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ONE-FORMS ON SPACES OF EMBEDDINGS: A FRAME WORK FOR CONSTITUTIVE LAWS IN ELASTICITY

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Dedicated to the memory of Professor Gottfried Köthe

0. INTRODUCTION

The present contribution to this volume is concemed with certain problems in non-linear functional analysis which are motivated by classica1 physics, specifically by elasticity theory: we are given a «body», i.e. a compact smooth manifold M' which moves and may be deformed in some $\mathbb{R}^n$ (equipped with a fixed inner product); we assume that the motion and deformation are such that the diffeomorphism type of M' does not change. Hence, M' is the image under a smooth embedding of some compact smooth manifold M (possibly with boundary ∂M) and the appropriate configuration space for the problem is the set $E(M, \mathbb{R}^n)$ of smooth embeddings $M \to \mathbb{R}^n$; this set is a smooth Fréchet manifold when endowed with its natura1 C^∞-topology.

The deformable medium is to be characterized by a «smooth one-form» on $E(M, \mathbb{R}^n)$, i.e. by a smooth real-valued function F which to each configuration $J \in E(M, \mathbb{R}^n)$ and distortion $L \in C^\infty_J(M, \mathbb{R}^n)$ assigns a number $F(J)(L)$, depending linearly on L, which is interpreted as the work caused by L at J, cf. section 4. An approach to elasticity along these lines is described e.g. in [E,S] and [Bi 4]; cf. also section 6 for more details where we also relate our treatment to the usual one such as given in [L,L].

If the deformations mentioned above are subject to smooth constraints or if the motion no longer takes place in $\mathbb{R}^n$, we will still assume that the ambient space is a smooth Riemannian manifold N and this forces us to introduce as a configuration space the manifold $E(M, N)$ of smooth embeddings $M \to N$. Since the tangent bundle $TE(M, N)$ no longer is trivial, in general, the treatment of one-forms on $E(M, N)$ becomes somewhat more complicated. In order to obtain «integral representations» of certain one-forms, we assume that both M and N are oriented. With this assumption, sections 2 and 3 introduce the basic geometric ingredients needed for integra1 rcpresentations of those one-forms which at each $J \in E(M, N)$ only depend on the one-jets of the vector fields L «along J».

We introduce the metrics $\mathfrak{B}$ and $\mathfrak{B}^\partial$ on $E(M, N)$ and $E(\partial M, N)$, respectively, which are continuous, symmetric and positive-definite bilinear forms on the respective tangent spaces. Both $\mathfrak{B}$ and $\mathfrak{B}^\partial$ are invariant under the group $Diff^+ M$ of orientation preserving diffcomorphisms of M and any group $\mathfrak{J}$ of orientation preserving isometries of N. Section 3 furthermore introduces the bundle $\mathfrak{A}^1_E(M, TN)$ of «smooth TN-valued one-forms on M» which cover embeddings $M \to N$, fibred over $E(M, N)$ by the Fréchet spaces

$\mathfrak{H}^1_J(M, TN) \cong A^1(M, J^*TN)$. On these fibres, a «dot metric» $\mathfrak{g}$ is defined by

$$\mathfrak{g}(J)(\mathfrak{a}, \mathfrak{b}) = \int_M \mathfrak{a} \cdot \mathfrak{b}\, \mu(J),$$

for $\mathfrak{a}$, b $\in \mathfrak{H}^1_J(M, TN)$. Here $\mathfrak{a}$. b is a smooth real-valued function on M which is symmetric and bilinear in $\mathfrak{a}$, b and whose construction is based on the classical «trace inner product» for bundle endomorphisms of the Riemannian bundle *TN.* If $\mathfrak{g}$ is restricted to the subspace $\mathfrak{T}_E(M, TN)$ introduced in section 3, one obtains a generalization of the classica1 Dirichlet integral (cf. [Bi 2]).

Section 4 deals with $\mathfrak{g}$ -representable one-forms F on $E(M, N)$, by which we mean the following:
There exists a smooth map $\mathfrak{a} : E(M, N) \to \mathfrak{H}^1_E(M, TN)$ such that for $J \in E(M, N)$ and L $\in C^\infty_J(M, TN)$,

$$(0.1) \qquad F(J)(L) = \int_M \mathfrak{a}(J) \cdot \nabla L \mu(J) = \mathfrak{g}(J)(\mathfrak{a}(J), \nabla L),$$

where VL is the covariant derivative of L along J induced by theLevi-Civita connection of N. In particular, this yields a more precise notion of the «dependence on the one-jets» of $F(J)(L)$ mentioned above.

A crucial step is the following result of section 4: for any $\mathfrak{g}$ -representable one-form F, there exists a smooth vector field $\mathfrak{h}$ on $E(M, N)$ such that

$$(0.2) \qquad F(J)(L) = \int_M \nabla\mathfrak{h}(J) \cdot \nabla L \mu(J) = \mathfrak{g}(J)(\nabla\mathfrak{h}(J), \nabla L)$$

holds for $J \in E(M, N)$ and $L \in C^\infty_J(M, TN)$. The existence of such a field $\mathfrak{h}$ follows from the fact that $\mathfrak{a}$ in (0.1) defines an elliptic boundary problem value (of the Neumann type) whose solvability is guaranted by [Ho 2]. The right-hand side of (0.2) may be rewritten in the form

$$\int_M \langle \Delta(J)\mathfrak{h}(J), L\rangle \mu(J) + \int_{\partial M} \langle \nabla_{\mathfrak{n}} \mathfrak{h}(J), L\rangle i_{\mathfrak{n}}\, \mu(J),$$

with $\mathfrak{n}$ the positively oriented m(J) -unit normal of ∂M. Here A(J) is the Laplacian of V and m(J) on M. In physical terms, if F describes the deformable medium in N, then A $(J)\mathfrak{h}(J)$ and $\nabla_{\mathfrak{n}}\mathfrak{h}(J)$ are the force densities acting on M, ∂M, respectively.

Section 5 deals with the special case N = IR" (with a fixed inner product $<,>$) and shows, e.g., that the «one-jet dependence» of F as formulated above is equivalent with the independence of F of the center of mass for each configuration J. Finally, section 6 indicates the reason for the description of the medium in $\mathbb{R}^n$ by means of a one-form on $E(M, \mathbb{R}^n)$: Classically, elasticity theory as described in [L,L] deals with the set of all those Riemannian structures on M which are pull-backs of $<,>$ under the elements of $E(M, \mathbb{R}^n)$ and this set is in general not a manifold in its C^∞-topology; lifting the description to $E(M, \mathbb{R}^n)$ provides a configuration space which is a manifold. Moreover, one of the reasons why we describe the deformable medium by a one-form F is that N generically does not admit non-trivial orientation preserving isometries and hence, one cannot simply work with a symmetric stress tensor. However, as a theorem in [S] shows, F *can be* replaced by a smooth symmetric tensor field provided that it is SC(n) -invariant and that infinitesimal rigid motions do not cause any work. Note lastly that it is shown in [Bi 4] that the description of elasticity of [L,L] is included in our current framework.

