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The intersection graph of ideals of a lattice

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Abstract. For a finite lattice L, we define and study the intersection graph of ideals, denoted by $G(L)$. We study the interplay of lattice-theoretic properties of L with graph-theoretic properties of $G(L)$.

Keywords: Lattice, Intersection graph, Clique number

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Introduction

1 Introduction

The investigation of graphs related to various algebraic structures is a very large and growing area of research. Several classes of graphs associated with algebraic structures have been actively investigated. For example, Cayley graphs have been studied in [4], [14], [15], [16], [18], intersection graphs have been investigated in [9], [10], [19], zero-divisor graphs have been studied in [5], [6], [7], [11], [13] and cozero-divisor graphs have been investigated in [1], [2], [3].

In this paper, we define and study the intersection graph of ideals of a finite lattice. First we recall some definitions and notations on graphs. We use the standard terminology of graphs contained in [8]. In a graph G, the *distance* between two distinct vertices a and b, denoted by $\mathrm{d}(a,b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a,b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \sup\{\mathrm{d}(a,b) : a \text{ and } b \text{ are distinct vertices of } G\}$. The *girth* of G, denoted by $\mathrm{g}(G)$, is the length of the shortest cycle in G, if G contains a cycle; otherwise, we set $\mathrm{g}(G) := \infty$. Also, for two distinct vertices a and b in G, the notation $a - b$ means that a and b

are adjacent. The set of vertices of G is denoted by $V(G)$. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. Also, we say that G is *totally disconnected* if no two vertices of G are adjacent. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset form an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of the largest independent set. The *degree* of a vertex x, denoted by $\deg(x)$, is the number of the edges in the graph incident with x. We denote the maximum degree of vertices in G by $\Delta(G)$. A *refinement* of a graph H is a graph G such that the vertex-sets of G and H are the same and every edge in H is an edge in G. Also a graph on n vertices such that $n-1$ of the vertices have degree one, and are all adjacent only to the remaining vertex a, is called a *star graph* with center a. Let G_1 and G_2 be subgraphs of G. We say that G_1 and G_2 are *disjoint* if they have no vertex and no edge in common. The *union* of two disjoint graphs G_1 and G_2, which is denoted by G_1+G_2, is a graph with $V(G_1+G_2)=V(G_1)\cup V(G_2)$ and $E(G_1+G_2)=E(G_1)\cup E(G_2)$.

2 Intersection graph of a lattice

In this section, firstly we recall some definitions and notations on lattices. Then we introduce the intersection graph of ideals of a lattice.

A *lattice* is a set L with two binary operations $\wedge$ and $\vee$ on L satisfying the following conditions: for all $a,b,c\in L$,

1. $a\wedge a=a,\ a\vee a=a,$
2. $a\wedge b=b\wedge a,\ a\vee b=b\vee a,$
3. $(a\wedge b)\wedge c=a\wedge(b\wedge c),\ a\vee(b\vee c)=(a\vee b)\vee c$, and
4. $a\vee(a\wedge b)=a\wedge(a\vee b)=a.$

The operations $\wedge$ and $\vee$ on L are called meet and joint, respectively.

In the next theorem, we recall an equivalent definition for a lattice with respect to a partial order relation. This definition will be used in the sequel of the paper.

Theorem 2.1. [17, Theorem 2.1] Let L be a lattice. One can define an order $\leqslant$ on L as follows:

For any $a, b \in L$, we set $a \leqslant b$ if and only if $a \wedge b = a$. Then $(L, \leqslant)$ is an ordered set in which every pair of elements has a greatest lower bound (g.l.b.) and a least upper bound (l.u.b.). Conversely, let P be an ordered set such that, for every pair $a, b \in P$, g.l.b.(a, b), l.u.b.$(a, b) \in P$. For each a and b in P, we define $a \wedge b :=$ g.l.b.(a, b) and $a \vee b :=$ l.u.b.(a, b). Then $(P, \wedge, \vee)$ is a lattice.

Note that in every lattice $a \wedge b = a$ implies that $a \vee b = b$.

The concept of intersection graph of a lattice was first introduced by Zelinka [19]. In his work, all proper sublattices of a lattice L are the vertices of this graph and two distinct vertices are adjacent whenever they have non-empty intersection. He defined three interesting types of intersection graph of a lattice, which he called algebraic intersection graph, set-theoretical intersection graph and interval intersection graph. To this end, he considered three types of sublattices of L as follows. An algebraic sublattice of a lattice L is a non-empty subset of L which is closed with respect to the operations of join and meet.

