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A STRICT TOPOLOGY FOR SOME WEIGHTED SPACES OF CONTINUOUS FUNCTIONS

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Dedicated to the memory of Professor Gottfried Köthe

In the classical case the strict topology β introduced by Buck [2] on the space $C^b(X)$ of bounded continuous scalar valued functions on the locally compact Hausdorff space X is given by the system W of all weights on X that vanish at infinity. The β-bounded subsets of $C^b(X)$ are exactly the norm bounded subsets, and β is the finest locally convex topology which coincides on the norm bounded subsets with the compact open topology (cf. Dorroh [4]). Especially we have that $C^b(X) = CW(X) = CW_0(X)$ holds algebraically.

In this note we want to describe for an arbitrary system of weights V an associated system of weights W such that at least in many cases, including the classical one, the connection between $CV(X)$ and $CW_0(X)$ is the same as in the classical case.

Notation. X will always denote a locally compact Hausdorff space, $C(X)$ the space of all continuous real or complex valued functions on X. An upper semicontinuous (u.s.c.) function $v : X \to \mathbb{R}^+$ is called a *weight.* A set V of weights on X is called a *Nachbin family or a system of weights* iff $(N1) \forall x \in X \exists v \in V$ such that $v(x) > 0$ and $(N2) \forall \lambda > 0$, $v_1, v_2 \in V \exists v_3 \in V$ such tht λ max $(v_1, v_2) \leq v_3$. The *weighted space* $CV(X)$ is then defined by $CV(X) = \{f \in C(X) | p_v(f) = \sup_{x \in X} |f(x)|v(x) < \infty \ \forall v \in V\}$. $CV(X)$ is endowed with the topology τ_V generated by the seminorms $p_v, v \in V$ (cf. Nachbin [10]).

$CV_0(X) = \{f \in CV(X) | v|f| \text{ vanishes at infinity } \forall v \in V\}$ is a subspace of $CV(X)$ always endowed with the restriction of τ_V.

If $\underline{1}$ denotes the function identical 1 on X then $V = \{\lambda \cdot \underline{1} | \lambda > 0\}$ is a system of weights and $C^b(X) = CV(X)$ topologically. The set $\mathcal{K}$ of all positive multiples of characteristic functions of compact subsets of X is also a system of weights and $C\mathcal{K}(X)$ is the space $C(X)$ with the compact open topology.

If V and V' are systems of weights on X we say $V \leq V'$ if $\forall v \in V \exists v' \in V'$ such that $v \leq v'$. V and V' are said to be *equivalent* $(V \cong V')$ if $V \leq V'$ and $V' \leq V$. If V is a system of weights so is $V' = \{\lambda v | \lambda > 0, v \in V\}$ and both are equivalent. So we will often assume that $\lambda v \in V \forall \lambda > 0$, $v \in V$. For properties of weighted spaces see for example Nachbin [10], Bierstedt, Meise, Summers [1] and Ernst, Schnettler [6].

All other notations not introduced here are taken from Köthe [9], Jarchow [7], Ernst, Schnettler [6].

If v is an arbitrary weight on X we denote by W_v the set of weights $w \not\equiv 0$ such that (i) $w \leq v$ and (ii) $\frac{w}{v}$ vanishes at infinity. Here and in the sequel we put «$\frac{0}{0} = 0$». If V is a system of weights on X we define the *associated system of weights* W_V by $W_V = \cup_{v\in V} W_v$. If no confusion occurs we simply write W instead of W_V.

Remark. Systems of weights similar to W have been considered previously in the literature (cf. Napalkov [11], Korobeinik [8]).

In the first lemmas we want to establish some properties of W.

Lemma 1. *If $v \not\equiv 0$ we have $W_v \neq \phi$, and if V is a system of weights W is also a system of weights.*

Proof. If $v(x_0) > 0$ put $w(x_0) = v(x_0)$ and $w(x) = 0$ for $x \neq x_0$. Then $w \in W_v$.