1. GEOMETRIC PRELIMINARIES AND THE FRECHET MANIFOLD $E(M, N)$

Let M be a compact, oriented, connected smooth manifold with (oriented) boundary ∂M and N be a connected, smooth and oriented manifold with a Riemannian metric $<,>$. The Levi-Civita connection of $<,>$ on N is denoted by V and by d in the euclidean case, i.e. if N = $\mathbb{R}^n$ and $<,>$ is assumed to be a fixed scalar product. For $J \in E(M, \mathbb{R}^n)$ we define a Riemannian metric on M by setting

$$(1.1) \qquad m(J)(X,Y) := \langle TJX, TJY \rangle, \quad \forall X,Y \in \Gamma(TM)$$

and one on ∂M via the formula

$$(1.2) \qquad m(j)(X,Y) := (TjX, TjY), \quad \forall X,Y \in \Gamma(T(\partial M))$$

(here j := $J|\partial M$). More customary are the notations $J^* <,>$ and $j^* <,>$ for m(J) and m(j) respectively.

We use Γ (TQ) to denote the collection of all smooth vector fields of any smooth manifold Q (with or without boundary). Moreover by π_Q : TQ $\to$ Q we mean the canonical projection.

Let $L : M \to TM$ be a smooth map. Then $f = \pi_N \cdot L \in C^\infty(M, \mathrm{N})$ and L is a «vector field along f». For a fixed f, the set of all such «vector fields along f» is precisely the tangent space at f to $C^\infty(M, \mathrm{N})$ (cf. [Bi,Sn,Fi] and also below at the end of this section).

Next, let V be a (linear) connection on N, i.e. in TN. There is the associated splitting of T^2 N = $T(TN)$ into the canonically detined vertical bundle $V(TN)$ and the horizontal bundle $H(TN)$ defined by V (cf. [G,H,V]). Since $V(TN) = ker(T\pi_N)$, the fibre $V_v(TN)$ at the point $v \in TN$ is $T_v(T_qN)$ with $q = \pi_N v$ and hence, there is a natural isomorphism $\zeta_v : V_v(TN) \to T_{\pi_N v}$N for every $v \in TN$. These isomorphisms yield a bundle map $\zeta : V(TN) \to TN$ covering the projection π_N. Lastly, let $P : T^2N \to V(TN)$ be the projection with kernel $H(TN)$.

The covariant derivative ∇L of L is now defined as follows: for $X \in \Gamma(TM)$, $TL . X$ is a map $M \to T^2N$ and we set

$$\nabla_X L := \zeta P(TL \cdot X).$$

In our applications, ∇ will be the Levi-Civita connection of the Riemannian manifold $(N, \langle , \rangle)$ and in this situation, the Levi-Civita connections of $(M, m(J))$, $(\partial M, m(j))$ respectively are obtained as follows:
$TN|J(M)$ splits into $TJ(TM)$ and its orthogonal complement $(TJ(TM))^\perp$ (the Riemannian normal bundle of J) and hence any $Z \in \Gamma(J(M), TN)$ has an orthogonal decomposition $Z = Z^\top + Z^\perp$, where the tangential component $Z^\top$ is a section of $TJ(TM)$ and so is of the form $Z^\top = TJ . U$ for a unique $U \in \Gamma(TM)$.

If now $Y \in \Gamma(TM)$, then TJY is a smooth map $M \to TN$ and therefore, the above covariant derivative $\nabla(TJY)$ is well-defined. We use this to define the vector field $\nabla(J)_X Y$ on M by the equation

$$(1.3) \qquad TJ\left(\nabla(J)_X Y\right) = \nabla_X(TJY) - \left(\nabla_X(TJX)\right)^\perp,$$

for all $X, Y \in \Gamma(TM)$. Moreover, if now $X, Y \in \Gamma(T\partial M)$, then

$$(1.4) \qquad Tj\left(\nabla(j)_X Y\right) = TJ\left(\nabla(J)_X Y\right) - m(j)(W(j)X, Y) . N(j)$$

defines a vector field $\nabla(j)_X Y$ on ∂M. Here $W(j)$, the Weingarten map, is defined as follows: by assumption, M is oriented and hence the normal bundle $(TM|\partial M)/T(\partial M)$ has a nowhere vanishing section s which is used to define the induced orientation of ∂M. Under the Riemannian structure $m(J)$, the normal bundle of ∂M is isomorphic to $T(\partial M)^\perp$ and as a consequence, this bundle now has a section n of unit length which corresponds to a multiple of s by a non-vanishing positive function. This n is the positive unit normal vector field along ∂M. With this, let $N(j) = TJ . n$ and now set

$$(1.5) \qquad TJ . W(j)Z = \left(\nabla_Z N(j)\right)^\top, \qquad \forall Z \in \Gamma(T\partial M).$$

As mentioned earlier, this determines $W(j)$ uniquely. Note here that $N = \mathbb{R}^n$, we may replace TJ and Tj by their «principal parts» dJ and dj respectively. In this particular case, we moreover define the *second fundamental form* $f(J)$ of J under the additional assumptions that $\partial M = \emptyset$ and $\dim(M) = n-1$, where now $N(j)$ is replaced by the positive unit normal field along J and W(j) is defined as in (1.5). The two-tensor f then is given by

$$f(j)(X,Y) = m(J)(W(j)X,Y),$$

for $J \in E(M, \mathbb{R}^n)$ and X, Y $\in \Gamma(T\partial M)$. Note finally that now H(j) := $trW(j)$ and $\kappa(j) = \det(W(j))$ are respectively *the* (unnormalized) ***mean*** *curvature* and *the Gaussian* ***curvature*** of $j(\partial M) \subset \mathbb{R}^n$. References for this section are e.g. [A,M,R], [Be,Go] and [G,H,V].

It is well-known that the set $C^\infty(M, N)$ of smooth maps from M into N endowed with Withney's C^∞-topology is a Fréchet manifold (cf. e.g. [Bi,Sn,Fi]). For a given $K \in C^\infty(M, N)$, the tangent space $T_K C^\infty(M, N)$ is the Fréchet space $C^\infty_K(M, TN) = \{L \in C^\infty(M, N) | \tau_N \circ L = K\} \cong \Gamma(K^*TN)$ and the tangent bundle $TC^\infty(M, N)$ is identified with $C^\infty(M, TN)$, the topology again being the C^∞-topology. In all this, M is assumed to be compact.

The set $E(M, N)$ of C^∞-embeddings $M \to N$ is open in $C^\infty(M, N)$ and thus is a Fréchet manifold whose tangent bundle we denote by $C^\infty_E(M, TN)$; it is an open submanifold of $C^\infty(M, TN)$, fibred over $E(M, N)$ by «composition with τ_N». Moreover, if $\partial M = \emptyset$, $E(M, N)$ is a principal Diff M-bundle under the obvious right Diff M-action and the quotient $U(M, N) = E(M, N)/\mathrm{Diff}(M)$ is the manifold of «submanifolds of type M» of N (cf. the above reference, ch. 5, and further literature quoted there).

Lastly, the set $\mathfrak{M}(M)$ of all Riemannian structures on M is a Fréchet manifold since it is an open convex cone in the Fréchet space of smooth, symmetric bilinear forms on M. Moreover, the maps

$$m: E(M,N) \to \mathfrak{M}(M)$$

and

$$m: E(\partial M,N) \to \mathfrak{M}(\partial M)$$

are smooth (cf. [Bi,Sn,Fi]).

By an *E-valued* one-form $\mathfrak{a}$ on M, where E is a vector bundle over N, we mean a smooth map

$$\mathfrak{a}: TM \to E$$

for which $\mathfrak{a}|T_pM$ is linear for all p $\in M$. We denote the set of such one-forms by $\mathfrak{A}^1(M, E)$ and now obtain the following description of its structure:
The requirement that $\mathfrak{a} \in \mathfrak{A}^1(M, E)$ should be linear along the fibres of TM means that there is a (smooth) map $f: M \to N$ such that $\mathfrak{a}|T_pM$ is a linear map into $E_{f(p)}$ for p $\in M$, in other words, that $\mathfrak{a}$ is a bundle map $TM \to E$ over f.

