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DEFORMATIONS OF ZERODIMENSIONAL INTERSECTION SCHEMES AND RESIDUES

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

Let K be an algebraically closed field and $\mathbb{A}_K^n$ = Spec $K[X_1, \ldots, X_n]$ the affine n-space over $K (n > 0)$. For a closed point $P \in \mathbb{A}_K^n$ let

$$\mathcal{O}_P = K\left[X_1, \ldots, X_n\right]_{\mathfrak{m}_P}$$

be the local ring of $\mathbb{A}_K^n$ at P where $\mathfrak{m}_P$ is the maximal ideal of $K[X_1, \ldots, X_n]$ corresponding to P. If non-constant polynomials $f_1, \ldots, f_n \in K[X_1, \ldots, X_n]$ are given, we denote by $S := f_1 \cap \ldots \cap f_n$ the intersection scheme of the hypersurfaces $f_i = 0$ $(i = 1, \ldots, n)$. We assume that these hypersurfaces have no common points at infinity which is equivalent with the fact that their degree forms $Gf_i (i = 1, \ldots, n)$ form a regular sequence of $K[X_1, \ldots, X_n]$. Moreover it follows then that S is zerodimensional. So S may be regarded as a finite set of closed points $P_j \in \mathbb{A}_K^n$ $(j = 1, \ldots, h)$, namely the set of common zeros of the $f_i (i = 1, \ldots, n)$, called the *support* of S, together with a local ring

$$\mathcal{O}_{S,P_j} := \mathcal{O}_{P_j} / (f_1, \ldots, f_n)$$

at each point $P_j \in$ Supp S. The rings $\mathcal{O}_{S,P_j}$ $(j = 1, \ldots, h)$ are finite-dimensional K-algebras and

$$\mu_{P_j}\left(f_1, \ldots, f_n\right) := \dim_K \mathcal{O}_{S,P_j}$$

is the intersection multiplicity of the hypersurfaces $f_i = 0$ $(i = 1, \ldots, n)$ at P_j. We set $\mu_P(f_1, \ldots, f_n) = 0$, if $P \notin$ Supp S. We may also look at $f_1 \cap \ldots \cap f_n$ as being a zerodimensional intersection of n projective hypersurfaces where the hyperplane at infinity is chosen to avoid all intersection points. The number $d := \prod_{i=1}^n \deg f_i$ is called the *degree* of $f_1 \cap \ldots \cap f_n$.

Definition 1.1. *We say that* $f_1 \cap \ldots \cap f_n$ *is a* transversal complete intersection *at* $P \in \mathbb{A}_K^n$ *(or that the hypersurfaces* $f_i = 0$ $(i = 1, \ldots, n)$ *have a* normal crossing *at* P*) if*

$\mu_P(f_1,\ldots,f_n) = 1$. *We also say that* $S = f_1 \cap \ldots \cap f_n$ *is a transversal complete intersection if it is so at each* $P \in$ *Supp* S.

It is known and easy to see that S is a transversal complete intersection at P if and only if one of the following conditions is satisfied:

a) $\{f_1,\ldots,f_n\}$ is a regular system of parameters of $\mathcal{O}_P$.

b) The Jacobian determinant

$$J := \frac{\partial (f_1,\ldots,f_n)}{\partial (X_1,\ldots,X_n)}$$

does not vanish at P.

Moreover by Bézout's theorem S is globally a transversal complete intersection if and only if Supp S consists of d distinct points. It is often advantageous to be in this situation. Therefore in algebraic geometry for $K = \mathbb{C}$ it is a standard method to change the coefficients of the polynomials f_i «a little» in order to pass from the general case to the case of a transversal complete intersection. Then continuity arguments may be applied to transfer result which are known in this situation to the general case.

For an arbitrary K this method is not directly applicable, however we will show in the present note how it can be modified to work for a general algebraically closed field K. As an illustration we then apply it to the theory of residues of differential n-forms on $\mathbb{A}^n_K$.

2. DEFORMATIONS

We write $K[X]$ for $K[X_1,\ldots,X_n]$ in the sequel. Let $S = f_1 \cap \ldots \cap f_n$ be a zero-dimensional intersection scheme as in the introduction. In particular $\{Gf_1,\ldots,Gf_n\}$ is a regular sequence and $\{f_1,\ldots,f_n\}$ a quasiregular sequence of $K[X]$ where the latter means that $\{f_1,\ldots,f_n\}$ is a regular sequence of $\mathcal{O}_P$ for each $P \in$ Supp S.

