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On a theorem of D. Ryabogin and V. Yaskin about detecting symmetry [i]

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Abstract. We give a simple deduction of a recent theorem of D. Ryabogin and V. Yaskin, about detecting symmetry of star bodies in $\mathbb{R}^n$ with C^1 radial functions — via their conical section functions — from an older theorem of us.

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1 Notions and notations

We will work in *Euclidean space* $\mathbb{R}^n$, where $n \geq 2$. Its *unit sphere* will be written as S^{n-1}. We say that $K \subset \mathbb{R}^n$ is a *star body* if it is of the form $K = \{\lambda u \mid u \in S^{d-1}, 0 \leq \lambda \leq \varrho_K(u)\}$, where $\varrho_K : S^{n-1} \to (0, \infty)$ is a continuous function, which is called the *radial function* of the star body K. A *convex body in* $\mathbb{R}^n$ is a compact convex set with interior points. If $K \subset \mathbb{R}^n$ is a convex body containing 0 in its interior, then it is a star body. Moreover, its radial function ϱ_K is Lipschitz (we consider S^{n-1} with its geodesic metric, and Lipschitz is meant with respect to it), cf. [4], first paragraph of §3. For

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$f : S^{n-1} \to \mathbb{R}$ a Lipschitz function, we denote by $L(f)$ its Lipschitz constant (with respect to the geodesic metric on S^{n-1}). Observe that if the radial function ϱ_K of a star body K is C^1 (for which we will shortly say that *the star body is* C^1), then it is Lipschitz (actually, this implication holds even for any function $S^{n-1} \to \mathbb{R}$).

For $\xi \in S^{n-1}$ we write $\xi^\perp$ for the linear $(n-1)$-subspace of $\mathbb{R}^n$ orthogonal to ξ. We will use also spherical polar coordinates, with north pole some $\xi \in S^{n-1}$. That is, we write each $x \in S^{n-1}$ as

$$x = \xi \sin\psi + \eta \cos\psi, \quad \text{where} \quad \eta \in S^{n-1} \cap \xi^\perp \quad \text{and} \quad -\pi/2 \le \psi \le \pi/2.$$

We call ψ the *geographic latitude* (which will be more convenient for us than the customarily used $\varphi = \pi/2 - \psi$), and will write

$$x = (\eta, \psi).$$

A function $f : S^{n-1} \to \mathbb{R}$ is *even* if $f(x) = f(-x)$, for all $x \in S^{n-1}$.

D. Ryabogin and V. Yaskin [6], p. 509, denoted, for $\xi \in S^{n-1}$ and $z \in (-1, 1)$, by $C(\xi, z)$ the cone $\{0\} \cup \{x \in \mathbb{R}^n \setminus \{0\} \mid \cos(\angle \xi 0 x) = z\}$. Then, for $K \subset \mathbb{R}^n$ a star body, [6], pp. 509-510, defined the *conical section function* $C_{K,\xi}(z)$ of K as

$$C_{K,\xi}(z) := \mathrm{vol}_{n-1}\left(K \cap C(\xi, z)\right),$$

where vol_{n-1} means $(n-1)$-*volume.*

2 Some results of D. Ryabogin-V. Yaskin and E. Makai, Jr.-H. Martini-T. Ódor

[6] proved the following geometrical theorem, by a relatively short proof, but using advanced methods, namely, Fourier transform techniques. (The converse implication in Theorem A is obvious.)

Theorem A. *(*[6]*, Theorem 1.1) Let $K \subset \mathbb{R}^n$ be a C^1 star body. Assume that, for all $\xi \in S^{n-1}$, the function $C_{K,\xi}(z)$ has a critical point at $z = 0$. Then the body K is* 0*-symmetric.*

Here, a *critical point of a function* is a point such that the derivative of the function at this point exists, and equals 0.

[6], in the remarks in the second paragraph after their Theorem 1.2, mentioned that, by the methods of [4], Theorem A can be extended to any convex body containing 0 in its interior. This also follows from our Theorem A′ below.

Theorem A follows from the following analytical theorem.