There is $f \in C^\infty(M, N)$ such that $\pi_E \circ \mathfrak{a} = f \circ \tau_M$ (where π_E, τ_M are the respective bundle projections). The set of such one-forms is naturally identified with the Fréchet space

$A^1(M, f^*E)$. This shows that

$$\mathfrak{A}^1(M,E) = \cup_f \{A^1(M,f^*E) \mid f \in C^\infty(M,N)\}$$

It is clear from the construction that there is a natural surjection

$$\beta : \mathfrak{A}^1(M,E) \to C^\infty(M,N)$$

whose fibres are the Fréchet spaces $A^1(M, f^*E)$.

The map β is (set-theoretically!) locally trivial: $f \in C^\infty(M, N)$ has an open neighbourhood U_f such that there exists a fibre-preserving, fibrewise linear bijection

$$\varphi_f : \beta^{-1}(U_f) \to U_f \times A^1(M,f^*E),$$

which also is topological on each fibre; thus, for each $g \in U_f$, the restriction of φ_f to $\beta^{-1}(g)$ is a linear and topological isomorphis onto $A^1(M, f^*E)$.

The assertion of loca1 triviality can be established along the following lines (cf. [A]): One chooses a neighbourhood U_f of f in $C^\infty(M, N)$ which is diffeomorphic to some open, *convex* neighbourhood of $0 \in T_f C^\infty(M, N) = \Gamma(f^*TN)$. By the very construction of the usual Fréchet manifold structure of $C^\infty(M, N)$, this is always possible (cf. e.g. [Bi,Sn,Fi], ch. 5 and its references). Accordingly, there now exists a smooth contraction of U_f onto $\{f\}$, i.e. a smooth map $c : \mathbb{R} \times U_f \to C^\infty(M, N)$, such that $c(1,.)$ is the identity of U_f, $c(t, U_f) \subset U_f$ for $0 \leq t \leq 1$, and $c(0, g) = f$ for every $g \in U_f$. In particular, every $g \in U_f$ is smoothly homotopic to f by a homotopy induced by c. Accordingly, the choice of a liinear connection ∇ in E induces an isomorphism $g^*E \cong f^*E$ as in [G,H,V]; the corresponding isomorphisms $A^1(M, g^*E) \cong A^1(M, f^*E)$ now yield the desired trivialization φ_f.

Suppose next that U_1, U_2 are neighbourhoods of $f_{1,2}$ chosen as above and that $U_{1,2} := U_1 \cap U_2 \neq \emptyset$; let φ_i be the corresponding trivializations. Firstly, then, $U_{1,2} \times A^1(M, f^*E)$, $i = 1,2$, will be open submanifolds of $U_i \times A^1(M, f^*E)$ and secondly, the compositions $\varphi_2\varphi_1^{-1}$, $\varphi_1\varphi_2^{-1}$ are diffeomorphisms of these two submanifolds. As a conscqucnce, there exist a unique topology and diffcrcntiable structure on $\mathfrak{A}^1(M, E)$ with the following properties:

The sets $\beta^{-1}(U_f)$ obtained as above are open *submanifolds*, diffeomorphic to $U_f \times A^1(M, f^*E)$ under the maps φ_f. Thus, the mode1 space for $\beta^{-1}(U_f)$ is the Fréchet space $T_fC^\infty(M, N) \times A^1(M, f^*E)$. Lastly, the construction shows that with this diffcrentiablc structure, $\mathfrak{A}^1(M, E)$ bccomes a smooth Fréchet *vector bundle* over $C^\infty(M, N)$ with bundle projection β.

2. THE METRIC $\mathfrak{B}$ ON $E(M, N)$

The Riemannian structure $<,>$ of N induces a «Riemannian structure» $\mathfrak{B}$ on $E(M, N)$ as follows: for $J \in E(M, N)$, let $\mu(J)$ be the Riemannian volume defined on M by the given orientation and the structure m(J). For any two tangent vectors $L_1, L_2 \in C_J^\infty(M, TN)$, we set

$$(2.1) \qquad \mathfrak{B}(J)\,(L_1, L_2) := \int_M \langle L_1, L_2 \rangle \mu(J).$$

It is clear, that $\mathfrak{B}(J)$ is a continuous, symmetric, positive-definite bilinear form on C_J^∞ (M, TN). In the same manner, one obtains the metric $\mathfrak{B}^\partial$ on $E(\partial M, N)$.

The metrics $\mathfrak{B}$ and $\mathfrak{B}^\partial$ possess some invariance properties which will become important later: let $\mathrm{Diff}^+ M$ be the group of orientation-preserving diffeomorphisms of M. As a subgroup of $\mathrm{Diff}\, M$, it operates (freely) on the right on $E(M, N)$ as well as on $E(\partial M, N)$ by

$$(2.2) \qquad \begin{aligned} E(M,N) \times \mathrm{Diff}^+ M &\xrightarrow{\phi} E(M,N) \\ (J,\varphi) &\to J \cdot \varphi \end{aligned}$$

forafixed φ, wealsowrite $R_\varphi J$ for $J \cdot \varphi$.

Similarly, if $\mathfrak{J}$ is any group of orientation-preserving isometries of N, then it operates on the left on $E(M, N)$ as well as $E(\partial M, N)$ by

$$(2.3) \qquad \begin{aligned} \mathfrak{J} \times E(M,N) &\to E(M,N) \\ (g, J) &\to g \cdot J \end{aligned}$$

for fixed g, we also write $L_g J$ for g . J.

The geometry of these actions will be dealt with elsewhere, but we need the following - rather obvious! - result for some basic invariance properties of one-forms on $E(M, N)$:

Proposition 2.1. *Both $\mathfrak{B}$ und $\mathfrak{B}^\partial$ are invarinat under $\mathrm{Diff}^+ M$ and $\mathfrak{J}$.*

Proof. The $\mathrm{Diff}^+ M$-invariance is usual invariance of integration over M:

$$(2.4) \qquad \begin{aligned} R_\varphi^* \mathfrak{B}(J)\,(L_1, L_2) &= \mathfrak{B}(J \cdot \varphi)\,(L_1 \cdot \varphi, L_2 \cdot \varphi) = \\ &= \int_{\varphi(M)} \langle L_1, L_2 \rangle \cdot \varphi \mu(J \cdot \varphi) = \\ &= \mathfrak{B}(J)\,(L_1, L_2). \end{aligned}$$

Next, if $g \in \mathfrak{I}$, then $\mu(g \cdot J) = \mu(J)$ and hence

$$(2.5) \qquad \begin{aligned} L_\varphi^* \mathfrak{B}(J)\left(L_1, L_2\right) &= \mathfrak{B}(g \cdot J)\left(Tg \cdot L_1, Tg \cdot L_2\right) = \\ &= \int_M \langle Tg \cdot L_1, Tg \cdot L_2 \rangle \mu(g \cdot J) = \\ &= \mathfrak{B}(J)\left(L_1, L_2\right). \end{aligned}$$

Similar arguments establish the claim for $\mathfrak{B}^\partial$.