Let $(R,\mathfrak{m})$ be a complete noetherian local K-algebra with residue field K, for example a power series algebra over K. Assume R is a domain and let L be the algebraic closure of the quotient field $Q(R)$ of R. The canonical epimorphism $R \to K$ modulo $\mathfrak{m}$ induces an epimorphism

$$\varphi : R\left[X_1,\ldots,X_n\right] \to K\left[X_1,\ldots,X_n\right] \qquad \left(\varphi\left(X_i\right) = X_i \quad (i = 1,\ldots,n)\right).$$

Choose $f_1^*,\ldots,f_n^* \in R[X]$ such that

$$\deg f_i^* = \deg f_i \text{ and } \varphi\left(f_i^*\right) = f_i \quad (i = 1,\ldots,n).$$

Then it is clear that $\varphi(Gf_i^*) = Gf_i$ $(i = 1,\ldots,n)$. Since $\{Gf_1, Gf_n\}$ was a regular sequence in $K[X]$, the same holds for $\{Gf_1^*,\ldots,Gf_n^*\}$ in $R[X]$ by the local flatness criterion ([M], Thm. 22.5 Cor.). Then $\{Gf_1^*,\ldots,Gf_N^*\}$ is likewise a regular sequence in $L[X]$. In particular the intersection scheme $S^* := f_1^* \cap \ldots \cap f_n^*$ in $\mathbb{A}^n_L$ is zerodimensional.

Definition 2.1. *The scheme S^* is called a* deformation *of S with base ring R.*

Contrary to what was said above about $K = \mathbb{C}$ the deformation S^* «lives» in general in an affine space different from $\boldsymbol{A}_K^n$. However S and S^* are closely connected as we are going to show now.

Let

$$A := K[X]/\left(f_1, \dots, f_n\right) \text{ and } A^* := L[X]/\left(f_1^*, \dots, f_n^*\right)$$

be the affine coordinate algebras of S resp. S^* and let

$$B := R[X]/\left(f_1^*, \dots, f_n^*\right).$$

By the Chinese remainder theorem

$$(1) \qquad A = \prod_{P \in \operatorname{Supp} S} \mathcal{O}_{S,P}, \; A^* = \prod_{Q \in \operatorname{Supp} S^*} \mathcal{O}_{S^*,Q}.$$

Further $A = B/\mathfrak{m} B$ and $A^* = L \otimes_R B$.

Proposition 2.2. *A^* is an L-algebra of dimension d and B is a free R-module of rank d.*

Proof. Since the hyperplanes $f_i^* = 0$ $(i = 1, \dots, n)$ have no common points at infinity the first assertion in connection with (1) is Bézout's theorem. In order to prove the second it suffices to show that B is generated as an R-module by d elements. Let $\operatorname{gr} B$ be the graded ring with respect to the degree filtration on $R[X]$. By [KK], 1.11 it is enough to prove that $\operatorname{gr} B = R[X]/(Gf_1^*, \dots, Gf_n^*)$ is generated as an R-module by d elements.

But $\operatorname{gr} B/\mathfrak{m} \operatorname{gr} B = K[X]/(Gf_1, \dots, Gf_n) = \operatorname{gr} A$ is a K-algebra of dimension d, the Gf_i forming a regular sequence of $K[X]$. By Nakayama it suffices to show that $\operatorname{gr} B$ is a finitely generated R-module.

This is certainly true for its homogeneous components $\operatorname{gr}^i B (i \in \mathbb{N})$. But $\operatorname{gr}^i B/\mathfrak{m} \operatorname{gr}^i B$ vanishes for large i being the homogeneous component of degree i of the finite-dimensional K-algebra $\operatorname{gr} B/\mathfrak{m} \operatorname{gr} B$. Again by Nakayama $\operatorname{gr}^i B$ vanishes for large i and $\operatorname{gr} B$ is finitely generated over R. Q.E.D.

Proposition 2.3. *Under the assumptions of 2.2 we have*
a) *For $P = (a_1, \dots, a_n) \in$ Supp S*

$$\mathfrak{M}_P := \left(\mathfrak{m}, X_1 - a_1, \dots, X_n - a_n\right)$$

is a maximal ideal of $R[X]$ with $f_i^* \in \mathfrak{M}_P$ $(i = 1, \dots, n)$ and the map

$$\operatorname{Supp} S \to \operatorname{Max} B \quad \left(P \mapsto \mathfrak{M}_P/\left(f_1^*, \dots, f_n^*\right)\right)$$

is a bijection.

b) *There is a canonical decomposition*

$$B = \prod_{P \in \operatorname{Supp} S} B_P$$

with $B_P := R[X]_{\mathfrak{M}_P}/(f_1^*, \dots, f_n^*)$, *and* B_P *is a free* R*-module of rank* $\mu_P(f_1, \dots, f_n)$.