A set-theoretical sublattice of L is a non-empty subset of L which is a lattice with respect to the ordering induced by the ordering of L. It can be easily seen that every algebraic sublattice of a lattice L is a set-theoretical sublattice, too. The converse is not true (cf. [19, Figure 1]). The third type of sublattices are the intervals $< a, b >= \{x \in L \mid a \leqslant x \leqslant b\}$, where $a, b \in L$ with $a \leqslant b$. The interval $< a, b >$ is evidently an algebraic sublattice of L.

Here we define the fourth type of intersection graph of a lattice, namely the intersection graph of ideals of a lattice. Let 0 denote an element of L such that $0 \wedge a = 0$, for all $a \in L$. The intersection graph of ideals of a lattice L with 0, denoted by $G(L)$, is a graph with all non-trivial ideals of L as the vertices, where two distinct vertices I and J are adjacent if and only if $I \cap J \neq \{0\}$. Clearly $G(L)$ is an induced subgraph of the algebraic intersection graph.

The lattice L is said to be *bounded* if there are elements 0 and 1 in L such that

$$0 \wedge a = 0 \quad \text{and} \quad a \vee 1 = 1$$

for all $a \in L$. Clearly every finite lattice is bounded. In the rest of the paper, we assume that $L = (L, \wedge, \vee)$ is a finite lattice.

Definition 2.2. [12, Definition 39] A non-empty subset I of a lattice L is called an ideal of L if and only if the following conditions are satisfied:

(i) For $a, b \in I$, $a \vee b \in I$.

(ii) For $a \in I$ and $c \in L$, $a \wedge c \in I$.

An ideal I of L is proper if $I \neq L$.

Theorem 2.3. [12, Theorem 59] In an ideal I of L, the following conditions are satisfied:

(i) If $a \in I$ and $b \leqslant a$, then $b \in I$.

(ii) If $a \vee b \in I$, then $a, b \in I$.

An ideal of L which consists of all $x \leqslant a$, for some $a \in L$, is called a principal ideal generated by a and it is denoted by $[a]^\ell$ (see [12, Definition 41]).

Also, for $a \in L$, we denote the set $\{x \in L \mid a \leqslant x\}$ by $[a]^u$.

Theorem 2.4. [12, Theorem 68] In a finite lattice every ideal is principal.

Lemma 2.5. Suppose that $a, b \in L$. Then we have $[a]^\ell \cap [b]^\ell = [a \wedge b]^\ell$.

Proof. Clearly $x \in [a]^\ell \cap [b]^\ell$ if and only if $x \leqslant a$ and $x \leqslant b$. This is equivalent to saying that $x \leqslant a \wedge b$, and this means that $x \in [a \wedge b]^\ell$. $\boxed{QED}$

Note that in a lattice L, the union of two ideals is not an ideal in general. For example consider two elements $a, b \in L$ such that $a \notin [b]^\ell$ and $b \notin [a]^\ell$. Now, we have $a, b \in [a]^\ell \cup [b]^\ell$, but $a \vee b \notin [a]^\ell \cup [b]^\ell$. Thus $[a]^\ell \cup [b]^\ell$ is not an ideal of L.

By Theorem 2.4, there is a one-to-one correspondence between the set of all non-trivial ideals of L and the elements in $L \setminus \{0, 1\}$. Now let I and J be distinct non-trivial ideals of L. By Theorem 2.4, there exist elements a and b in L such that $I = [a]^\ell$ and $J = [b]^\ell$. Also, by Lemma 2.5, we have $I \cap J \neq \{0\}$ if and only if $a \wedge b \neq 0$. Therefore the intersection graph of ideals of the lattice L is isomorphic to the graph with vertex-set $L \setminus \{0, 1\}$ in which two distinct vertices a and b are adjacent if and only if $a \wedge b \neq 0$, and in the sequel of this note, we deal with this graph as the intersection graph of the lattice L.

Definition 2.6. 1. In a poset $(P, \leqslant)$, we say that a covers b or b is covered by a, in notation $b \prec a$, if and only if $b < a$ and there is no element p in P such that $b < p < a$.