3. THE FIBRED SPACE $\mathfrak{L}_E(M, TN)$ AND ITS DOT METRIC

To begin with, denote by $\mathfrak{A}_E^1(M, TN)$ the subset of $\mathfrak{A}_V^1(M, TN)$ consisting of all TN-valued one-forms covering embeddings $M \to N$. This is the inverse image of $E(M, N)$ under the projection $\beta : \mathfrak{A}^1(M, TN) \to C^\infty(M, N)$, hence is an open submanifold and, in fact, is itself a (Fréchet) vector bundle whose fibre at J we denote by $\mathfrak{A}_J^1(M, TN)$.

By construction of m(J), TJ is fibrewise isometric and accordingly, the linear algebra outlined in appendix 3.1 (cf. below) may be used to write $\mathfrak{a} \in \mathfrak{A}_J^1(M, TN)$ in the form

$$(3.1) \qquad \mathfrak{a} = c(\mathfrak{a}, TJ) \cdot TJ + TJ \cdot A(\mathfrak{a}, TJ)$$

for suitable bundle endomorphisms $c(\mathfrak{a}, TJ)$ of $TN|J(M)$ and $A(\mathfrak{a}, TJ)$ of TM; these endomorphisms are smooth and continuous linear functions of $\mathfrak{a}$. The second summand on the right can also be written as $\widehat{A}(\mathfrak{a}, TJ)TJ$ (cf. appendix 3.2), and so $\mathfrak{a} = c(\mathfrak{a}, TJ) + \widehat{A}(\mathfrak{a}, TJ)$. The usual «trace inner product» for endomorphisms of TN then yields the dot product

$$(3.2) \qquad \mathfrak{a} \cdot \mathfrak{b} := -\frac{1}{2} tr c(\mathfrak{a}, TJ) \cdot c(\mathfrak{b}, TJ) + tr A(\mathfrak{a}, TJ) \cdot A^*(\mathfrak{b}, TJ),$$

A^* the adjoint of A formed fibre-wise with respect to m(J), and we define

$$(3.3) \qquad \mathfrak{g}(TJ)(\mathfrak{a}, \mathfrak{b}) := \int_M \mathfrak{a} \cdot \mathfrak{b}\, \mu(J).$$

This yields a smooth and continuous, symmetric and positive-definite bilinear form on the Fréchet space $\mathfrak{A}_J^1(M, TN)$, the «dot metric».

We shall also need a subfibration of $\mathfrak{A}_E^1(M, TN)$, defined by

$$(3.4) \qquad \mathfrak{L}_E(M, TN) := \{\nabla L \mid L \in C_E^\infty(M, TN)\},$$

whose fibres we denote by $\mathfrak{T}_J(M, TN)$ $(= \mathfrak{T}_E(M, \mathrm{TN}) \cap \mathfrak{H}^1_J(M, \mathrm{TN}))$; evidently these are subspaces of the Fréchet spaces $\mathfrak{H}^1_J(M, \mathrm{TN})$; for more information, cf. appendix 3.2.

Next, we introduce the Laplacean A(J) which will depend on J via m(J) ; cf. [Ma] and sew remarks in appendix 3.2:

For $K \in C^\infty_J(M, \mathrm{TN})$, we define the covariant divergence by

$$\nabla^*(J)K := 0, \tag{3.5}$$

as usual, while following [Ma], V*($J)\mathfrak{a}$ for $\mathfrak{a} \in \mathfrak{H}^1_J(M, \mathrm{TN})$ is given locally by

$$\nabla^*(J)\mathfrak{a} := -\sum_{r=1}^{n} \nabla_{E_r}(\mathfrak{a})(E_r), \tag{3.6}$$

(E_r) a local orthonormal frame with respect to $m(J)$; $\nabla_X \mathfrak{a}$ = V($J)_X\mathfrak{a}$ is defined in the standard manner by

$$\left(\nabla(J)_X\mathfrak{a}\right)(Y) = \nabla_X(\mathfrak{a}Y) - \mathfrak{a}\left(\nabla(J)_X Y\right), \qquad \forall X, Y \in \Gamma(TM).$$

To see that this definition does not depend on the moving frames chosen we write $\mathfrak{a}$ as a finite sum

$$\mathfrak{a} = \sum \gamma^i \otimes s_i, \tag{3.7}$$

with 7' $\in A^1$ (M, R) and $s_i \in T_J E(M, N)$. Moreover, let $a(\gamma^i, J)$ be the smooth strong bundle endomorphism of TM such that

$$\nabla(J)_X(\gamma^i)(Y) = m(J)\left(a\left(\gamma^i, J\right)X, Y\right), \tag{3.8}$$

holds for all pairs X, Y $\in \Gamma(TM)$ and for each i . In addition let Y $^i \in \Gamma$ (TM) for each i be such that

$$\gamma^i(X) = m(J)\left(Y^i, X\right), \qquad vx \in 1\text{-}(2\text{-}M). \tag{3.9}$$

With these data it is a matter of routine to show that

$$\nabla^*(J)\mathfrak{a} = -\sum\left(tr\, a\left(\gamma^i, J\right) . s_i + \nabla_{Y^i}s_i\right), \tag{3.10}$$

an expression independent of any moving frame.

Clearly if $\gamma \in A^1(M, \mathbb{R})$ and $\nabla = d$ then

$$d^*\gamma = -\operatorname{div}_j Y,$$

provided that

$$\gamma(X) = m(J)(Y, X), \qquad \forall X, Y \in \Gamma(TM).$$

$\Delta(J)$ is then defined by

(3.11) $$\Delta(J) := \nabla\nabla^*(J) + \nabla^*(J)\nabla.$$

The Laplacean $\Delta(J)$ is elliptic for any $J \in E(M, N)$ (cf. [Pa]). As we will see below it is self-adjoint with respect to $\mathfrak{B}(J)$ if $\partial M = 0$. For each $K \in T_J E(M, N)$ equation (3.6) yields

(3.12) $$\Delta(J)K = \nabla^*(J)\nabla K = -\sum_{r=1}^{n} \nabla_{E_r}(\nabla K)\left(E_r\right).$$

Remark 3.1. Suppose that $\gamma \in A^1(M, \mathbb{R})$ and $\nabla = d$. Define the vector field Y on M by $\gamma(X) = m(J)(Y, X)$ $(\forall X \in \Gamma(TM))$. Then it is clear that $d^*\gamma = -\operatorname{div}_J Y$, div_J the classical divergence operator with respect to $\mu(J)$.

The following theorem will be a basic tool in our studies of one forms on $E(M, N)$:

Theorem 3.2. *For any $J \in E(M, N)$, any $\alpha \in \mathfrak{H}^1_E(M, TN)$ and two $L_1, L \in C^\infty(M, TN)$ the following two relations hold*

(3.13) $$\mathfrak{g}(J)(\alpha, \nabla L) = \mathfrak{B}(J)\left(\nabla^*(J)\alpha, L\right) + \mathfrak{B}^\partial(j)(\alpha(\mathfrak{n}), l),$$

and

(3.14) $$\mathfrak{g}(J)\left(\nabla L_1, \nabla L\right) = \mathfrak{B}(J)\left(\Delta(J)L_1, L\right) + \mathfrak{B}^\partial(j)\left(\nabla_{\mathfrak{n}} L_1, l\right),$$

where $j := J|\partial M$ and $l := L\,|\,\partial M$. Here ∇ denotes the Levi-Civita connection of the metric $\langle , \rangle$ on N. Let $\mathfrak{K}_J := \{L \in C^\infty_J(M, TN) | \nabla L = 0\}$ for any $J \in E(M, N)$, then