Theorem A′. *Let $f : S^{n-1} \to \mathbb{R}$ be a Lipschitz function. Assume that, for almost all $\xi \in S^{n-1}$, we have that the integral of f on the set $S^{n-1} \cap C(\xi, z)$, as a function of z, has a critical point at $z = 0$. Then f is an even function.*

We obtain Theorem A, by applying Theorem A′ to the C^1, hence Lipschitz function $f := \varrho_K^{n-1}/(n-1)$. Since the radial function ϱ_K of a convex body $K \subset \mathbb{R}^d$, containing 0 in its interior, is Lipschitz (cf. §1), the above mentioned extension of Theorem A to convex bodies, containing 0 in their interiors, follows from Theorem A′ similarly.

Remark. To justify the hypotheses of Theorem A′, we recall from [4], Lemma 3.5 and its proof, and Lemma 3.6, the following. For $f : S^{n-1} \to \mathbb{R}$ being a Lipschitz function, for almost all $\xi \in S^{n-1}$ we have that, for almost all $x \in S^{n-1} \cap \xi^\perp$, the function f is differentiable. Further, for almost all $\xi \in S^{n-1}$ we have that, for $z = 0$,

$$\frac{d}{dz} \int_{S^{n-1} \cap (\xi^\perp + z\xi)} f(x) dx$$

exists. Moreover, it equals

$$\int_{S^{n-1} \cap \xi^\perp} \frac{\partial f}{\partial \psi}(x) dx,$$

where $\psi = \psi_\xi$ is the geographic latitude, with the north pole at ξ (hence the partial derivative $\partial f / \partial \psi$ is taken along a meridian, in the direction toward the north pole ξ), and where also the second integral exists, for almost all $\xi \in S^{n-1}$. (These readily imply that, also in Theorem A′, the converse implication holds.)

Now we cite a theorem from [4].

Theorem B. *([4], Lemma 3.6, Theorem 3.8) Let $f : S^{n-1} \to \mathbb{R}$ be a Lipschitz function. Assume that, for almost all $\xi \in S^{n-1}$, we have*

$$\int_{S^{n-1} \cap \xi^\perp} \frac{\partial f}{\partial \psi}(x) dx = 0.$$

Then f is an even function.

To justify the hypotheses of Theorem B, recall the Remark above. (The above remark readily implies that, also in Theorem B, the converse implication holds.)

We have used Theorem B in [4] to prove another geometrical theorem. This theorem was proved for $n = 2$ by [2], Theorem 1; for $n \geq 3$ it was first proved by [4], Corollary 3.4, Lemma 3.5, Theorem 3.8, by using spherical harmonics, and the Funk-Hecke formula. It was reproved, for $n \geq 3$, by a relatively short proof, however, using advanced methods, namely, Fourier transform techniques, by [6], Theorem 1.2, for the C^1 case. This geometrical theorem states the following.

Let $K \subset \mathbb{R}^n$ be a star body with Lipschitz radial function. Then, for almost all $\xi \in S^{n-1}$, the function $z \mapsto \mathrm{vol}_{n-1}\big(K \cap (\xi^\perp + z\xi)\big)$ (vol_{n-1} meant here as $(n-1)$-dimensional Lebesgue measure) is differentiable at 0. Let, for almost all $\xi \in S^{n-1}$, this function have a critical point at $z = 0$. Then the body K is 0-symmetric. (The last but two sentence readily implies that, also in this geometrical theorem, the converse implication holds.)

An infinitesimal variant of the last mentioned geometrical theorem, for the case of a convex body (infinitesimally) close to the unit ball, not for the $(n-1)$-volumes of the intersections $K \cap (\xi^\perp + z\xi)$, but for the ($(n-2)$-dimensional) surface area, and also for the lower (but positive) dimensional quermassintegrals (cf. [1], §32, [7], §§ 4.1, 4.2) of these intersections, has been proved, in the sufficiently regular case, in [3], Theorem. Details cf. there.

In [5] we have proved an (almost) generalization of Theorem B, when $\partial f/\partial\psi$ in the hypothesis of Theorem B was replaced by $(\partial/\partial\psi)^m f$, for $m \geq 2$ an integer. Details cf. there.