If $w \in W$ there is $v \in V$ such that $w \in W_v$. For $\lambda > 0$ take $v' \in V$ such that $\lambda v \leq v'$. Then $\lambda w \in W_{v'}$, hence $\lambda w \in W$. For $w_1, w_2 \in W$ choose $v_1, v_2, v \in V$ such that $w_i \in W_{v_i}$, $i = 1,2$ and $v_i \leq v$, $i = 1,2$. Put $w = \max(w_1, w_2)$. Then $w \in W_v$ and W satisfies $(N2)$.

Finally, if $x_0 \in X$ choose $v \in V$ such that $v(x_0) > 0$. Then the function w from the first part of the proof shows that W also satisfies $(N1)$. ∎

Remark. By definition we always have $W \leq V$. If in addition $\mathcal{K} \leq V$ there is for $\lambda > 0$ and any compact set $K \subset X$ a weight $v \in V$ such that $\lambda_{\chi_K} \leq v$ (χ_K denotes the characteristic function of K). Hence $\lambda_{\chi_K} \in W_v$ and $\mathcal{K} \leq W$.

Lemma 2. *Let $v \neq 0$ be any weight on X. Then for any $w \in W_v$ there is a continuous function μ on X such that the following holds: (i) $0 \leq \mu \leq 1$, (ii) $w \in W_{\mu v}$, and (iii) $\mu v \in W_v$.*

Proof. Choose a sequence (K_n) of compact subsets of X such that $\frac{w(x)}{v(x)} \leq \frac{1}{2^{2n}}$ for $x \notin K_n$. We may assume $K_n \subset \dot{K}_{n+1} \forall n \in \mathbb{N}$. Choose $\varphi_n \in C(X)$ such that $0 \leq \varphi_n \leq 1$, $\varphi_n(x) = 1 \forall x \in K_n$, and $\operatorname{supp}\varphi_n \subset K_{n+1} \forall n \in \mathbb{N}$. Put $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\varphi_n$. Then $\mu \in C(X)$ and (i) is satisfied. For $x \notin \cup_{n=1}^{\infty} K_n$ we have $\mu(x) = w(x) = 0$. For $x \in K_{n+1} \setminus K_n$ we have $\frac{1}{2^n} \leq \mu(x) \leq \frac{1}{2^{n-1}}$, $w(x) \leq \frac{1}{2^{2n}} v(x) \leq \mu(x)v(x)$, and hence $\frac{w(x)}{\mu(x)v(x)} \leq \frac{2^n}{2^{2n}} = \frac{1}{2^n}$. Thus (ii) is satisfied. (iii) is obvious. ∎

Remark. Lemma 2 has some nice implications: (i) $W_{W_V} \cong W_V$, (ii) $W \cong C_0^+(X) \cdot V$ where $C_0^+(X)$ denotes the positive continuous functions on X vanishing at infinity. Hence in the classical case W_V exactly generates the strict topology. and (iii) $V \subset C(X)$ implies $W \cong (W \cap C(X))$, i.e. in this case we can assume the functions in W to be continuous.

Next we prove a technical lemma which will be useful in the sequel.

Lemma 3. *Let v be a weight on X and $\beta : X \to \mathbb{R}^+$ any function which is bounded on the compact subsets of X. If $v\beta$ is unbounded on X there is $w \in W_v$ such that $w\beta$ is unbounded on X.*

Proof. Since $v\beta$ is unbounded choose a sequence $(x_n) \subset X$ such that $v(x_n)\beta(x_n) \geq \geq n^2 \forall n \in \mathbb{N}$. Put $Y = \{x_n | n \in \mathbb{N}\}$. Then Y and any of its subsets are closed. To prove this take $M \subset Y$ and $x_0 \in X \setminus M$. If U is a compact neighbourhood of x_0 the set $U \cap M$ is finite (or empty) since β and v hence βv are bounded on U. Hence $X \setminus M$ is open.