(3.15) $$L \in \mathfrak{K}_J \Longleftrightarrow \left(\Delta(J)L = 0 \text{ and } \nabla_{\mathfrak{n}} L = 0\right).$$

In fact $\dim \mathfrak{K}_J < \infty$. *Equation (3.14) implies in turn a Green's equation*

(3.16) $$\int_M \langle \Delta(J)K, L\rangle\mu(J) - \int_M \langle K, \Delta(J)L\rangle\mu(J) = \\ = \int_{\partial M} \langle \nabla_{\mathfrak{n}} L, K\rangle i_{\mathfrak{n}}\, \mu(J) - \int_{\partial M} \langle \nabla_{\mathfrak{n}} K, L\rangle i_{\mathfrak{n}}\, \mu(J).$$

Here $i_n\, p(J)$ *is the volume element on* ∂M *defined by* $p(J)$ *. Moreover, if* $\partial M = \emptyset$ *then* $\mathfrak{a}$ *is* $\mathfrak{g}$ *-orthogonal to all of* $\mathfrak{H}^1_E(M, TN)$*, iff* $\nabla^*(J)\mathfrak{a} = 0$ *.*

Proof. Writing any $L \in C^\infty(M, TN)$ relative to a given $J \in E(M, N)$ in the form

$$L = TJX(L, J) + L^\perp, \tag{3.17}$$

with aunique $X(L, J) \in \Gamma(TM)$ (and $L^\perp$ being such that $L^\perp(p)$ is the component normal to TJT_pM for all $p \in M$), we have the following formula at hand:

$$\nabla_X L = TJ\nabla_X X(L, J) + (\nabla_X L)^\perp, \qquad \forall X \in \Gamma(TM). \tag{3.18}$$

From this equation we read off the coefticients in the decomposition (3.1):

$$c(\nabla L, T)TJ = (\nabla L)^\perp, \tag{3.19}$$

as well as

$$A(\nabla L, TJ) = \nabla X(L, J) + W(J, L), \qquad \forall L \in C^\infty(M, TN) \text{ and } \forall J \in E(M, N). \tag{3.20}$$

Here $W(J, L)$ is given by $TJW(J, L)X = (\nabla L^\perp X)^\top$, where, once again, $\perp$ denotes the component in $TN \mid J(M)$ orthogonal to $TJ(TM)$, while $\top$ is the component tangential to j(M), i.e. the component in $TJ(TM)$.

For each $\mathfrak{a} \in \mathfrak{H}^1(M, TN)$ and for each $J \in E(M, N)$, we write on the other hand

$$\mathfrak{a} = \overline{A}(\nabla\mathfrak{a}, TJ)TJ, \tag{3.21}$$

with $\overline{A}(\mathfrak{a}, TJ) : TN|J(M) \to TN \mid J(M)$ the smooth bundle endomorphism introduced above. Then for any moving frame (E_r) on M, orthonormal with respect to m(J), we deduce

$$\mathfrak{a} \cdot \nabla L = \sum_{r=1}^{m} \langle \overline{A}^*(\mathfrak{a}, TJ) \cdot \overline{A}(\nabla L, TJ)TJE_r; TJE_r \rangle =$$

$$= \sum_{r=1}^{m} \langle \overline{A}^*(\mathfrak{a}, TJ) \cdot \nabla_{E_r} L, TJE_r \rangle,$$

$\overline{A}^*(\mathfrak{a}, TJ)$ being the adjoint of $\overline{A}(\mathfrak{a}, TJ)$ formed with respect to $<,>$. Hence

$$\mathfrak{a}\cdot\nabla L = \sum_{r=1}^{m}\langle\nabla_{E_r}\left(\overline{A}^*(\mathfrak{a},TJ)L\right), TJE_r\rangle + \\ -\sum_{r=1}^{m}\langle L, \nabla_{E_r}\left(\overline{A}(\mathfrak{a},TJ)\right)TJE_r\rangle$$

yields

$$\mathfrak{a}\cdot\nabla L = \sum_{i=1}^{m}\langle\nabla_{E_r}\left(\overline{A}^*(\mathfrak{a},TJ)L\right), TJE_r\rangle + \\ + \langle\nabla^*(J)\mathfrak{a}, L\rangle + \sum_{i=1}^{m}\langle\overline{A}(\mathfrak{a},TJ)\nabla_{E_r}(TJ)E_r, L\rangle$$

Since $(\overline{A}^*(\mathfrak{a}, TJ)L)^{\perp} = TJZ(\mathfrak{a}, L, J)$ for some welldefined $Z(\mathfrak{a}, L, J)$ and since $\nabla_{E_r}(TJ)E_r$ is pointwise norma1 to $TJTM$ the following series of equations are immediate:

$$\mathfrak{a}\cdot\nabla L = -\sum_{i=1}^{m}\langle\nabla_{E_r}(c(\mathfrak{a},TJ)L), TJE_r\rangle + \operatorname{div}_J Z(\mathfrak{a}, L, J) + \\ + \langle\nabla^*(J)\mathfrak{a}, L\rangle + \sum_{i=1}^{m}\langle c(\mathfrak{a},TJ))\nabla_{E_r}(TJ)E_r, L^{\top}\rangle = \\ = -\sum_{i=1}^{m}\langle\nabla_{E_r}\left(c(\mathfrak{a},TJ)L^{\perp}\right), TJE_r\rangle + \\ -\sum_{i=1}^{m}\langle\nabla_{E_r}\left(c(\mathfrak{a},TJ)L^{\top}\right), TJE_r\rangle + \\ + \operatorname{div}_J Z(\mathfrak{a}, L, J) + \langle\nabla^*(J)\mathfrak{a}, L\rangle + \\ + \sum_{i=1}^{m}\langle c(\mathfrak{a},TJ)\nabla_{E_r}(TJ)E_r, L^{\top}\rangle = \\ = -\sum_{i=1}^{m}\langle\nabla_{E_r}\left(c(\mathfrak{a},TJ)L^{\perp}\right), TJE_r\rangle + \\ + \operatorname{div}_J Z(\mathfrak{a}, L, J) + (\nabla^*(J)\mathfrak{a}, L), \tag{3.22}$$

where div_J is the divergence operator associatcd with m(J). Writing $c(\mathfrak{a}, TJ)L^{\perp} = TJ\,U(\mathfrak{a}, L, J)$, for some well defined $U(\mathfrak{a}, L, J) \in \Gamma(TM)$, we obtain

$$\mathfrak{a}\cdot\nabla L = -\operatorname{div}_J U(\mathfrak{a}, L, J) + \operatorname{div}_J Z(\mathfrak{a}, L, J) + (\nabla^*(J)_{\mathfrak{a}}, L). \tag{3.23}$$

In case a = VK, then (3.23) turns imo

$$\text{(3.24)} \qquad \mathbf{VL\ .\ VL} = -\operatorname{div}_J U(K, L, J) + \operatorname{div}_J Z(K, L, J) + (\mathbf{A(J)}\ K, L).$$

Integrating (3.23) and (3.24) and applying the theorem of Gauss yields the desired equations (3.13) and (3.14). Since **A(J)** is elliptic (cf. appendix 3.2) dim $\mathfrak{K}_J < \infty$ as shown, e.g. in [Pa] and [Ho 2]. The rest of the routine arguments in this proof are left to the reader.

