcup_{j=1}^{k(i)} I_{\sigma^{-1}j}(\pi)$. As $I^0_{\sigma^{-1}(k+1)} \neq I_{\sigma^{-1}(k+1)}(\pi)$, by definition, we have $r(\sigma^{-1}[1,k(i)]) \leq$ $\leq r(\tau^{-1}[1,i]) \leq r(\sigma^{-1}[1,k(i)]) + 1$. When the left equality holds, $G(\tau)$ and $G(\sigma)$ intersect the horizontals $y = i + \frac{1}{2}$ and $y = k(i) + \frac{1}{2}$, respectively, the same number of times. When the right equality holds, $G(\tau)$ and $G(\sigma)$ intersect the horizontals $y = i + \frac{1}{2}$ and $y = k(i) + \frac{1}{2} + 1$, respectively, either the same number of times or else l times for $G(\tau)$ and $l-2$ times for $G(\sigma)$. (The latter occurs when $I_{\sigma^{-1}(k+1)}(\pi)$ joins the left side of some $I_{\sigma^{-1}j}(\pi)$, $j \leq k(i)$.) Now (3.3) is proved.

Proposition 3.4. *For each $\pi \in \mathbf{N}^{\mathbf{N}}$ and $\alpha \in [0,1] Q_\pi$ has norm at most 2^α on $m(\alpha)$.*

Proof. Fix $\pi \in \mathbf{N}^{\mathbf{N}}$, and below let $\tau = \tau(\sigma, \pi)$ as σ varies in $\sum$. Lemma 3.2 implies

$$\begin{aligned} \| Q_\pi x \| &= \sup_{\sigma \in \sum} \frac{\| T_\sigma Q_\pi x \|}{\| \sigma \|^\alpha} = \\ &= \sup_{\sigma \in \sum} \frac{\| Q_{\sigma\pi} T_\tau x \|}{\| \sigma \|^\alpha} \leq \\ &\leq \sup_{\sigma \in \sum} \left(\frac{\| \sigma \| + 2}{\| \sigma \|} \right)^\alpha \frac{\| T_\tau x \|}{\| \tau \|^\alpha}. \end{aligned} \tag{3.5}$$

Since $\sigma = \mathrm{Id}$ implies $\tau = \mathrm{Id}$, and since $\sigma \neq \mathrm{Id}$ implies $\| \sigma \| \geq 3$, the proposition follows.
If $x \in m$ and $F \in \mathcal{F}_0$ (i.e., $|F| < \infty$), define $|x|(F) = \sum_{j \in F} |x_j|$.

Lemma 3.6. *There exists a constant $c < \infty$ such that if $x \in m(\alpha)$, then*

$$\sup_{\substack{F \in \mathcal{F}_0 \\ F \neq \emptyset}} \frac{|x|(F)}{|F|^\alpha} \leq c \| x \|_\alpha . \tag{3.7}$$

Proof. Order the positive terms in x as $x_{n_1} \geq x_{n_2} \geq \ldots$ For each N define $\sigma_N \in \sum$ to have support $[1,N] \cup \{n_1, \ldots, n_N\}$ and to satisfy $\sigma_N n_j = j$. As $\| \sigma_N \| = O(N)$ is clear, we have

$$N x_{n_N} \leq \sum_{j=1}^{N} x_{n_j} \leq \| T_{\sigma_N} x \| \leq \| \sigma_N \|^\alpha \| x \|_\alpha = O(N^\alpha) \| x \|_\alpha .$$

It follows $x_{n_N} = O(N^{\alpha-1})$. A similar argument for the negative terms and the fact $\sum_{j=1}^k j^{\alpha-1} = O(k^\alpha)$ establishes (3.7). The lemma is proved.