In what follows, we show that our Theorem B implies Theorem A′ (and thus also Theorem A).

3 Proof of the implication Theorem B $\Longrightarrow$ Theorem A′.

Proof. We have, writing $\sin\psi := z$ (where $-\pi/2 < \psi < \pi/2$),

$$\int_{S^{n-1}\cap C(\xi,z)} f(x)dx = \int_{S^{n-1}\cap C(\xi,z)} f(\eta,\psi)d(\eta,\psi). \tag{1}$$

By the Lipschitz property of f we have, for $(\eta,\psi) \in S^{n-1}$,

$$|f(\eta,\psi) - f(\eta,0)| \leq L(f)\cdot|\psi|. \tag{2}$$

By (2) we have, for $\psi \neq 0$, that

$$\left\{ \begin{array}{l} |f(\eta,\psi) - f(\eta,0)|/|\sin\psi| = [|f(\eta,\psi) - f(\eta,0)|/|\psi|]\cdot[\psi/\sin\psi] \\ < L(f)\cdot(\pi/2)/1 =: c. \end{array} \right. \tag{3}$$

By [4], Lemma 3.5 and its proof, and Lemma 3.6 (cf. also our Remark), for almost all $\xi \in S^{n-1}$, we have that for almost all $x \in S^{n-1}\cap\xi^\perp$ the function f is differentiable, and thus, in particular, $(\partial f/\partial\psi)(x)$ (taken along a meridian, in the direction toward ξ) exists. Moreover, for these (almost all) ξ's, and for

$z \in (-1,1) \setminus \{0\}$ and $z \to 0$ we have, using in the first equality (1),

$$
\begin{cases}
\left[\int_{S^{n-1}\cap C(\xi,z)} f(x)dx - \int_{S^{n-1}\cap C(\xi,0)} f(x)dx\right] / z = \\
\left[\int_{S^{n-1}\cap C(\xi,z)} f(\eta,\psi)d(\eta,\psi) - \int_{S^{n-1}\cap C(\xi,0)} f(\eta,0)d(\eta,0)\right] / z = \\
\left[\int_{S^{n-1}\cap \xi^\perp} f(\eta,\psi)d\eta \cdot \cos^{n-2}\psi - \int_{S^{n-1}\cap \xi^\perp} f(\eta,0)d\eta\right] / \sin\psi = \\
\int_{S^{n-1}\cap \xi^\perp} \left[[f(\eta,\psi)d\eta - f(\eta,0)]/\psi] \cdot [\psi/\sin\psi]\right] d\eta + \\
\int_{S^{n-1}\cap \xi^\perp} f(\eta,\psi)d\eta \cdot (\cos^{n-2}\psi - 1)/\sin\psi \to \\
\int_{S^{n-1}\cap \xi^\perp} (\partial f/\partial\psi)(\eta) \cdot 1 \cdot d\eta + 0,
\end{cases}
\tag{4}
$$

by $\psi/\sin\psi \to 1$, and $|\int_{S^{n-1}\cap\xi^\perp} f(\eta,\psi)d\eta| \cdot (1-\cos^{n-2}\psi)/|\sin\psi| \leq \cos^{n-2}(\psi) \times \mathrm{vol}_{n-2}(S^{n-2}) \cdot \max\{|f(x)| \mid x \in S^{n-1}\} \cdot (1-\cos^{n-2}\psi)/|\sin\psi| = O(|\psi|) \to 0$, for $\psi \to 0$ (vol_{n-2} meaning $(n-2)$-volume). Still we used for the convergence of the summand in the fourth line of (4) Lebesgue's dominated convergence theorem with integrable majorant c, cf. (3), for each $\xi \in S^{n-1}$ for which for almost all $x \in S^{n-1} \cap \xi^\perp$ the function f is differentiable, thus for almost all $\xi \in S^{n-1}$.

By the hypothesis of Theorem A$'$, the last expression in (4) vanishes for almost all $\xi \in S^{n-1}$, thus the hypothesis of Theorem B is satisfied. Hence also the conclusion of Theorem B is satisfied, i.e., f is even, which is the conclusion of Theorem A$'$ as well. ■

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