We close this section by showing that the metric 5 on the fibres of $\mathfrak{L}_E(M, TN)$ also possesses the *invariance* under Diff$^+ M$ and any group orientation-preserving isometries on N :

For any choice $\varphi \in$ Diff' M, $J \in E(M, N)$ and $L \in C^\infty(M, TN)$ *we* form

$$\text{(3.25)} \qquad \nabla(L \cdot \varphi) = \nabla L \cdot \varphi$$

and represent $\nabla(L\ .\ \varphi)$ with respect to $T(J\ .\ \varphi)$ yielding

$$\text{(3.26)} \qquad \nabla(L \cdot \varphi) = c(\nabla(L \cdot \varphi), T(J \cdot \varphi)) \cdot T(J \cdot \varphi) A(\nabla(L \cdot \varphi), T(J \cdot \varphi)).$$

Multiplying V$(L\ .\ \varphi)$ with $(T\varphi)^{-1}$ and comparing the resulting coefficients of (3.26) with those of (3.1) shows

$$c(\nabla L, TJ) \cdot \varphi = c(\nabla(L \cdot \varphi), T(J \cdot \varphi))T(J \cdot \varphi))$$

and

$$A(\nabla L, TJ) \cdot \varphi = T\varphi A(\nabla(L \cdot \varphi), T(J \cdot \varphi)) \cdot (T\varphi)^{-1}.$$

Now we verify

$$\text{(3.27)} \qquad \begin{aligned} &\mathfrak{g}(J \cdot \varphi)\left(\nabla\left(L_1 \cdot \varphi\right), \nabla\left(L_2 \cdot \varphi\right)\right) = \\ &= -\frac{1}{2}\int_M trc\left(\nabla L_1, TJ\right) \cdot c\left(\nabla L_2, TJ\right) \cdot \varphi\mu(J \cdot \varphi) + \\ &+ \int_M trA\left(\nabla L_1, TJ\right). A^*\left(\nabla L_2, TJ\right). \varphi\mu(J\ .\ \varphi) = \\ &= 5\,(J)\left(\nabla L_1, \nabla L_2\right), \end{aligned}$$

proving the Diff$^+$ M-invariance of 5 at *TJ.* To show the $\mathfrak{J}$-invariance we let g $\in \mathfrak{J}$ and only need to remark that

$$\text{(3.28)} \qquad \nabla(Tg \cdot L) = Tg \cdot \nabla L$$

holds. The rest is obvious. Therefore we have:

Proposition ***3.3. The metric 5 on*** $\mathfrak{L}_E(M, TN)$ ***is invariant under Diff$^+$ M and any group*** $\mathfrak{J}$ *of* ***orientation-preserving isometries on*** N **.**

APPENDIX 3.1.

As indicated earlier, we present here some of the linear algebra used in the construction of the dot product used in this section. The arguments may be interpreted as fibrewise considerations for bundle maps or, with some obvious changes in the formulation, as considerations at the levei of section modules.

The aim is to show that the dot product «essentially» is induced by the classical trace inner product in endomorphism rings of euclidean spaces and to this end, we now consider euclidean spaces E, F with irmer products $<,>$ and a fixed *isometry* α of E onto the subspace $E_1 \subset F$. For the sake of convenience, we write the elements of F as columns $\binom{e_1}{e_2}$ with respect to the direct sum decomposition $F = E_1 \oplus E_1^{\perp}$; here, $e_1 \in E_1$ and $e_2 \in E_1^{\perp}$; let also $p_1 : F \to E_1$, $p_2 : F \to E_1^{\perp}$ be the respective orthogonal projections.

Any endomorphism D of F now is represented by a 2 x 2-matrix

$$D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix},$$

where $D_{11} \in L(E_1)$, $D_{22} \in L(E_1^{\perp})$, $D_{12} \in L(E_1^{\perp}, E_1)$ and $D_{21} \in L(E_1, E_1^{\perp})$; *the* matrix acts on a column $\binom{e_1}{e_2}$ by the usual rules of matrix algebra.

Next, let $\varphi \in L(E, F)$. We are going to write φ in the form

$$\varphi = c\alpha + \alpha A = c\alpha + \hat{A}\alpha \tag{3.29}$$

for suitable choices of c $\in$ L(F) and $A \in$ L(E) (or $\hat{A} \in$ L(F)), both of them linear functions of φ:
For $e \in E$, *write* $\varphi e = \binom{\varphi_1 e}{\varphi_2 e}$; *thus*, $\varphi_1 = p_1\varphi$ and $\varphi_2 = p_2\varphi$. Firstly, since $E_1 = \mathrm{im}(\alpha)$, the expression $<\varphi e, \alpha f>$ (with e, $f \in E$) reduces to $<\varphi_1 e, \alpha f>$ and this bilinear form on E now can be written in the form $<Ae, f>$ fora unique $A \in$ L(E) ; in fact, since α is an isometry,

$$A = \alpha^{-1}\varphi_1 = \alpha^{-1}p_1\varphi.$$

There is a corresponding endomorphism A_1 of E_1 , namely $A_1 = p_1\varphi\alpha^{-1}$ and the endomorphism $\hat{A}$ of F now is the extension by 0 of this map; in other words:

$$\hat{A} = \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.30}$$

Secondly, we wish to write $\varphi_2 = p_2\,\varphi$ in the forrn $c\alpha$ for some $c \in L(F)$ and it is clear that c is not automatically uniquely deterrnined by this condition (unless $E_1 = F$), so that in the course of the construction, certain choices will have to be made. In a first step, let $c_1 : E_1 \to E_1^{\perp}$ be defined by $c_1 = P_2\varphi\alpha^{-1}$. Any extension of c_1 to an endomorphism of F then is of the form,

$$\begin{pmatrix} \beta & 7 \\ c_1 & \delta \end{pmatrix}$$

and its action on αe is the map $\alpha e \to \binom{\beta\alpha e}{c_1\alpha e}$; this forces the choice $\beta = 0$, but leaves γ, δ undetermined. The obvious choice for δ is 0 and with this, there now are three options for $\gamma : \gamma = 0,\ \gamma = c_1^*$ or $\gamma = -c_1^*$ (where «*» is the euclidean adjoint) and in all three cases, c will remain a linear function of φ. At this point, we make the choice $7 = -c_1^*$, so that we obtain

$$c = \begin{pmatrix} 0 & -c_1^* \\ c_1 & 0 \end{pmatrix}, \tag{3.31}$$

a skew-symmetric endomorphism of $F : c^* \to -c$. In part, this choice is motivated by the usual splitting $so(F) = so(E_1) \oplus so(E_1^{\perp}) \oplus L(E_1, E_1^{\perp})$, cf. section 5.

Let now $\psi : E \to F$ be a second linear map, written in the forrn $\psi = (D + \widehat{B})\,\alpha$ under the construction just outlined. A simple calculation shows that

$$(c + A)(D + B)^* = -cD + \widehat{A}\widehat{B}^* + (c\widehat{B}^* - \widehat{A}^*D),$$

where the terrn in parentheses is *trace free*. Moreover, the trace of $\widehat{A}\widehat{B}^*$ (in F) is easily seen to coincide with $tr_E(AB^*)$ since α is an isometry. Accordingly, the «trace inner product» in L(F) now reduces to $\cdot\, tr_F(cD) + tr_E(AB^*)$. Thus, we see that the dot product $\varphi \cdot \psi$ in $L(E, F)$ essentially is the inner product induced by the classical trace inner product under the construction $\varphi \to c + \widehat{A}\ \cdot$ up to the factor $\frac{1}{2}$ in the first summand. We shall add some remarks on this point below, but firstly now indicate the application of the linear algebra outlined here to the actual constructions used in this section:
Pointwisse, the role of α is played by TJ, that of φ by $\mathfrak{a} \in \mathfrak{H}^1_J(M, TN)$; accordingly $c(\mathfrak{a}, TJ) = c$ and $A(\mathfrak{a}, TJ) = a$. Note that this also shows that the bundle endomorphisms used above depend linearly on $\mathfrak{a}$.