Proof. a) Since B is finite over R by 2.2 the maximal ideals of B lie over $\mathfrak{m}$, hence Max$B$ and Max A are in one-to-one correspondence. Further Max A and Supp S are in natural one-to-one correspondence, hence a) follows.

b) Since B is finite over a complete local ring we have by the Chinese remainder theorem

$$B = B_1 \times \dots \times B_h$$

where the B_j are the localizations of B at its maximal ideals. If the maximal ideal is $\mathfrak{M}_{P_j}/(f_1^*, \dots, f_n^*)$ for $P_j \in \operatorname{Supp} S$, then $B_j = B_{P_j}$.

As a homomorphic image of B each B_P ($P \in \operatorname{Supp}(S)$) is finite over R. Let $\mathfrak{m}_P$ be the maximal ideal of P in A. Since $B_P/\mathfrak{m} B_P = A_{\mathfrak{m}_P} = \mathcal{O}_{S,P}$ we obtain by Nakayama that B_P is generated by $\mu_P(f_1, \dots, f_n)$ elements. But $B = \prod_{P \in \operatorname{Supp} S} B_P$ is free of rank $d = \sum_{P \in \operatorname{Supp} S} \mu_P(f_1, \dots, f_n)$. Hence each factor B_P must be free of rank $\mu_P(f_1, \dots, f_n)$.

For the next proposition we need a lemma

Lemma 2.4. *The integral closure* $\overline{R}$ *of* R *in* L *is a local ring with residue field* K.

Proof. Write $\overline{R} = \cup S$ where S runs over all rings with $R \subset S \subset \overline{R}$ such that S is finite over R. By the Chinese remainder theorem each such S is a direct product of its localizations at the maximal ideals. Being a domain S can therefore have only one maximal ideal, call it $\mathfrak{m}_S$.

Each $\mathfrak{M} \in \operatorname{Max} \overline{R}$ lies over a maximal ideal of S, hence $\mathfrak{M} \cap S = \mathfrak{m}_S$ and $\mathfrak{M} = \cup \mathfrak{m}_S$. Therefore $\overline{R}$ has only one maximal ideal. Its residue field is an algebraic extension of K. Since K is algebraically closed it must be K.

Proposition 2.5. *Under the assumptions of 2.2 let* $Q = (\alpha_1, \dots, \alpha_n) \in \operatorname{Supp} S^*$. *Then*

a) $\alpha_1, \dots, \alpha_n \in \overline{R}$.

b) *If* a_i *denotes the residue class of* α_i *in* K $(i = 1, \dots, n)$ *then* $P := (a_1, \dots, a_n) \in \operatorname{Supp} S$, *hence there is a natural map*

$$\eta : \operatorname{Supp} S^* \to \operatorname{Supp} S \qquad ((\alpha_1, \dots, \alpha_n) \mapsto (a_1, \dots, a_n))$$

c) *η is surjective and for each $P \in$ Supp S there is a canonical isomorphism of L-algebras*

$$L \otimes_R B_P \cong \prod_{Q \in \eta^{-1}(P)} A^*_Q$$

where $A^*_Q = \mathcal{O}_{S^*,Q}$ is the localization of A^* at the maximal ideal corresponding to Q.

Proof. a) Let $\mathfrak{M}_Q \in \operatorname{Max} A^*$ be the maximal ideal which corresponds to Q and let $\varepsilon_Q : A^* \to A^*/\mathfrak{M}_Q$ be the canonical epimorphism. Clearly $A^*/\mathfrak{M}_Q = L$. Let x^*_i denote the residue class of X_i in A^*. Then $\varepsilon_Q(x^*_i) = \alpha_i$ $(i = 1, \dots, n)$. Since x^*_i is integral over R by 2.2 its image $\alpha_i \in L$ must be integral over R, too, hence $\alpha_i \in \overline{R}$ $(i = 1, \dots, n)$.
b) Consider the commutative diagram

$$\begin{array}{ccccc}
R[X] & \xrightarrow{\bmod (f^*_1, \dots, f^*_n)} & B & \xrightarrow{\varepsilon_Q} R[\alpha_1, \dots, \alpha_n] \subset & \overline{R} \\
\varphi \downarrow & & \overline{\varphi} \downarrow & & \varepsilon \downarrow \\
K[X] & \xrightarrow[\bmod (f_1, \dots, f_n)]{} & A & \xrightarrow[\overline{\varepsilon}_Q]{} & K
\end{array}$$

where ε is the canonical epimorphism onto the residue field, $\overline{\varphi}$ is induced by φ and $\overline{\varepsilon}_Q$ is induced by ε_Q. Let x_i denote the image of X_i in A. Then $\overline{\varepsilon}_Q(x_i) = a_i$ $(i = 1, \dots, n)$.
c) By 2.3c) we have

$$A^* = L \otimes_R B = \prod_{P \in \text{Supp } S} L \otimes_R B_P.$$

Here $L \otimes_R B_P$ is an L-algebra of dimension $\mu_P(f_1, \dots, f_n) \neq 0$, hence it has at least one maximal ideal. Let $\mathfrak{M}_Q \in \operatorname{Max} A^*$ be the corresponding maximal ideal of A^*.