As a corollary to the estimate $x_{n_N} = O(N^{\alpha-1})$ we have

Proposition 3.8. *If* $\alpha < 1$, *then*

$$m(\alpha) \subseteq \bigcap_{p > \frac{1}{1-\alpha}} l^p. \tag{3.9}$$

Proposition 3.4 and Lemma 3.6 imply that the norm $|||\cdot|||'_\alpha$ is dominated by $\|\cdot\|_\alpha$, where

$$|||x|||'_\alpha = \sup_{\substack{\pi \in \mathbf{N}^{\mathbf{N}} \\ \emptyset \neq F \in \mathcal{F}_0}} \frac{|Q_\pi x|(F)}{|F|^\alpha}. \tag{3.10}$$

By the closed graph theorem the equivalence of $|||\cdot|||'_\alpha$ and $\|\cdot\|_\alpha$ is a consequence of

Lemma 3.11. *If* $x \in m$ *is such that* $|||x|||'_\alpha < \infty$, *then* $\| x \|_\alpha < \infty$.

Proof. Let $\sigma \in \sum$ be such that $\| T_\sigma x \| > \| x \|$. Choose n such that $\| T_\sigma x \| = |x(\sigma^{-1}[1,n])|$ and suppose $\sigma^{-1}[1,n] = \bigcup_{j=1}^r I_j$, where $r = r(\sigma^{-1}[1,n])$. By (2.7) $r = O(\| \sigma \|)$. Choose $\pi \in \mathbf{N}^{\mathbf{N}}$ with $I_j = I_{l_j}(\pi)$, $1 \leq j \leq r$, and let $F = \{l_1, \ldots, l_r\}$. We have

$$\| T_\sigma x \| = |x\left(\sigma^{-1}[1,n]\right)| \leq |Q_\pi x|(F) \leq |F|^\alpha |||x|||'_\alpha = r^\alpha |||x|||'_\alpha = O(\| \sigma \|^\alpha) |||x|||'_\alpha.$$

Therefore, $\| x \|_\alpha < \infty$ (and $\| x \|_\alpha \leq C|||x|||'_\alpha$ for a universal constant $C < \infty$). The lemma is proved.

As noted earlier, Lemma 3.6 and 3.11 imply

Proposition 3.12. *The norms* $\|\cdot\|_\alpha$ *and* $|||\cdot|||'_\alpha$ *are equivalent on* $m(\alpha)$.

The norm $|||\cdot|||'_\alpha$ is closely related to a norm $|||\cdot|||_\alpha$ which we define by

$$|||x|||_\alpha = \sup_{\substack{F \in \mathcal{F}_0 \\ F \neq \emptyset}} \frac{|x(F)|}{r(F)^\alpha}. \tag{3.13}$$

In fact,

Proposition 3.14. *If* $0 \leq \alpha \leq 1$, *then*

$$\frac{1}{2}|||x|||'_\alpha \leq |||x|||_\alpha \leq |||x|||'_\alpha. \tag{3.15}$$

Proof. To prove the right hand inequality, let $\emptyset \neq F = \bigcup_{j=1}^{r} I_j \in \mathcal{F}_0$, with $r = r(F)$, and select $\pi \in \mathbf{N}^{\mathbf{N}}$ with $I_j = I_{l_j}(\pi)$, $1 \le j \le r$. Let $E = \{l_1, \dots, l_r\}$, and observe

$$|x(F)| \le |Q_\pi x|(E) \le |E|^\alpha |||x|||'_\alpha = r(F)^\alpha |||x|||'_\alpha .$$

Therefore, $|||x|||_\alpha \le |||x|||'_\alpha$. To prove the left-hand inequality let $\pi \in \mathbf{N}^{\mathbf{N}}$ and $F \in \mathcal{F}_0$, and select $E \subseteq F$ such that $(Q_\pi x)_j$, $j \in E$, has constant sign and

$$|Q_\pi x(E)| \ge \frac{1}{2} |Q_\pi x|(F).$$

In general, $r(E) < |E|$, but we may alter π without changing $|Q_\pi x|(F)$ or $Q_\pi x(E)$ so that $r(E) = |E|$; simply collapse maximal segments of E to points using a «coarser» π. This alteration decreases $|F|$. We now have

$$\frac{|Q_\pi x|(F)|}{|F|^\alpha} \le 2 \frac{|Q_\pi x(E)|}{|E|^\alpha} = 2 \frac{|Q_\pi x(E)|}{r(E)^\alpha} \le 2 |||x|||_\alpha .$$

Now sup over $F \in \mathcal{F}_0$. The proposition is proved.