Let us turn to the factor $\frac{1}{2}$ in equation (3.2); it appears because of the following reason. The endomorphism

$$-c(\mathfrak{a}, TJ) \cdot c(\mathfrak{b}, TJ)(J(p)) : T_{J(p)}N \to T_{J(p)}N \tag{3.32}$$

of $T_{J(p)}$ N splits for each $p \in M$ into a direct sum of the two linear maps

$$-c(\mathfrak{a}, TJ) \cdot c(\mathfrak{b}, TJ)|(J(p)) : T_J T_p M$$

and

$$-c(\mathfrak{a}, TJ) \cdot c(\mathfrak{b}, TJ)|(TJT_pM)^{\perp},$$

both endomorphisms of TJT_pM and $(TJT_pM)^{\perp}$ respectively. Their traces are identical. Thus the factor $\frac{1}{2}$ allows us to take only the pointwise formed trace of

(3.33) $$-c(\mathfrak{a}, TJ) . c(\mathrm{b}, TJ)|TJT_pM$$

into account. The endomorphism (3.33) can be pulled back to TM in the obvious manner. Hence in the dot product (3.2) contribute traces of endomorphisms of TM only.

APPENDIX 3.2.

It is pointed out earlier that the fibres $C^\infty_J(M, TN)$ of $C^\infty(M, TN) = TC^\infty(M, N)$ *are* naturally isomorphic to the section spaces $\Gamma\,(J^*TN)$; similarly, $A^1_J(M, TN)$ is isomorphic to $A^1(M, J^*TN)$. On the other hand, if V denotes e.g. the Levi-Civita connection of N, then there is the induced «pull-back connection» $J^*\nabla$ in J^*TN, obtained in the usual manner. It now is routine to verify that the following diagram commutes:

$$\begin{array}{ccc} C^\infty_J(M,TN) & \simeq & 1\text{-}(J^*TN) \\ \nabla\downarrow & & \downarrow \text{J'V} \\ A^1_J(M,TM) & \simeq & A^1_J(M,TM) \end{array}$$

V simply «is» the induced connection in J^*TN.

In addition, J^*TN carries a natural Riemannian structure given by $<,>$ in TN ; the connection $J^*\nabla$ is compatible with this metric. The Riemannian structure of J^*TN together with $\mu(J)$ now is used to obtain a pre-Hilbert space structure in $\Gamma(J^*TN)$ as well as in $A^1(M, J^*TN)$, etc., and hence under the isomorphisms in the above diagram, one obtains a forma1 adjoint V(J) of V. This operator coincides with the operator $\nabla^*(J)$ of this section and this shows that $\nabla^*(J)$ again is a first-order operator. Accordingly, the Laplacean A *(J)* as defined in the text now is seen to be a second-order elliptic operator. This will be true «at all levels», i.e. on the spaces $A^p_J(M, TN)$, $p \geq 1$, detined in the obvious manner. We omit the details here, but point out that the ellipticity of A *(J)* will be crucial later on.

At «level 0», the symbol of V is injective and one concludes now that the range $\mathfrak{T}_J(M, TN)$ of this V is closed in $\mathfrak{A}^1_J(M, TN)$, hence itsself a Fréchet space. In fact, one can argue that it is a split subspace and that $\mathfrak{T}_E(M, TN)$ is a Fréchet subbundle of $\mathfrak{A}^1_E(M, TN)$. *The* technical details of these claims will be dealt with elsewhere.

4. ONE FORMS ON $E(M, N)$

Recall that the tangent bundle of $E(M, N)$ is identified with $C^\infty_E(M, TN)$; accordingly, we define 1-forms on $E(M, N)$ as follows:

A *(scalar) 1-form* on $E(M, N)$ is a smooth function

$$F: C^\infty_E(M, TN) \to \mathbf{R}$$

with the property that for each $J \in E(M, N)$, the restriction $F(J) = F|C^\infty_J(M, TN)$ is linear in $L \in C_J(M, TN)$. In particular, $F(J)$ is a continuous linear form on this fibre, i.e. an element of the topological dual $C^\infty_J(M, TN)' \simeq \Gamma(J^*TN)'$. Loosely speaking, then, F is a smooth section of the «cotangent bundle» $\bigcup_J C_J(N, TN)'$ of $E(M, N)$, but this point-of-view will not be pursued any further here; cf. however below.

For our purposes, it will be sufficient to limit attention to a smaller class of such one-forms; in particular, their values will depend only on the one-jets of the elements of $C^\infty_E(M, TN)$. More precisely:

Definition 4.1. ***The one-form F on E(M, N) is said to be $\mathfrak{g}$-representable if there exists a smoothsection $\mathfrak{a}: E(M, N) \to \mathfrak{H}^1_E(M, TN)$ of the bundle $(\mathfrak{H}^1_E(M, TN), \beta, E(M, N))$ such that***

$$F(J)(L) = \int_M \mathfrak{a}(J) \cdot \nabla L \mu(J) = \mathfrak{g}(J)(\mathfrak{a}(J), \nabla L) \tag{4.1}$$

for $J \in E(M, N)$ and $L \in C^\infty_J(M, TN)$. The section $\mathfrak{a}$ is called the ($\mathfrak{g}$-)kernel of F.

For instance, suppose that $\mathfrak{h}$ is a smooth section of $C^\infty_E(M, TN)$ over $E(M, N)$, i.e. a smooth vector field. Then $\mathfrak{a}(J) = \nabla\mathfrak{h}(J)$ will provide a $\mathfrak{g}$-kemel and the right-hand side of (4.1) then will detine a representable one-form. In fact, this example can be shown to characterize the representable one-forms, cf. below. Let us denote by $A^1_{\mathfrak{g}}(E(M, N), \mathbf{R})$ the collection of all smooth $\mathfrak{g}$-representable one-forms on $E(M, N)$.

Remark 4.2. Clearly, the existence of non-trivial 1-forms, in particular that of $\mathfrak{g}$-representable ones depends on the existence of not identically vanishing smooth sections of the bundles in question. Both $\mathfrak{H}^1_E(M, TN)$ and $C^\infty_E(M, TN) = TE(M, N)$ admit local sections since they are locally trivial over $E(M, N)$. Moreover, the mode1 spaces $\Gamma(J^*TN)$ of $E(M, N)$ are *nuclear* Fréchet spaces obtained as countable inverse limits of Hilbert spaces, namely e.g. the H^s-completions of $\Gamma(J^*TN)$ for $s \in \mathbf{N}$. This implies that $E(M, N)$ admits enough «bump functions». Given the open ncighbourhoods U, V of J with $\overline{V} \subset U$, there exist an open neighbourhcod W of J and a smooth function f on $E(M, N)$ such that $\overline{W} \subset V$, together with $0 \le f \le 1$, $f|\overline{W} = 1$ and $f = 0$ on the complement of V . With this existence of non-zero sections of the above bundles is clear. The paracompactness of $E(M, N)$ (as subspace of the paracompact and locally metrizable, hence metrizable space $C^\infty(M, N)$!) can be used to obtain smooth partitions of unity, but we omit the details here and return to all these matters elsewhere.