If $Q = (\alpha_1, \dots, \alpha_n)$ and $P = (a_1, \dots, a_n)$, then ε_Q induces an R-epimorphism $B_P \to R[\alpha_1, \dots, \alpha_n]$, since all factors of A^* different from $L \otimes_R B_P$ are mapped to 0 by ε_Q. Then $\overline{\varepsilon}_Q$ induces a K-epimorphism $\mathcal{O}_{S,P} = B_P/\mathfrak{m} B_P \to K$. But there is only one such epimorphism, the canonical epimorphism onto the residue field. It maps x_i to a_i $(i = 1, \dots, n)$, hence $\eta(Q) = P$ and η is surjective.

It is now clear that the maximal ideals of $L \otimes_R B_P$ are of the form $\mathfrak{M}_Q$ for $Q \in \eta^{-1}(P)$. Therefore the product decomposition of c) follows again from the Chinese remainder theorem.

Corollary 2.6. *For each $P \in$ Supp S*

$$\mu_P(f_1, \dots, f_n) = \sum_{Q \in \eta^{-1}(P)} \mu_Q(f^*_1, \dots, f^*_n).$$

3. SMOOTHING INTERSECTION SCHEMES

We shall show now that $S = f_1 \cap \dots \cap f_n$ can be «smoothed» in the following sense:

Theorem 3.1. *For $S = f_1 \cap \ldots \cap f_n$ as in the introduction there exists a deformation $S^* = f_1^* \cap \ldots \cap f_n^*$ of S such that*
a) *S^* is a transversal complete intersection.*
b) $Gf_i^* = Gf_i (i = 1, \ldots, n)$.

Proof. We may assume without loss of generality that deg $f_i > 1$ for $i = 1, \ldots, m$, deg $f_i = 1$ for $i = m+1, \ldots, n$. Necessarily $\{f_{m+1}, \ldots, f_n\}$ are linearly independent over K, so we may also assume that $f_i = X_i$ for $i = m+1, \ldots, n$. In case $m = 0$ the proof is trivial with $f_i^* = f_i = X_i (i = 1, \ldots, n)$. So let $m > 0$.

Choose an indeterminates $u, u_1, \ldots, u_m$, let $R = K[[u, u_1, \ldots, u_m]]$ be the power series algebra over K in these indeterminates, $\mathfrak{m}$ its maximal ideal and L the algebraic closure of $Q(R)$. Set

$$f_i^* = f_i + uX_i + u_i f_i^* = f_i + uX_i + u_i \quad (i = 1, \ldots, m)$$

$$f_i^* = f_i = X_i \qquad (i = m+1, \ldots, n).$$

Then $Gf_i^* = Gf_i$ for $i = 1, \ldots, n$, as deg $f_i > 1$ for $i = 1, \ldots, m$. Since $\{Gf_1, \ldots, Gf_n\}$ was a regular sequence in $K[X]$ it is clear that $\{Gf_1^*, \ldots, Gf_n^*\}$ is a regular sequence in $L[X]$ and $S^* := f_1^* \cap \ldots \cap f_n^*$ is a deformation which already satisfies 3.1 b). In order to prove a) we wish to show that the Jacobian determinant $J^* := \frac{\partial(f_1^*, \ldots, f_n^*)}{\partial(X_1^*, \ldots, X_n^*)} = \frac{\partial(f_1^*, \ldots, f_m^*)}{\partial(X_1^*, \ldots, X_m^*)}$ does not vanish at any $Q \in \text{Supp } S^*$.

For $P = (a_1, \ldots, a_n) \in \text{Supp } S$ let $\mathfrak{M}_P := (\mathfrak{m}, X_1 - a_1, \ldots, X_n - a_n) \subset R[X]$ and let $B_P = R[X]_{\mathfrak{M}_P} / (f_1^*, \ldots, f_n^*)$. Remember that by 2.3 b) the ring B_P is finite over R, hence B_P is a complete local ring. After a translation we may assume that $a_1 = \ldots = a_n = 0$. Then

$$B_P = \widehat{B_P} = R[[X_1, \ldots, X_n]] / (f_1^*, \ldots, f_n^*).$$

But $f_i^* = u_i + \varphi_i^*$ with $\varphi_i^* \in K[[u, X_1, \ldots, X_n]]$ for $i = 1, \ldots, m$ and $f_i^* = X_i$ for $i = m+1, \ldots, n$. It follows that the canonical homomorphism

$$K[[u, X_1, \ldots, X_m]] \to B_P$$

is an isomorphism. In particular B_P is a domain. By 2.3 c) we know that B is the direct product of the B_P for $P \in \text{Supp } S$.