We conclude this section with a remark. Let $0 < \alpha < 1$, and define $\lambda(\alpha) \subseteq m(\alpha)$ by

(3.16) $$\lambda(\alpha) = \left\{ x \in m | Q_\pi x \in l^{\frac{1}{1-\alpha}}, \text{ all } \pi \in \mathbf{N}^{\mathbf{N}} \right\}.$$

It is easy to see directly that each $x \in \lambda(\alpha)$ is a convergent series, and this plus a sliding hump argument implies $\| x \|_{\lambda(\alpha)} < \infty$, $x \in \lambda(\alpha)$, where

(3.17) $$\| x \|_{\lambda(\alpha)} \sup_{\pi \in \mathbf{N}^{\mathbf{N}}} \| Q_\pi x \|_p \qquad \left(p = \frac{1}{1-\alpha} \right).$$

(On the right side of (3.17) the $\| \cdot \|_p$ -norm is the traditional l^p-norm.) Hölder's inequality implies $||| \cdot |||'_\alpha$ is dominated by $\| \cdot \|_{\lambda(\alpha)}$, but the two are not equivalent. Also, the sliding hump argument yields

$$\lim_{N \to \infty} \sup_{\pi \in \mathbf{N}^{\mathbf{N}}} \sum_{j=N}^{\infty} | (Q_\pi x)_j |^p = 0 \qquad (x \in \lambda(\alpha)).$$

The reader can check that this implies $\lambda(\alpha) \subseteq c(\alpha)$. The results of this section, including Propositions 3.4 and 3.8, imply

(3.18) $$\lambda(\alpha) \subseteq c(\alpha) \subseteq m(\alpha) \subseteq \bigcap_{\beta > \alpha} \lambda(\beta).$$

All inclusions are strict. For example, $x \sim \sum_{j=1}^{\infty} (-1)^j j^{\alpha-1}$, $\alpha < 1$, belongs to $c(\alpha)$ but not to $\lambda(\alpha)$. And $x \sim \sum_{j=1}^{\infty} (-1)^j (\log) j^{\alpha-1}$, $\alpha < 1$, belongs to $\lambda(\beta)$, $\beta > \alpha$, but not to $m(\alpha)$.

4. DUALITY

The objects $\|\cdot\|_\alpha^*$, $Z(\alpha)$ and R have been defined in (1.9)-(1.11). The formal adjoint of R is

$$(4.1)\qquad \left(R^*x\right)(F) = \sum_{j=1}^{\infty} \chi_F(j)x_j = x(F).$$

The natural domain of R^* is $c(1) = c$. In this section we shall establish the expected relation

$$(4.2)\qquad (x, Rz) = \left[R^*x, z\right] \qquad (x \in m(\alpha), z \in Z(\alpha), \alpha < 1$$

where

$$(4.3)\qquad \begin{aligned} (x,y) &= \sum_{j=1}^{\infty} x_j y_j \\ [w,z] &= \sum_{F\in\mathcal{F}} w(F)z(F). \end{aligned}$$

When $\alpha = 1$, (4.2) is true for $x \in c(1) = c$ and $z \in Z(1)$.

Let $N \geq 1$, and define $\mathcal{F}_N = \{F \in \mathcal{F} | F \cap [1,N] \neq \emptyset\}$. $\mathcal{F}_N \nearrow \mathcal{F} - \{\emptyset\}$ and $\mathcal{F}_N^c \searrow \{\emptyset\}$ as $N \to \infty$. Recalling that $x(\emptyset) = 0 = z(\emptyset)$, we shall ignore $\emptyset$.