Now define a function $w : X \to \mathbb{R}^+$ by

$$w(x) = \begin{cases} \dfrac{v(x_n)}{n} & \text{for} \quad x = x_n \\ 0 & \text{for} \quad x \in X \setminus Y. \end{cases}$$

Obviously $0 \leq w \leq v$ and for any $\varepsilon > 0$ the set $\{x \in X | \frac{w(x)}{v(x)} \geq \varepsilon\}$ is finite hence compact. Since for $\varepsilon > 0$ the sets $\{x \in X | w(x) \geq \varepsilon\}$ are contained in Y they are closed and hence w is u.s.c. and $w \in W_v$. By construction $\beta(x_n)w(x_n) \geq n$ and βw is unbounded on X. ∎

Now we can prove a connection between $CV(X)$ and $CW_0(X)$.

Proposition 1. *If V is a system of weights on X then the identities $CW_0(X) = CW(X) = = CV(X)$ hold algebraically.*

Proof. First we show $CV(X) \subset CW_0(X)$. If $f \in CV(X)$ and $w \in W$ choose $v \in V$ such that $w \in W_v$. Put $M = \sup_{x \in X} |f(x)|v(x)$. Then $|f(x)|w(x) = |f(x)|v(x)\frac{w(x)}{v(x)} \leq \leq M\frac{w(x)}{v(x)} \; \forall x \in X$. For $\varepsilon > 0$ choose a compact subset $K \subset X$ such that $\frac{w(x)}{v(x)} \leq \leq \frac{\varepsilon}{M} \; \forall x \in X \setminus K$. Then $|f(x)|w(x) \leq \varepsilon \; \forall x \in X \setminus K$ and $|f(x)|w(x)$ is bounded on K. Hence $f \in CW_0(X)$.

Since $CW_0(X) \subset CW(X)$ it suffices to show $CW(X) \subset CV(X)$. If $f \in C(X) \setminus CV(X)$ there is $v \in V$ such that $|f|v$ is unbounded on X, but $|f|v$ is bounded on compact subsets of X. By Lemma 3 there is $w \in W$ such that $|f|w$ is unbounded. Hence $f \notin \notin CW(X)$ and the proof is complete. ∎

Next we want to prove a connection between the topologies τ_W and the restriction of $\tau_{\mathcal{K}}$ to $CW_0(X)$, which will again be denoted by $\tau_{\mathcal{K}}$.

Proposition 2. *If $\mathcal{K} \leq V$ the following holds:*

a) The bounded sets in $CV(X)$ and $CW_0(X)$ are the same and

b) $\tau_{\mathcal{K}}$ and τ_W induce the same topology on τ_V-bounded sets.

Proof. a) Since $\tau_W \leq \tau_V$ we only have to show that each τ_W-bounded subset B is also τ_V-bounded. Since $\mathcal{K} \leq V$ we have $\tau_{\mathcal{K}} \leq \tau_W$ by the remark following Lemma 1. Put $\beta(x) = \sup_{f\in B}|f(x)|$. The boundedness of B with respect to τ_V is equivalent to the boundedness of $v\beta$ for all $v \in V$ (put $0 \cdot \infty = 0$). Since B is also $\tau_{\mathcal{K}}$-bounded β is bounded on the compact subsets of X. Assume B is not τ_V-bounded. Then choose $v \in V$ such that $v\beta$ is unbounded. Then by Lemma 3 there is $w \in W$ such that $w\beta$ is unbounded. Hence B is not τ_W-bounded which is a contradiction.

b) By a well known lemma due to Grothendieck it suffices to show that the two topologies induce the same neighbourhoods of zero on the absolutely convex τ_W-bounded subsets. Let B be an absolutely convex τ_W-bounded set and put $\beta(x) = \sup_{f\in B}|f(x)|$. First we want to show that βw vanishes at infinity for all $w \in W$. If $w \in W$ take $v \in V$ such that $w \in W_v$. B is τ_V-bounded by Proposition 1, hence $M = \sup_{x\in X}\beta(x)v(x)$ is finite. Since $\beta w = \beta v \frac{w}{v} \leq M\frac{w}{v}$ the function βw vanishes at infinity.