In the Jacobian determinant J^* the indeterminates u_i $(i = 1, \ldots, m)$ do not occur and as a function of the u_i $(i = 1, \ldots, m)$ it is a polynomial in $K[X_1, \ldots, X_n][u]$ whose degree form with respect to u is u^m. Hence the image of J^* in B_P does not vanish for any $P \in \text{Supp } S$ and the image of J^* in B is not a zerodivisor. Then also its image in

$$A^* := L[X] / (f_1^*, \ldots, f_n^*) = L \otimes_R B$$

is not a zerodivisor, L/R being a flat extension. Call this image j^*.

We can write $A^* = \prod_{Q\in \text{Supp } S^*} A^*_Q$ with $A^*_Q = \mathcal{O}_{S^*,Q}$. The A^*_Q are artinian local rings. Since j^* is not a zerodivisor in A^* it follows that the image of j^* in A^*_Q is a unit for any $Q \in \text{Supp } S^*$, hence the image of j^* in the residue field of A^*_Q does not vanish. But this image is just $J^*(Q)$.

Corollary 3.2. *(«Dynamical description» of the intersection multiplicity). Let S^* be as in the theorem and let $\eta : \text{Supp } S^* \to \text{Supp } S$ be the map described in 2.5. Then $\eta^{-1}(P)$ consists for each $P \in \text{Supp } S$ of exactly $\mu_P(f_1, \ldots, f_n)$ distinct points.*

4. RESIDUES

For $S = f_1 \cap \ldots \cap f_n$ as in the introduction, a differential form $\omega = g dX_1 \ldots dX_n \in \Omega^n_{K[X]/K}$ ($g \in K[X]$), and a closed point $P \in \mathbb{A}^n_K$ we consider now the residue

$$\text{Res}_P \begin{bmatrix} \omega \\ f_1, \ldots, f_n \end{bmatrix}.$$

We may regard these residues as generalizations of the intersection multiplicity $\mu_P(f_1, \ldots, f_n)$. At least in characteristic 0 the intersection multiplicity is the residue for $\omega = df_1 \ldots df_n$ (see (9) below). As ω varies the residues describe the behavior of the intersection S at P more closely then the intersection multiplicity does alone.

We write $f = \{f_1, \ldots, f_n\}$ in the sequel. Let us recall the construction of the residue (due to [SS1], [SS2]) and its principal properties. The «canonical module» $\omega_{A/K} := \text{Hom}_K(A, K)$ of A/K is a free A-module of rank 1. A basis element $\sigma : A \to K$ of $\omega_{A/K}$ is called a «trace» of A/K. To the presentation $A = K[X]/(f)$ there is always associated a trace which is obtained as follows ([SS1],§ 4; see also [K], app. F). Let x_i be the image of X_i in $A (i = 1, \ldots, n)$, set $A^e := A \otimes_K A$, and let I be the kernel of $A^e \to A$ $(a \otimes b \mapsto ab)$. Then $\text{Ann}_{A^e} I$ is in a natural manner an A-module and there exists a canonical isomorphism of A-modules

$$\phi : \text{Ann}_{A^e} I \xrightarrow{\sim} \text{Hom}_A \left(\omega_{A/K}, A\right)$$

with $\phi(\sum a_i \otimes b_i)(\ell) = \sum \ell(a_i) b_i$ for all $\sum a_i \otimes b_i \in \text{Ann}_{A^e} I$ and all $\ell \in \omega_{A/K}$. In $A[X] := A[X_1, \ldots, X_n]$ there are equations

$$(2) \qquad f_i = \sum_{j=1}^{n} a_{ij} \left(X_j - x_j\right) \qquad (i = 1, \ldots, n; a_{ij} \in A[X]).$$

Let Δ_x^f denote the image of $\det(a_{ij})$ in $A^e = A[X]/(f)A[X]$. Then $\operatorname{Ann}_{A^e} I = A \cdot \Delta_x^f$. The (unique) element $\tau_f^x \in \omega_{A/K}$ with $\phi(\Delta_x^f)(\tau_f^x) = 1$ is then a trace of A/K.

Its relation to the standard trace $\sigma_{A/K}$ of the algebra A/K (which in general is not a «trace» in our present sense) is as follows: let $\frac{\partial f}{\partial x}$ denote the image of the Jacobian determinant J in A. Then in $\omega_{A/K}$

(3) $$\sigma_{A/K} = \frac{\partial f}{\partial x} \cdot \tau_f^x$$

(see [SS1], 4.2 or [K], F. 23).