If $F \in \mathcal{F}_N$ define $A_F = F \cap [1,N]$ and $B_F = F \cap [N+1,\infty)$. If $x \in m(\alpha)$ ($c(1)$ when $\alpha = 1$) and $z \in Z(\alpha)$, $[R^*x, z]$ exists as an absolutely convergent sum. Moreover, because $x(F) = x(A_F) + x(B_F)$ with $r(A_F)$, $r(B_F) \leq r(F)$, we can write

$$(4.4)\qquad \begin{aligned} \sum_{F\in\mathcal{F}_N} x(F)z(F) &= \sum_{F\in\mathcal{F}_N} x\left(A_F\right)z(F) + \sum_{F\in\mathcal{F}_N} x\left(B_F\right)z(F) = \\ &= \sum_{A\subseteq[1,N]}\sum_{F\in\mathcal{F}_N} x(A)z(F)\delta_{AA_F} + \sum_{F\in\mathcal{F}_N} x\left(B_F\right)z(F) = \\ &= \sum_{j=1}^{N} x_j(Rz)_j + \sum_{F\in\mathcal{F}_N} x\left(B_F\right)z(F). \end{aligned}$$

Let $\epsilon > 0$ and $r < \infty$ be fixed. Since x above is a convergent series, N can be chosen so that

$$(4.5)\qquad |x(B)| < \epsilon \qquad \left(B \in \mathcal{F}_N^c, r(B) \leq r\right).$$

We divide the sum on the right in (4.4) according to $r(B_F) \le r$ and $r(B_F) > r$. The contribution from the first grouping is dominated by $\epsilon \| z \|_\alpha^*$. As for the second grouping

$$(4.6) \qquad \Big| \sum_{\substack{F\in\mathcal{F}_N \\ r(B_F)>r}} x(B_F)\, z(F) \Big| \le |||x|||_\alpha \sum_{\substack{F\in\mathcal{F}_N \\ r(F)>r}} |z(F)| r(F)^\alpha$$

which tends to zero as $r \to \infty$. Finally, $[R^*x, z]$ differs from the left side of (4.4) by a sum over $\mathcal{F}_N^c$:

$$\Big| \sum_{F\in\mathcal{F}_N^c} x(F) z(F) \Big| \le |||x|||_\alpha \sum_{F\in\mathcal{F}_N^c} |z(F)| r(F)^\alpha = o(1)$$

where $o(1)$ is as $N \to \infty$. Collecting results, we have proved

Proposition 4.7. *If* $0 \le \alpha < 1$ *and* $x \in m(\alpha)$, $z \in Z(\alpha)$, *or if* $\alpha = 1$ *and* $x \in c(1)$, $z \in Z(1)$, *then*

$$(4.8) \qquad (x, Rz) = [R^*x, z]$$

where $(\cdot,\cdot)$ *and* $[\cdot,\cdot]$ *are defined by (4.3).*

5. CHARACTERIZATION OF $l(\alpha)$

Recall that $l(\alpha)$ is defined to be the β-dual of $c(\alpha)$. If $\varphi \in c(\alpha)^*$, define $y_i = \varphi(e_i)$, $e_i \sim \sum \delta_{ij}$. Since $x = \sum_{i=1}^\infty x_i e_i$, the series converging in the $c(\alpha)$ norm, we have

$$(5.1) \qquad \varphi(x) = \sum_{i=1}^\infty x_i y_i = (x, y) \qquad (x \in c(\alpha)).$$

Conversely, every $y \in l(\alpha)$ defines $\varphi \in c(\alpha)^*$ by the uniform boundedness principle. In what follows we shall characterize $l(\alpha)$ as $RZ(\alpha)$.

Lemma 5.2. *If* $x \in m(\alpha)$, *then* $x \in c(\alpha)$ *if, and only if,*

$$(5.3) \qquad \lim_{r(F)\to\infty} \frac{x(F)}{r(F)^\alpha} = 0.$$

Proof. If $F = \bigcup_{j=1}^r I_j$, $r = r(F)$, is a union of maximal segments, then $x(F) = (T_\sigma x)[1, n]$ for an appropriate n and $\sigma \in \sum(\sigma^{-1}[1, n] = F)$. As we have seen before $\| \sigma \|$ and $r(F)$ are comparable, and therefore (5.3) is true if $x \in c(\alpha)$. Conversely, if x satisfies (5.3), and if $\sigma \in \sum$ is such that $\| T_\sigma x \| > \| x \|$, select n such that $\| T_\sigma x \| = |x(\sigma^{-1}[1, n])|$. Then $\| \sigma \|$ dominates $r(\sigma^{-1}[1, n])$, and therefore (5.3) implies $\lim_{\|\sigma\|\to\infty} \frac{\|T_\sigma x\|}{\|\sigma\|^\alpha} = 0$. The lemma is proved.