Since $\tau_{\mathcal{K}} \leq \tau_W$ it suffices to show that for each τ_W-neighbourhood of zero $U_w = \{f \in$ $\in CW_0(X) \mid \sup_{x\in X}|f(x)|w(x) \leq 1\}$ there is a $\tau_{\mathcal{K}}$ neighbourhood of zero U such that $U \cap B \subset U_w$. By the first part of the proof there is a compact subset $K \subset X$ such that $\beta(x)w(x) \leq 1$ $\forall x \in X \setminus K$. Put $R = \sup_{x\in K} w(x)$ and $U = \{f \in CW_0(X) \mid |f(x)| \leq$ $\leq \frac{1}{R}\forall x \in K\}$. Then R is finite and U is a $\tau_{\mathcal{K}}$-neighbourhood of zero. If $f \in B \cap U$ we have

$$w(x)|f(x)| \leq \begin{cases} 1 & \text{for } x \in X \setminus K \\ RR^{-1} & \text{for } x \in K \end{cases}. \quad \text{Hence } U \cap B \subset U_w. \qquad \blacksquare$$

From Proposition 2 naturally the question arises, whether τ_W is the finest locally convex topology on $CV(X)$ which coincides on the τ_V-bounded subsets with $\tau_{\mathcal{K}}$. Unfortunately we do not know the exact anwer. However, in [3] Cook shows (by completely different methods) that the finest locally convex topology on $C(X)$ which coincides on the order intervalls of $C(X)$ with $\tau_{\mathcal{K}}$ is again a weighted topology. This may give a hint that, at least in some cases, the finest locally convex topology that coincides on τ_V-bounded sets with $\tau_{\mathcal{K}}$ is a weighted topology, hopefully τ_W. However our situation is different from Cook's: first $CV(X)$ is in general a proper subspace of $C(X)$, and secondly the order intervalls are bounded but the converse is not true in general. It is true if and only if $CV(X)$ has a fundamental system $\mathcal{B}$ of bounded sets (i.e. each bounded set is contained in a member of $\mathcal{B}$) such that $\beta(x) =$ $= \sup_{f\in B}|f(x)|$ is continuous $\forall B \in \mathcal{B}$. At the end of this note we give an example which (among other things) shows that even in good situations one can not expect the bounded sets to be contained in order intervalls.

If $CV(X)$ has a countable fundamental system of bounded sets, i.e. a fundamental sequence $\mathcal{B} = (B_n)$ of bounded sets, the situation is much better since we can apply results from [6]. Since X is locally compact $CV(X)$ satisfies condition (*) from [6] saying, $\forall x \in X \exists f \in CV(X)$ such that $f(x) \neq 0$. Let $\tilde{\mathcal{K}}$ denote $C_c^+(X)$. Then $\tilde{\mathcal{K}}$ is a system of weights and we have $\mathcal{K} \cong \tilde{\mathcal{K}}$ and $\tau_{\mathcal{K}} = \tau_{\tilde{\mathcal{K}}}$. In this situation we know from [6] that the finest locally convex topology $\tau_{\mathcal{K}}^{\mathcal{B}}$ which coincides on each B_n with $\tau_{\mathcal{K}}$ is a weighted topology. A system of weights for this topology can be described in terms of the functions $\beta_n(x) = \sup_{f \in B_n} |f(x)|$ and the weights in $\tilde{\mathcal{K}}$. If (φ_n) is any sequence in $\tilde{\mathcal{K}}$ put $s'(x) = \inf_{n \in \mathbf{N}_0}(\beta_n(x) + \frac{1}{\varphi_n(x)})$, where $B_0 = \{0\}$ and $\beta_0 \equiv 0$ .Then the sets $U_{(\varphi_n)} = \{f \in CV(X) | |f| \leq s'\}$ form a basis of $\tau_{\mathcal{K}}^{\mathcal{B}}$-neighbouhoods of zero as (φ_n) runs through all sequences in $\tilde{\mathcal{K}}$. A system of weights describing this topology is given by the weights u^{-1}, where $u(x) = \sup_{f \in U_{(\varphi_n)}} |f(x)|$.