2. An element a in L is called an atom if $0 \prec a$. Similarly, a is called a co-atom if $a \prec 1$. We denote the set of all atoms in a lattice L by $A(L)$, and co-atoms by $C(L)$.

3 Basic properties of $G(L)$

We begin this section with the following lemma which is needed in the rest of the paper.

Lemma 3.1. The vertices x and y are adjacent in $G(L)$ if and only if there exists an atom $a \in A(L)$ such that $x, y \in [a]^u$.

Proof. First suppose that x is adjacent to y. So $x \wedge y \neq 0$. Hence there exists an atom $a \in A(L)$ such that $a \leqslant x \wedge y$. Thus $x, y \in [a]^u$.

The converse statement is clear. $\boxed{QED}$

The following proposition follows from Lemma 3.1.

Proposition 3.2. Let $A(L) = \{a_1, \ldots, a_k\}$ and L be a lattice which is the union of the sets $[a_1]^u, \ldots, [a_k]^u$ such that $[a_i]^u \cap [a_j]^u = \{1\}$, for $i \neq j$. Then

$$G(L) = K_{n_1} + \cdots + K_{n_k},$$

where $n_i = |[a_i]^u| - 1$.

Corollary 3.3. The intersection graph $G(L)$ is complete if and only if $|A(L)| = 1$.

In the next lemma, we describe the neighborhood of a vertex x in $G(L)$. Recall that, for a vertex x, the neighborhood of x, denoted by $N(x)$, is the set of vertices adjacent to x.

Lemma 3.4. Let x be a vertex in $G(L)$. Then we have that

$$N(x) = \bigcup_{a \in A(L), a \leqslant x} [a]^u \setminus \{x, 1\}.$$

In particular, if b is an atom of L, then we have that $N(b) = [b]^u \setminus \{b, 1\}$.

Proof. Suppose that $y \in N(x)$. By Lemma 3.1, $x, y \in [a]^u$, for some $a \in A(L)$. Thus we have that

$$N(x) \subseteq \bigcup_{a \in A(L), a \leqslant x} [a]^u \setminus \{x, 1\}.$$

Also one can easily check that the converse inclusion holds, too. $\boxed{QED}$

In the following theorem, we study the increasing property of the degrees of vertices in $G(L)$.

Theorem 3.5. Let x and y be vertices in $G(L)$. If $x \leqslant y$, then $\deg(x) \leqslant \deg(y)$.

Proof. By Lemma 3.4, we have that

$$N(x) = \bigcup_{a \in A(L), a \leqslant x} [a]^u \setminus \{x, 1\}$$

and

$$N(y) = \bigcup_{a \in A(L), a \leqslant y} [a]^u \setminus \{y, 1\}.$$

Since $x \leqslant y$, $a \leqslant x$ implies that $a \leqslant y$, for all $a \in A(L)$. So $N(x) \subseteq N(y)$, and hence $\deg(x) \leqslant \deg(y)$. $\boxed{QED}$

The following corollary is an immediate consequence of Theorem 3.5.

Corollary 3.6. In the intersection graph $G(L)$ we have the following equality:

$$\Delta(G(L)) = \max\{\deg(m) \mid m \in C(L)\}.$$

Proposition 3.7. Assume that $|C(L)| = 1$. Then $G(L)$ is a refinement of a star graph.

Proof. Let $C(L) = \{m\}$. Clearly, for all atoms a in L, we have that $a \leqslant m$. Now the result follows by Lemma 3.4. $\boxed{QED}$

Proposition 3.8. The intersection graph $G(L)$ is totally disconnected if and only if $L = A(L) \cup \{0, 1\}$. Moreover, the vertex x is an isolated vertex in $G(L)$ if and only if x is an atom with $|[x]^u| = 2$.

Proof. First suppose that $G(L)$ is totally disconnected. If there exists a vertex x of $G(L)$ that is not an atom and such that, for some atom a, $x \in [a]^u \setminus \{1\}$, then the vertices a and x are adjacent, which is impossible. So we have $L = A(L) \cup \{0, 1\}$. The converse statement is clear.

The second statement follows from Lemma 3.4. $\boxed{QED}$

In the next proposition, we determine the girth of the intersection graph $G(L)$.

Proposition 3.9. If there exists an atom a in L such that $|[a]^u| \geqslant 4$, then $\mathrm{g}(G(L)) = 3$. Otherwise, $\mathrm{g}(G(L)) = \infty$.