Consider now the decomposition $A = \prod_{P \in \operatorname{Supp} S} A_P$ and read (2) as equations in $A_P[X]$. Then in the same way as $\tau_f^x : A \to K$ a trace $(\tau_f^x)_P : A_P \to K$ can be constructed. It is easily seen that $(\tau_f^x)_P$ is the restriction of τ_f^x to the direct factor A_P. Hence for $a = (\{a_P\}_{P \in \operatorname{Supp} S})$, $a_P \in A_P$ we have

(4) $$\tau_f^x(a) = \sum_{P \in \operatorname{Supp} S} \left(\tau_f^x\right)_P (a_P).$$

Moreover for the standard trace $\sigma_{A_P/K}$

(5) $$\sigma_{A_P/K} = \left(\frac{\partial f}{\partial x}\right)_P \cdot \left(\tau_f^x\right)_P$$

where $(\frac{\partial f}{\partial x})_P$ is the image of J in A_P. We mention that for these statements K need not be a field but can be an arbitrary commutative ring. However it is necessary that f is a quasi-regular sequence in $K[X]$ and A is at least a finitely generated projective K-module (see [SS1] and [K], app. F).

Now for $\omega = g dX$ as above let γ (resp. γ_P) be the image of g in A (in A_P). Then

$$\int \begin{bmatrix} \omega \\ f \end{bmatrix} := \tau_f^x(\gamma)$$

is called the *integral* of ω with respect to (the «integration path») f and for $P \in \operatorname{Supp} S$

$$\operatorname{Res}_P \begin{bmatrix} \omega \\ f \end{bmatrix} := \left(\tau_f^x\right)_P (\gamma_P)$$

is called the *residue* of ω at P with respect to f. We set $\operatorname{Res}_P[{\omega \atop f}] = 0$ if $P \notin \operatorname{Supp} S$.

It can be shown that integral and residue do not depend on the coordinates $X_1, \dots, X_n$ used in their construction. They are K-linear in ω and there is a «transformation formula» ([SS2], 1.1) or [K], F. 26) which describes how they depend on f. Clearly by (4)

$$(6) \qquad \int \begin{bmatrix} \omega \\ f \end{bmatrix} = \sum_P \operatorname{Res}_P \begin{bmatrix} \omega \\ f \end{bmatrix}$$

and by construction for $\omega = g\,dx$

$$(7) \qquad \int \begin{bmatrix} \omega \\ f \end{bmatrix} = 0 \quad \text{for} \quad g \in (f)$$

$$(8) \qquad \operatorname{Res}_P \begin{bmatrix} \omega \\ f \end{bmatrix} = 0 \quad \text{for} \quad g \in (f) A_P.$$

Moreover by (5)

$$\operatorname{Res}_P \begin{bmatrix} df_1 \dots df_n \\ f \end{bmatrix} = \left(\tau_f^x\right)_P \left(\left(\frac{\partial f}{\partial x}\right)_P\right) = \sigma_{A_P/K}(1) = \dim_K A_P \cdot 1_K$$

hence

$$(9) \qquad \operatorname{Res}_P \begin{bmatrix} df \\ f \end{bmatrix} = \mu_P(f) \cdot 1_K.$$

For transversal complete intersections there is a simple formula for the residue.

Proposition 4.1. *Let S be a transversal complete intersection at P. Then*

$$\operatorname{Res}_P \begin{bmatrix} \omega \\ f \end{bmatrix} = \frac{g(P)}{J(P)}.$$

Proof. The assumption implies that $A_P = K$, $\sigma_{A_P/K} = \mathrm{id}_K$, and that $J(P) \neq 0$. Hence the definition of the residue in connection with (5) shows that

$$\operatorname{Res}_P \begin{bmatrix} g dX \\ f \end{bmatrix} = \frac{1}{J(P)} \sigma_{A_P/K}(g(P)) = \frac{g(P)}{J(P)}.$$

By smoothing $f_1 \cap \dots \cap f_n$ we shall deduce a similar formula for the residue in the general case. Let $S^* = f_1^* \cap \dots \cap f_n^*$ be a deformation of S as in section 2. We keep the notations introduced there. The previous constructions can be applied to $B = R[X]/(f^*) = \prod_{P \in \mathrm{Supp}\, S} B_P$, hence there are traces

$$\tau^x_{f^*} : B \to R, \left(\tau^x_{f^*}\right)_P : B_P \to R.$$

For $\beta \in B (\beta \in B_P)$ let $\beta^\sim \in A (\beta^\sim \in A_P)$ denote the image by the reduction modulo $\mathfrak{m} B$. In particular $\beta^\sim \in K$ for $\beta \in R$. It is known that the above traces are compatible with base change ([K], F. 27) which implies in our present situation the following formulas:

$$(10) \qquad \begin{aligned} \tau^x_f(\beta^\sim) &= \left(\tau^x_{f^*}\right)(\beta)^\sim \quad \text{for } \beta \in B \\ \left(\tau^x_f\right)_P(\beta^\sim) &= \left(\tau^x_{f^*}\right)_P(\beta)^\sim \quad \text{for } \beta \in B_P. \end{aligned}$$