Since $\mathcal{K}$ and $\tilde{\mathcal{K}}$ are equivalent the topology $\tau_{\mathcal{K}}^{\mathcal{B}}$ can also be constructed from the functions $s(x) = \inf_{n \in \mathbf{N}_0}(\beta_n(x) + \frac{1}{\lambda_n \chi_{K_n}(x)})$, where $\lambda_n > 0 \ \forall n \in \mathbf{N}_0$ and (K_n) is any sequence of compact subsets of X such that $K_n \subset \dot{K}_{n+1} \ \forall n \in \mathbf{N} \Lambda$, $K_0 = \emptyset$. To prove $\tau_W = \tau_{\mathcal{K}}^{\mathcal{B}}$ we need a further condition on V and the $\beta_n' s$. V is said to satisfy condition (g), if for any sequence (K_n) of compact subsets of X such that $K_n \subset \dot{K}_{n+1} \ \forall n \in \mathbf{N}_0$ there is $v \in V$ such that for $\gamma_n = \inf_{x \in K_n \setminus \dot{K}_{n-1}} v(x)\beta_n(x) (K_{-1} = \phi)$ holds: (i) $\gamma_n > 0 \ \forall n \in \mathbf{N}$ and (ii) $\gamma_n \to \infty$.

If (B_n) is a fundamental sequence of bounded sets in $CV(X)$ we can always assume $B_0 = \{0\}$, $2\beta_{n+1}$, and $B_n = \{f \in CV(X) | |f| \leq \beta_n\}$.

Lemma 4. *Let V be a system of weights such that $CV(X)$ has a fundamental sequence $\mathcal{B} = (B_n)$ of bounded sets and such that V satisfies (g). If $s = \inf_{n \in \mathbf{N}_0}(\beta_n + \frac{1}{\lambda_n \chi_{K_n}})$, $\forall_n > 0$ and (K_n) a sequence of compact subsets of X with $K_n \subset \dot{K}_{n+1} \forall n \in \mathbf{N}_0$, there are $\mu_n \geq \lambda_n$ such that for the function $t = \inf_{n \in \mathbf{N}_0}(\beta_n + \frac{1}{\mu_n \chi_{K_n}})$ the following holds:*
(i) $t \leq s$.
(ii) $\forall n \in \mathbf{N}\, t(x) = \beta_n(x) \frac{1}{\mu_n}$ for $x \in K_n \setminus K_{n-1}$.
(iii) $\exists v \in V$ such that $\frac{1/t}{v}$ vanishes at infinity.

Proof. For the sequence (K_n) choose $v \in V$ as in (g). v is bounded on the compact sets $K_n \setminus \dot{K}_{n-1}$, hence by $\gamma_n > 0$ we have $\mathrm{d}_n = \inf_{x \in K_n \setminus \dot{K}_{n-1}} \beta_n(x) > 0$. Now put $\mu_n = \max(\lambda_n, \frac{1}{\mathrm{d}_n})$. If $x \in K_n \setminus K_{n-1}$ we have $\inf_{1 \leq k \leq n-1}(\beta_k(x) + \frac{1}{\mu_k \chi_{K_k}(x)}) = \infty$ and $\beta_n(x) + \frac{1}{\mu_k \chi_{K_k}(x)} \leq \beta_n(x) + \mathrm{d}_n \leq 2\beta_n(x) \leq \beta_{n+1}(x) \leq \beta_{n+j}(x) + \frac{1}{\mu_{n+j}\chi_{K_{n+j}}(x)}$ $\forall j \in \mathbf{N}$. Hence (ii) is satisfied. To prove (iii) note that $t(x) = \infty \forall x \in X \setminus \cup_{n=1}^{\infty} K_n$ and then $\frac{1/t(x)}{v(x)} = 0$ by the convention «$\frac{0}{0} = 0$». If $x \in K_n \setminus K_{n-1}$ we have $t(x) \neq$ $\neq 0 \neq v(x)$ and $t(x)v(x) \geq \beta_n(x)v(x) \geq \gamma_n$, hence (iii). ∎