Set up the space $\mathcal{F}\times\mathbb{N}$, and embed $\mathcal{F}-\{\emptyset\}$ in the product as the graph of $r(\cdot)$. Define X to be the closure of this graph when $\mathcal{F}$ is endowed with the product topology as a subset of $\{0,1\}^N$. X is precisely the set

$$X=\{(F,r)\in\mathcal{F}\times\mathbb{N}\,|\,F=\emptyset \text{ and } r\geq 1 \text{ or } F\neq\emptyset \text{ and } r\geq r(F)\} \tag{5.4}$$

X is a countable, locally compact metric space.

Each $x\in m(\alpha)$, $\alpha<1$, determines a function f_x on $\mathcal{F}\times\mathbb{N}$, where

$$f_x(F,r)=x(F). \tag{5.5}$$

Lemma 5.6. *The map $x\to f_x|_X$ sends $m(\alpha)$, $\alpha<1$, to a subspace of $C(X)$.*

Proof. It is sufficient to prove that if $\lim_k(F_k,r(F_k))=(F,r)$, then $\lim_k x(F_k)=x(F)$. Clearly, $r(F_k)=r$, large k, and $r(F)\leq r$. This means one or more segments of F_k may «slide» to ∞. Since x is a convergent series and $r(F_k)$ is bounded, $x(F_k)\to x(F)$ (=0 if $F=\emptyset$). The lemma is proved.

Define $B_\alpha(X)$ to be the set of continuous functions f on X such that $f(\emptyset,r)=0$ and

$$\|f\|_{\alpha,\infty}=\sup_{\substack{(F,r)\in X\\ F\neq\emptyset}}\frac{|f(F,r)|}{r^\alpha}<\infty. \tag{5.7}$$

Define $C_\alpha(X)$ to be the closed supspace of $B_\alpha(X)$ consisting of f such that

$$\lim_{(F,s)\to\infty}\frac{f(F,s)}{s^\alpha}=0. \tag{5.8}$$

The map $f\to\frac{f(F,s)}{s^\alpha}$ sends $C_\alpha(X)$ isometrically onto a subspace of the continuous functions vanishing at ∞ on X. It follows readily that $C_\alpha(X)^*$ is identified with the space of functions w on X such that

$$\|w\|_\alpha^*=\sup_{\substack{(F,r)\in X\\ F\neq\emptyset}}|w(F,r)|r^\alpha<\infty \tag{5.9}$$

(and $w(\emptyset,r)=0$, $r\geq 1$).

It is clear that if $\alpha<1$, the map $x\to f_x$ is an isometry between $(m(\alpha),|||\cdot|||_\alpha)$ and a subspace of $(B_\alpha(X),\|\cdot\|_{\infty,\alpha})$ which sends $c(\alpha)$ to a closed subspace of $C_\alpha(X)$ (even when $\alpha=1$).

Theorem 5.10. *Let $0 < \alpha \leq 1$, and let $Z(\alpha)$ be as in section 1. Then*

$$RZ(\alpha) = l(\alpha) \tag{5.11}$$

where $l(\alpha)$ is the β-dual of $c(\alpha)$.

Proof. Proposition 4.7 implies $RZ(\alpha) \subseteq l(\alpha)$. For the reverse inclusion let $y \in l(\alpha)$ determine $\varphi \in c(\alpha)^*$. Regard $c(\alpha)$ as a closed subspace of $C_\alpha(X)$, and make a Hahn-Banach extension Φ of φ. Φ is represented by w with $\| w \|_\alpha^* < \infty$ ((5.9)). We have setting $z(F) = \sum_{(F,s)\in X} w(F,s)$ (a sum over $s \geq r(F)$ if $F \neq \emptyset$)

$$(x,y) = \varphi(x) = \Phi\left(f_x\right) = \sum_{(F,r)\in X} x(F)w(F,r) = \sum_{F\in\mathcal{F}} x(F)z(F).$$

Now $|z(F)|r(F)^\alpha = |\sum_{s\geq r(F)} w(F,s)r(F)^\alpha|$, and therefore $\| z \|_\alpha^* \leq \| w \|_\alpha^* = \| \Phi \|$. ($\| z \|_\alpha^*$ refers to the norm on $Z(\alpha)$.) It follows $y = Rz$, $z \in Z(\alpha)$, and the theorem is proved.