Moreover $L \otimes_R \tau^x_{f^*}$ is the trace of A^*/L associated to the presentation $A^* = L[X]/(f^*)$. We denote this trace again by $\tau^x_{f^*}$ and write $(\tau^x_{f^*})_P$ for $L \otimes (\tau^x_{f^*})_P$. Then $(\tau^x_{f^*})_P$ is the restriction of $\tau^x_{f^*}$ to the direct factor $L \otimes_R B_P$ of A^*. Since

$$L \otimes_R B_P = \prod_{Q \in \eta^{-1}(P)} A^*_Q$$

we have for $\{\alpha_Q\} \in \prod A^*_Q (\alpha_Q \in A^*_Q)$

$$(11) \qquad \left(\tau^x_{f^*}\right)_P\left(\left\{\alpha_Q\right\}\right) = \sum_{Q \in \eta^{-1}(P)} \left(\tau^x_{f^*}\right)_Q\left(\alpha_Q\right).$$

Now combining (10) and (11) with the definition of residues we obtain

Theorem 4.2. *Let $S^* = f^*_1 \cap \ldots \cap f^*_n$ be a deformation of S. For $\omega = g dX_1 \ldots dX_n$ let $g^* \in R[X]$ be a preimage of g and let $\omega^* := g^* dX_1 \ldots dX_n \in \Omega^n_{L[X]/L}$. Then*

$$\int \begin{bmatrix} \omega^* \\ f^* \end{bmatrix} \in R, \quad \sum_{Q \in \eta^{-1}(P)} \mathrm{Res}_Q \begin{bmatrix} \omega^* \\ f^* \end{bmatrix} \in R$$

and

$$\int \begin{bmatrix} \omega \\ f \end{bmatrix} = \int \begin{bmatrix} \omega^* \\ f^* \end{bmatrix}^\sim, \quad \mathrm{Res}_P \begin{bmatrix} \omega \\ f \end{bmatrix} = \left(\sum_{Q \in \eta^{-1}(P)} \mathrm{Res}_Q \begin{bmatrix} \omega^* \\ f^* \end{bmatrix} \right)^\sim.$$

Proof. Let γ_P^* (resp. γ_Q^*) be the image of g^* in B_P (in A_Q^*). Then $(\gamma_P^*)^\sim$ is the image of g in A_P and by (10) and (11)

$$\operatorname{Res}_P\begin{bmatrix}\omega\\ f\end{bmatrix} = \left(\tau_f^x\right)_P((\gamma_P^*)^\sim) = \left(\left(\tau_{f^*}^x\right)_P(\gamma_P^*)\right)^\sim = \left(\sum_{Q\in\eta^{-1}(P)}\left(\tau_{f^*}^x\right)_Q\left(\gamma_Q^*\right)\right)^\sim =$$

$$= \left(\sum_{Q\in\eta^{-1}(P)}\operatorname{Res}_Q\begin{bmatrix}\omega^*\\ f^*\end{bmatrix}\right)^\sim .$$

The statement about the integral follows now from (6).

Corollary 4.3. *If S^* is a transversal complete intersection and $J^* := \frac{\partial(f_1^*,\dots,f_n^*)}{\partial(X_1,\dots,X_n)}$, then*

$$\operatorname{Res}_P\begin{bmatrix}\omega\\ f\end{bmatrix} = \left(\sum_{Q\in\eta^{-1}(P)}\frac{g^*(Q)}{J^*(Q)}\right)^\sim \quad \textit{and} \quad \int\begin{bmatrix}\omega\\ f\end{bmatrix} = \left(\sum_{Q\in \operatorname{Supp} S^*}\frac{g^*(Q)}{J^*(Q)}\right)^\sim .$$