With this lemma we can prove

Proposition 3. *If $\mathcal{K} \leq V$, if $CV(X)$ has a fundamental sequence $\mathcal{B} = (B_n)$ of bounded sets, and if V satisfies (g), then τ_W is the finest locally convex topology on $CV(X)$ that coincides with $\tau_{\mathcal{K}}$ on the τ_V-bounded sets.*

Remark. The proposition says in fact that in this case $CW_0(X)$ is a (gDF)-space (cf. [6]).

Proof of Proposition 3. By Proposition 2 we know already $\tau_W \leq \tau_{\mathcal{K}}^{\mathcal{B}}$. It remains to show $\tau_{\mathcal{K}}^{\mathcal{B}} \leq \tau_W$. Let U be a $\tau_{\mathcal{K}}^{\mathcal{B}}$-neighbourhood of zero. There is a sequence (C_n) of compact subsets of X and $\lambda_n > 0$ such that for the function $s' = \inf_{n \in \mathbf{N}_0} (\beta_n + \frac{1}{\lambda_n \chi_{C_n}})$ we have $U_{s'} = \{f \in CV(X) | |f| \leq s'\} \subset U$. Enlarging the $C_n' s$, if necessary, we obtain a sequence (K_n) of compact subsets of X such that $K_n \subset \dot{K}_{n+1}$ and $C_n \subset K_n \forall n \in \mathbf{N}_0$. Hence $s = \inf_{n \in \mathbf{N}_0} (\beta_n + \frac{1}{\lambda_n \chi_{K_n}}) \leq s'$ and $U_s = \{f \in CV(X) | |f| \leq s\}$ is a $\tau_{\mathcal{K}}^{\mathcal{B}}$-neighbourhood of zero such that $U_s \subset U_{s'} \subset U$. Now let t denote the function constructed in Lemma 4. Then U_t is a $\tau_{\mathcal{K}}^{\mathcal{B}}$-neighbourhood of zero and $U_t \subset U_s$. Put $\mu(x) = \sup_{f \in U_t} |f(x)|$. Then $U_\mu = U_t$ (cf. [6]), μ is l.s.c. and $\mu > 0$ by condition (*). By construction we have $\mu(x) = t(x) = \beta_n(x) + \frac{1}{\mu_n}$ for $x \in \dot{K}_n \setminus K_{n-1}$. If x is in the thin set $K_n \setminus \dot{K}_n$, $\mu(x)$ is either $\beta_n(x) + \frac{1}{\mu_n}$ or $\beta_{n+1}(x) + \frac{1}{\mu_{n+1}}$. Hence $\frac{1/\mu}{v}$ vanishes at infinity by (iii) of Lemma 4. Finally putting $w = \frac{1}{\mu}$ we have shown $w \in W_{\lambda v}$ for some $\lambda > 0$, $v \in V$ and $U_t = \{f \in CV(X) | \sup_{x \in X} |f(x)| w(x) \leq 1\}$ is a τ_W-neighbourhood of zero. ∎

Remark. Proposition 3 obviously already contains the classical result of Dorroh and Sentilles, that $C^b(X)$ with the strict topology is a (gDF)-space.

One may ask now whether condition (g) in Proposition 3 is necessary or completely superfluous. The answer to both questions is no. A hint that some condition on V is necessary gives the following consideration. If V_1, V_2 are systems of weights on X such that $V_1 \leq V_2$, $V_1 \not\geq V_2$, and $CV_1(X)$ and $CV_2(X)$ have the same fundamental sequence $\mathcal{B}$ of bounded sets, one would expect $\tau_{W_1} \neq \tau_{W_2}$. Thus at most τ_{W_2} can equal $\tau_{\mathcal{K}}^{\mathcal{B}}$. Hence we have the best chance for $\tau_{\mathcal{K}}^{\mathcal{B}}$ to equal τ_W if V is the finest system of weights on X such that $CV(X)$ has $\mathcal{B}$ as fundamental sequence of bounded sets. At the end of this note we give an example for this situation.