Proof. Suppose that there exists $a \in L$ with $|[a]^u| \geqslant 4$. Then we have the cycle $a - x - y - a$, where x and y are distinct elements in $[a]^u \setminus \{a, 1\}$. Hence $\mathrm{g}(G(L)) = 3$.

Now assume that $|[b]^u| \leqslant 3$, for all atoms b in L. Suppose for a contradiction that there exists a cycle $x_1 - x_2 - \cdots - x_n - x_1$ in $G(L)$, where $n \geq 3$. If $x_1, x_2 \notin A(L)$, then there exists an atom $a \in A(L)$ such that $\{x_1, x_2, a, 1\} \subseteq [a]^u$, which is impossible. Hence we have $x_1 \in A(L)$ or $x_2 \in A(L)$. Without loss of generality, we may assume that $x_1 \in A(L)$. Since $x_2, x_n \in N(x_1)$, we have $x_2, x_n \in [x_1]^u$. Therefore $|[x_1]^u| \geq 4$, which is a contradiction. Thus in this situation $G(L)$ doesn't contain any cycle, and so $\mathrm{g}(G(L)) = \infty$. $\boxed{QED}$

In the following theorem, we study the clique number of the intersection graph of a lattice L.

Theorem 3.10. In the intersection graph $G(L)$, we have the following inequality:

$$\omega(G(L)) \geqslant \max\{|[a]^u| - 1 \mid a \in A(L)\}.$$

Proof. In view of Lemma 3.4, we have $N(a) = [a]^u \setminus \{1, a\}$. Also, by Lemma 3.1, the induced subgraph on the vertex-set $[a]^u \setminus \{1\}$ is complete. So the result holds. $\boxed{QED}$

In the next result, we determine the independence number of $G(L)$.

Theorem 3.11. In the intersection graph $G(L)$, we have that $\alpha(G(L)) = |A(L)|$.

Proof. Clearly an induced subgraph of $G(L)$ with vertex-set $A(L)$ is totally disconnected, and so $A(L)$ is an independent set in $G(L)$. Hence $\alpha(G(L)) \geqslant |A(L)|$. Now, if there exists an independent set S with more than $|A(L)|$ elements, then there are elements $x, y \in S$ such that $x, y \in [a]^u$, for some atom a in L. In view of Lemma 3.1, x is adjacent to y, which is impossible. Therefore we have $\alpha(G(L)) = |A(L)|$. $\boxed{QED}$

In the following proposition, we show that the intersection graph of a lattice is well-behaved under a lattice isomorphism. We recall that for two lattices L and S, the one-to-one correspondence $f : L \longrightarrow S$ is a lattice isomorphism if, for any $a, b \in L$, $f(a \vee b) = f(a) \vee f(b)$ and $f(a \wedge b) = f(a) \wedge f(b)$.

Proposition 3.12. Assume that $f : L \longrightarrow S$ is a lattice isomorphism. Then there exists a graph isomorphism between the intersection graphs $G(L)$ and $G(S)$.

Proof. Define the map $\overline{f} : V(G(L)) \longrightarrow V(G(S))$ such that $\overline{f}(x) = f(x)$, for each $x \in V(G(L))$. Suppose that the vertices x and y are adjacent in $G(L)$. So $x \wedge y \neq 0$. By the lattice isomorphism definition, we have that $x \wedge y \neq 0$ if and only if $f(x) \wedge f(y) \neq 0$. Hence $\overline{f}(x)$ is adjacent to $\overline{f}(y)$ in $G(S)$. Thus the result holds. $\boxed{QED}$

In the following example, we state that the converse of Proposition 3.12 is not true, in general. However, by Theorem 3.11, if the intersection graphs of the lattices L and S are isomorphic, then $|A(L)| = |A(S)|$.

Example 3.13. Suppose that S and L are the lattices given in Figure 1.

Then, it is easy to see that $G(L)$ and $G(S)$ are isomorphic to K_3, but the lattices L and S are not isomorphic.

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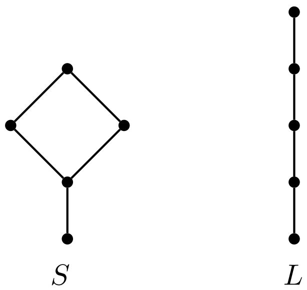

Figure 1.

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