For $f_1\cap\dots\cap f_n$ as at the beginning write $Gf_i = \sum_{j=1}^n d_{ij}X_j$ $(i = 1,\dots,n;\ d_{ij}\in K[X]$ homogeneous of degree $\deg f_i - 1)$ and let d_X^{Gf} denote the image of $\det(d_{ij})$ in $\operatorname{gr} A$. It is known ([KK], 2.8) that this image generates $\operatorname{gr}^\rho A$ where $\rho := \sum \deg f_i - n$, and that $\operatorname{gr}^\rho A$ is the socle of $\operatorname{gr} A$ and a one-dimensional vector space over K. Therefore for each $h\in K[X]$ with $\deg h = \rho$ there is a unique $\kappa\in K$ such that

$$(12) \qquad Gh \equiv \kappa\cdot\det\left(d_{ij}\right) \bmod \left(Gf_1,\dots,Gf_n\right).$$

We set $\kappa = 0$ if $\deg h < \rho$. Moreover, since $\operatorname{gr}^\rho A$ is the homogeneous component of highest degree of $\operatorname{gr} A$, it follows from [KK], 1.11a) that for each $g\in K[X]$ there is an $h\in K[X]$ with

$$(13) \qquad \deg h\le\rho \text{ and } g\equiv h \bmod \left(f_1,\dots,f_n\right).$$

A classical theorem of Jacobi states

Theorem 4.4. *If $f_1\cap\dots\cap f_n$ is a transversal complete intersection and $h\in K[X]$ a polynomial with deg $h\le\rho$, then*

$$\sum_{P\in \operatorname{Supp} S}\frac{h(P)}{J(P)} = \kappa .$$

The terms on the left hand side of this formula are residues by 4.1. The same is true for κ as well:

Proposition 4.5. *Let O denote the origin of $\mathbb{A}^n_K$. Then*

$$\kappa = \operatorname{Res}_O \begin{bmatrix} GhdX_1 \ldots dX_n \\ Gf_1, \ldots, Gf_n \end{bmatrix}.$$

Proof. It is known that $\tau^x_{Gf}(d^{Gf}_X) = 1$ and τ^x_{Gf} is homogeneous of degree $-\rho$ ([KK], 2.8). Hence by (12) and the definition of the residue

$$\operatorname{Res}_O \begin{bmatrix} GhdX \\ Gf \end{bmatrix} = \tau^x_{Gf}\left(\kappa \cdot d^{Gf}_X\right) = \kappa$$

in case deg $h = \rho$, and the residue vanishes if deg $h < \rho$.

Now we have shown that the Jacobian formula is a special case of

Residue Theorem 4.6. *For $\omega = gdX$ choose $h \in K[X]$ with* deg $h \leq \rho$ *and* $g \equiv h \bmod (f)$. *Then*

$$(14) \qquad \int \begin{bmatrix} \omega \\ f \end{bmatrix} = \int \begin{bmatrix} hdX \\ f \end{bmatrix} = \sum_P \operatorname{Res}_P \begin{bmatrix} hdX \\ f \end{bmatrix} = \operatorname{Res}_O \begin{bmatrix} GhdX \\ Gf \end{bmatrix}.$$

Conversely let $S^* := f_1^* \cap \ldots \cap f_n^*$ be a deformation of $f_1 \cap \ldots \cap f_n$ as in 3.1, in particular a transversal complete intersection. For h as in 4.6 choose h^* in 4.3 to be h. Then by 4.3 the Jacobian forumula for $f_1^* \cap \ldots \cap f_n^*$ implies the residue theorem 4.6: the first equality in (14) follows from (7). Further

$$(15) \qquad \int \begin{bmatrix} hdX \\ f \end{bmatrix} = \left(\sum_Q \frac{h(Q)}{J^*(G)}\right)^{\sim} = \left(\operatorname{Res}_O \begin{bmatrix} GhdX \\ Gf \end{bmatrix}\right)^{\sim} = \operatorname{Res}_O \begin{bmatrix} GhdX \\ Gf \end{bmatrix}$$

where we have used that $Gf^* = Gf$ and that $\operatorname{Res}_O[\frac{GhdX}{Gf}]$ considered as a residue in $\mathbb{A}^n_L$ is the same as the residue in $\mathbb{A}^n_K$ (base change). Thus the proof of the residue theorem can be reduced to that of the Jacobian formula by smoothing $f_1 \cap \ldots \cap f_n$. However the proof of the residue theorem given in [KK], 4.8 by deforming S to Spec (gr A) does not require to show that Jacobian formula in advance. The present residue theorem is also a very special case of a general residue theorem in Grothendieck duality theory ([HK]).

In (15) we have used the Jacobian formula for S^*. It has turned out that $\sum_Q \frac{h(Q)}{J^*(G)}$ is already an element of K. Thus we obtain the following formula for the computation of the integral.

Proposition 4.7. *For ω and h as in 4.6, and S^* as above*

$$\int \begin{bmatrix} \omega \\ f \end{bmatrix} = \sum_{Q \in \operatorname{Supp} S^*} \frac{h(Q)}{J^*(G)}.$$

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