That (g) is not necessary can be seen from the following example. Let V denote the set of all weights on $\mathbf{R}$. Then $W = V$ and $CW_0(\mathbf{R}) = CV(\mathbf{R}) = C_c(\mathbf{R})$ algebraically. A fundamental sequence of bounded sets B_n is given by the functions $\beta_n = 2^{n-1} \chi_{(-n,n)}$, $n \in \mathbf{N}$. By [6] $CV(X)$ is a (gDF)-space and τ_V and $\tau_{\mathcal{K}}$ coincide on the $B_n' s$. On the other hand V does not satisfy (g) for the sequence of $K_n = [-(n+1), n+1]$.

To exhibit further examples where Proposition 3 is applicable we consider a construction given by Bierstedt, Meise, Summers [1]: let (v_n) denote a sequence of strictly positive weights on X such that $v_n \geq 2v_{n+1} \forall n \in \mathbb{N}$. If V is the system of weights containing all weights $\overline{v}$ of the form $\overline{v} = \inf_{n\in\mathbb{N}} \alpha_n v_n$, $\alpha_n > 0$, then $C\overline{V}(X) = \cup_{n=1}^{\infty} Cv_n(X)$ is a space with a fundamental sequence of bounded sets which can be described by the functions $\beta_n = \frac{2^{n-1}}{v_n}$, i.e. $B_n = \{f \in C\overline{V}(X) \mid |f| \leq \beta_n\}$. With this notation we can prove

Proposition 4. *If for each compact subset $K \subset X$ and each $n \in \mathbb{N}$ $\inf_{x\in K} v_n(x) > 0$ then $\overline{V}$ satisfies condition (g).*

Remark. The condition on the sequence (v_n) is obviously satisfied if all v_n are continuous.

Proof of Proposition 4. Let (K_n) be a sequence of compact subsets of X such that $K_n \subset \subset \mathring{K}_{n+1}$ $\forall n \in \mathbb{N}$. We have to construct a sequence (α_n) of strictly positive numbers such that $\overline{v} = \inf_{n\in\mathbb{N}} \alpha_n v_n$ satisfies (g). Put $\mathrm{d}_n = \inf_{x\in K_n} v_n(x)$ and $M_n = \sup_{x\in K_n} v_n(x)$, $n \in \mathbb{N}$. Hence $0 < \mathrm{d}_n \leq M_n < \infty$ $\forall n \in \mathbb{N}$. Now put $\alpha_1 = 1$ and $\alpha_n = M_1 \frac{M_1 \ldots M_{n-1}}{\mathrm{d}_1 \ldots \mathrm{d}_n}$ for $n \geq 2$. For any $n, k \in \mathbb{N}$ and $x \in K_n$ holds

$$\alpha_{n+k} v_{n+k}(x) \geq \alpha_{n+k} \mathrm{d}_{n+k} = M_1 \frac{M_1 \ldots M_{n+k-1}}{\mathrm{d}_1 \ldots \mathrm{d}_{n+k-1}} \geq \alpha_n v_n(x).$$

Hence $\overline{v}(x) = \min(\alpha_1 v_1(x), \ldots, \alpha_n v_n(x)) \geq v_n(x)$ $\forall x \in K_n$ and $\beta_n(x)\overline{v}(x) \geq \geq \beta_n(x) v_n(x) = 2^{n-1}$ $\forall x \in K_n$. ∎