We remark that because $\| z \|_\alpha^* \geq \| \varphi \| = \| \Phi \| = \| w \|_\alpha^*$ above, it must be that $w(F,s) = = 0$, $s > r(F)$. Also, we have proved that $||| \cdot |||_\alpha$ and $||| \cdot |||_\alpha^*$ ((1.10)) are dual norms.

6. $l(\alpha)^* = l_0(\alpha)^* \oplus \mathbf{R}\,\psi_0 = m(\alpha) \oplus \mathbf{R}\,\psi_0$.

We begin with the observation that the Banach dual to $(Z(\alpha), \| \cdot \|_\alpha^*)$ can be identified with the set $W(\alpha)$ of functions $w(\cdot)$ on $\mathcal{F}$ such that $w(\phi) = 0$ and $\| w \|_\alpha^{**} < \infty$ where

$$\| w \|_\alpha^{**} \sup_{\substack{F\in\mathcal{F}\\ F\neq\emptyset}} \frac{|w(F)|}{r(F)^\alpha}. \tag{6.1}$$

Remark 6.2. If $\alpha < 1$ and $x \in m(\alpha)$, or if $\alpha = 1$ and $x \in c(1)$, then

$$\| R^* x \|_\alpha^{**} = |||x|||_\alpha. \tag{6.3}$$

It is only necessary to observe that R^*x is defined on $\mathcal{F}_0 \cup \mathcal{F}_0^c = \mathcal{F}$ and the definition (3.13) is unchanged when $\mathcal{F}_0$ is replaced by $\mathcal{F}$.

Lemma 6.4. *Let $z(\cdot)$ be summable on $\mathcal{F}$, and assume $Rz = 0$. Then*

$$\sum_{F\in\mathcal{F}_0^c} z(F) = 0. \tag{6.5}$$

Proof. Define $(R_0 z)_j = \sum_{F\in\mathcal{F}_0} x_F(j) z(F)$. Clearly, $\lim_{j\to\infty}(R_0 z)_j = 0$. For each N we have

$$0 = (Rz)_N = (R_0 z)_N + \sum_{F\in\mathcal{F}_0^c} \chi_F(N) z(F) = o(1) + \sum_{F\in\mathcal{F}_0^c} \chi_F(N) z(F).$$

As $\lim_{N\to\infty} \chi_F(N) = 1$, $F \in \mathcal{F}_0^c$, (6.5) follows from the bounded convergence theorem.

Now we shall analyze $l(\alpha)^*$, assuming $\alpha < 1$. To this end fix $\psi \in l(\alpha)^*$, and define $\Psi = \psi \circ R \in Z(\alpha)^*$. As noted in connection with (6.1), Ψ is represented by an element $w(\cdot) \in W(\alpha)$, and w satisfies

$$(6.6) \qquad [w, z] = 0 \qquad (z \in Z(\alpha) \cap \ker R).$$

To exploit (6.6) let $F_1, F_2 \in \mathcal{F}$, and construct a test element $z \in \ker R$, supported on the four points $F_1, F_2, F_1 \cup F_2$ and $F_1 \cap F_2$, by

$$z(F) = \begin{cases} 1 & F = F_1 \cup F_2, F_1 \cap F_2 \\ -1 & F = F_1, F_2 \\ 0 & \text{otherwise} \end{cases}.$$

Since $Rz = 0$, (6.6) implies

$$(6.7) \qquad w(F_1 \cup F_2) = w(F_1) + w(F_2) - w(F_1 \cap F_2).$$

Since $w(\phi) = 0$, $w(\cdot)$ is additive on the ring $\mathcal{F}_0$. Define $x_j = w(\{j\})$, and observe that by definition (3.13)

$$(6.8) \qquad |||x|||_\alpha \leq \| w \|_\alpha^{**} = \| \Psi \| = \| \psi \|.$$