Now assume V is a system of weights on $X, \mathcal{K} \leq V$ and such that $CV(X)$ has a fundamental sequence (B_n) of bounded sets given by the functions β_n. Then there is a finest system of weights V_b on X such that $V \leq V_b$, $CV(X) = CV_b(X)$ algebraically and that all B_n's are bounded in $CV_b(X)$. V_b is given by the weights u of the form $u = \inf_{n\in\mathbb{N}} \frac{\alpha_n}{\beta_n}$, $\alpha_n > 0$ $\forall n \in \mathbb{N}$ (cf. [6]). Since $\mathcal{K} \leq V$ we have that $\sup_{x\in K} \beta_n(x)$ is finite $\forall n \in \mathbb{N}$ and any compact subset $K \subset X$. If we furthermore have $\beta_n > 0 \forall_{n\in\mathbb{N}}$ the functions $v_n = \frac{1}{\beta_n}$ satisfy the hypothesis in Proposition 4. In this case we have $\overline{V} = V_b$ and hence

Corollary. *Let V be a system of weights on X with $\mathcal{K} \leq V$ and assume $CV(X)$ has a fundamental sequence $\mathcal{B} = (B_n)$ of bounded sets such that the functions $\beta_n(x) = \sup_{f\in B_n} |f(x)|$ are strictly positive. Then the finest locally convex topology $\tau_{\mathcal{K}}^{\mathcal{B}}$ on $CV(X)$ that coincides on the τ_V-bounded sets with $\tau_{\mathcal{K}}$ is given by the system of weights W_{V_b} associated with V_b.*

This Corollary shows how to find an example where $\tau_W \neq \tau_{\mathcal{K}}^{B}$: choose a system of weights V such that V is strictly coarser than V_b and hope that W_V will also be strictly coarser than W_{V_b}. This will be done in the following example (cf. [5], [6]):

$$\text{Take } X = \mathbb{R}^+, \beta : \mathbb{R}^+ \to \mathbb{R}^+ : \beta(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ 2^{(n^2)} & \text{for } x \in (n, n+1] \, n \in \mathbb{N} \end{cases},$$

$B = \{f \in C(\mathbb{R}^+) \mid |f| \leq \beta\}$, $\widetilde{V} = \{\widetilde{v} \in C^+(\mathbb{R}^+) | \beta \leq \widetilde{v}\}$, $V = \{\frac{\lambda}{\widetilde{v}} | \lambda > 0, \ \widetilde{v} \in \widetilde{V}\}$. Then V is a system of strictly positive continuous weights on $\mathbb{R}^+$, $CV(\mathbb{R}^+)$ has $(2^{n-1} B)$ as fundamental sequence of bounded sets, $V \neq V_b$, and B is not contained in any order intervall of $CV(\mathbb{R}^+)$. τ_{V_b} is a normed topology with closed unit ball B. Hence $V_b = \overline{V} = \{\frac{\lambda}{\beta} | \lambda > 0\}$ and $\tau_{\mathcal{K}}^{B} = \tau_{W_{\overline{V}}}$. By the remark following Lemma 2 we may assume that W_V contains continuous functions only. To prove $\tau_{W_V} \neq \tau_{W_{\overline{V}}}$ it suffices to find a function $\overline{w} \in W_{\overline{V}}$ which is not dominated by any function in W_V.

If $\mu(x) = \begin{cases} 1 & \text{for } x \in [0,1] \\ n+1 & \text{for } x \in (n, n+1], n \in \mathbb{N} \end{cases}$ then $\overline{w} = \frac{1}{\mu\beta} \in W_{\overline{V}}$. Assume there is $w \in W_V$ such that $\overline{w} \leq \lambda w$ for some $\lambda > 0$. Then $\mu\beta \geq \frac{1}{\lambda w}$. On the other hand $sup_{x \in X} w(x)\beta(x)$ is finite, hence there is $\gamma > 0$ such that $\beta \leq \gamma \cdot \frac{1}{w}$. Taking $\delta = \max(\lambda, \gamma)$ we obtain $\beta \leq \delta \frac{1}{w} \leq \delta^2 \mu\beta$. Since the jumps of β increase very rapidly as n tends to infinity, there is no continuous function between β and $\delta^2 \mu\beta$. Hence $\tau_{W_V} \neq \tau_{W_{\overline{V}}}$.

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