It follows that $x \in m(\alpha)$, and since by assumption $\alpha < 1$, $R^*x \in W(\alpha)$. Let $w_0 = w - R^*x$. By construction, w_0 is supported on $\mathcal{F}_0^c$. If $F \in \mathcal{F}_0^c$, then $F^c \in \mathcal{F}_0$, and (6.7) implies

$$(6.9) \qquad w_0(\mathbf{N}) = w_0(F) + w_0(F^c) = w_0(F).$$

That is, w_0 is constant on $\mathcal{F}_0^c$, and we have for $y = Rz$

$$(6.10) \quad \psi(y) = \Psi(z) = [w, z] = [R^*x, z] + w_0(\mathbf{N}) \sum_{F\in\mathcal{F}_0^c} z(F) = (x, y) + w_0(\mathbf{N})\psi_0(y)$$

where

$$\psi_0(y) = \sum_{F \in \mathcal{F}_0^c} z(F) \qquad (y = Rz). \tag{6.11}$$

Lemma 6.4 implies (6.11) is well-defined on $l(\alpha)$ and therefore $l(\alpha)^* = m(\alpha) \oplus \mathbf{R}$ as exhibited in (6.10)-(6.11). We define

$$l_0(\alpha) = \{y \in l(\alpha) | \psi_0(y) = 0\}. \tag{6.12}$$

Theorem 6.13. *Let* $0 \leq \alpha < 1$. *The* β*-dual of* $l(\alpha)$ *is* $m(\alpha)$. $m(\alpha)$ *is the Banach dual of* $l_0(\alpha)$ *while* $m(\alpha) \oplus \mathbf{R}\psi_0$ *is the Banach dual of* $l(\alpha)$.

Proof. We already know from Proposition 4.7 that $m(\alpha) \subseteq \beta$-dual $l(\alpha)$, $\alpha < 1$. Conversely, suppose $(\overline{x}, y)$ exists for all $y \in l(\alpha)$. The uniform boundedness principle and the preceding discussion imply there exist $x \in m(\alpha)$ and $t \in \mathbf{R}$ such that

$$(\overline{x}, y) = (x, y) + t\psi_0(y) \qquad (y \in l(\alpha)). \tag{6.14}$$

We must show that $t = 0$ and $\overline{x} = x$. The proof of Lemma 6.4 shows that

$$\lim_{n\to\infty} y_n = \psi_0(y) \qquad (y = Rz \in l(\alpha))$$

and therefore if we replace $\overline{x}$ by $\overline{x} - x$ and reletter, the question reduces to the nature of a sequence $\overline{x}$ such that

$$(\overline{x}, y) = t \lim_{n\to\infty} y_n \qquad (y \in l(\alpha)).$$

Considering $y = e_i \sim \sum \delta_{ji}$, we find $\overline{x} = 0$, and therefore $t = 0$. It follows $m(\alpha)$ is indeed the β-dual of $l(\alpha)$. The relation (6.10) identifies $m(\alpha)$ with $l_0(\alpha)^*$ and $m(\alpha) \oplus \mathbf{R}\psi_0$ with $l(\alpha)^*$. The theorem is proved.

Remark. If $\alpha = 1$, $l(1)$ is identified with $c(1)^*$ by Abel's formula as in the first paragraph of section 1. $l(1)$ is the image of $l^1 \oplus \mathbf{R}$ under the map $(u, t) \mapsto y, y_k = t + \sum_{j=k}^{\infty} u_j$. Given $\psi \in l(1)^*$, associate to ψ and $\Psi = \psi \circ R$ an element $w \in W(1)$, and use w to determine an element $x \in m(1)$ just as in the proof of Theorem 6.13. However, R^*x does not exist naturally as a function on $\mathcal{F}$ unless $x \in c(1)$. In fact, x belongs to the β-dual of $l(1)$ only if $x \in c(1)$. It remains the case that the β-dual of $l_0(1) \cong \{(u, t) | t = 0\}$ is $m(1)$ (Remark 1.16), and the dual of $l(1)$ is isomorphic to $m(1) \oplus \mathbf{R}\psi_0 \cong l_0(1)^* \oplus \mathbf{R}\psi_0$.

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