We now show that any $\mathfrak{g}$-kemel $\mathfrak{a}$ of a smooth one-form F can be presentcd by $\nabla\mathfrak{h}$, where

$$\mathfrak{h} : E(M,N) \to C^\infty_E(M,TN)$$

is a smooth vector field. This means that for any $J \in E(M, N)$

$$\int_M \mathfrak{a}(J) \cdot \nabla L\mu(J) = \int_M \nabla\mathfrak{h}(J) \cdot \nabla L\mu(J) \tag{4.2}$$

or equivalently

$$\mathfrak{g}(J)(\mathfrak{a}(J), \nabla L) = \mathfrak{g}(J)(\nabla\mathfrak{h}(J), \nabla L) \tag{4.3}$$

has to hold for all $L \in C^\infty_J(M, TN)$. To do so we are required to solve

$$\Delta(J)\mathfrak{h}(J) = \nabla^*\mathfrak{a} \tag{4.4}$$

and

$$\nabla_{\mathfrak{n}}\mathfrak{h}(J) = \mathfrak{a}(\mathfrak{n}). \tag{4.5}$$

This is for each $J \in E(M, N)$ an elliptic boundary value problem (cf. [Pa] or [Ho 2] as well as appendix 3.2) and admits according to [Ho 2] a smooth solution $\mathfrak{h}(J)$ for each $J \in E(M, N)$. Since the solutions are smooth with respect to small perturbations of the system (cf. [Ho 2]), we may state:

Theorem 43. *Any $F \in A^1_{\mathfrak{g}}(E(M, N), \mathbb{R})$ admits a smooth vector field*

$$\mathfrak{h} : E(M,N) \to C^\infty_E(M,TN)$$

for which

$$F(J)(L) = \int_M \nabla\mathfrak{h}(J) \cdot \nabla L\mu(J) \tag{4.6}$$

holds for all variables of F.

The following corollary is an easy consequence of proposition 2.1:

Corollary *4.4. Let G and K be a groups acting on M and on N for a given $J \in E(M, N)$ via the homomorphism*

$$\Phi : G \to Diff^{+} M \quad and \quad \Psi : K \to \mathfrak{J}$$

respectively. where $\mathfrak{J}$ is an isometry group of N preserving the orientation. If $F \in A^1(E(M, N), \mathbf{R})$ is $\mathfrak{g}$-representable and invariant at J, under ϕ and Ψ respectively, then there is a smooth vectorjield $\mathfrak{h} : E(M, N) \to C^\infty_E(M, TN)$ such that

$$F(J)(L) = \int_{JM} \nabla \mathfrak{h}(J) \cdot \nabla L \mu(J)$$

and

$$\mathfrak{h}(J \cdot \Phi(f)) = \mathfrak{h}(J) \dot{\Phi}(g), \qquad \forall g \in G \tag{4.7}$$

as well as

$$\mathfrak{h}(\Psi(k) \cdot J) = T\Psi(k) \cdot \mathfrak{h}(J), \qquad \forall k \in K \tag{4.8}$$

hold for all variables of F .

5. THE SPECIAL SITUATION N = $\mathbf{R}^n$

In this section we will show that in case of N = $\mathbf{R}^n$ (with a fixed inner product $<,>$), the spaces $\mathfrak{X}_E(M, T\mathbf{R}^n)$ allow a considembly simpler and more detailed description; in particular, formula (4.1) takes on a more concrete form of importance in the applications. The simplifications are due to the triviality of $T\mathbf{R}^n = \mathbf{R}^n \times \mathbf{R}^n$ and to the fact that the natural operation of $\mathbf{R}^n$ as the group of translations of the vector space $\mathbf{R}^n$ together yield a «splitting» of $\mathfrak{X}_E(M, T\mathbf{R}^n)$. Firstly, since $T\mathbf{R}^n$ is trivial, so is the pull-back $J^*T\mathbf{R}^n$ for each J, i.e. $J^*T\mathbf{R}^n \cong M \times \mathbf{R}^n$ and hence the fibre $\mathfrak{X}_J(M, T\mathbf{R}^n)$ may be identified with the space $\{dL | L \in C^\infty(M, \mathbf{R}^n)\} := A_0$ of exact $\mathbf{R}^n$-valued one-forms on M; thus,

$$\mathfrak{X}_E(M, T\mathbf{R}^n) = E(M, \mathbf{R}^n) \times \{dL | L \in C^\infty(M, \mathbf{R}^n)\}, \tag{5.1}$$

and this is easily seen to be a Fréchet manifold: under the differentiation operator d , $C^\infty(M, \mathbf{R}^n)$ maps onto A_0 with kernel the subspace $\dot{\mathbf{R}}^n$ of constant maps (since M is connected). Accordingly, for each $J \in E(M, \mathbf{R}^n)$, $\mathfrak{X}_J(M, T\mathbf{R}^n) \cong C^\infty(M, \mathbf{R}^n)/_{\mathbf{R}^n}$ is a Fréchet space. $\mathfrak{X}_E(M, T\mathbf{R}^n)$ now inherits the product structure of the right-hand side of (5.1); this also shows that $\mathfrak{X}_E(M, T\mathbf{R}^n)$ is a trivial Fréchet bundle over $E(M, T\mathbf{R}^n)$.

Secondly, the operation of $\mathbb{R}^n$ as the group of translations of R ^{n} provides an action $E(M,T\mathbb{R}^n)$ x $\mathbb{R}^n \to E(M,T\mathbb{R}^n)$ b y $(J,u) \to J + u$, where u is interpreted as the constant map $M \to \{u\}$, cf. below. We next wish to determine the orbit space of this action and to this end, we introduce

$$E_0(M,\mathbb{R}^n) := \left\{ J_0 \in E(M,\mathbb{R}^n) \,\middle|\, \int_M J_0\mu\ (J_0)\ = \right\}0$$

$E_0(M, \mathbf{R}^n)$ meets every equivalence class mod translations in exactly one point and thus determines a section of the equivalence relation.

Observe that $\mu(J + u) = \mu(J)$ since translations are (orientation-preserving) isometries. Hence,if $J_0, J_0' \in E_0(M,\mathbb{R}^n)$ and $J_0' + u$, then

$$0 = \int_M J_0'\mu\,(J_0') = \int_M (J_0 + u)\,\mu\,(J_0) = \operatorname{vol}_{J_0}(M)\cdot u,$$

whence $u = 0$ becauseof $\operatorname{vol}_{J_0}(M) = \int_M \mu(J_0) > 0$.

Next, let $J \in$ E(M, R”) be arbitrary and define

$$u_J := \left(1/\operatorname{vol}_J(M)\right)\int_M J\mu(J) \in \mathbb{R}^n;$$

u_j is the barycenter of J(M) for the uniform massdistribution $\rho = 1$. Then

$$\int_M (J - u_J)\ \mu\ (J - u_J) = \int_M (J - u_J)\ \mu(J) = 0$$

and SO $J - U_J \in E_0(M, \mathbf{R}^n)$: every equivalence class meets $E_0(M, \mathbb{R}^n)$. We conclude that the map $(J_0, u) \to J_0 + u$ is a bijection of $E_0(M,\mathbb{R}^n) \times \mathbf{R}^n$ onto $E(M,\mathbf{R}^n)$

$$E(M,\mathbb{R}^n) = E_0\ (M,\mathbb{R}^n)\ x\ \mathbb{R}^n,$$

cf. also below.

Next, it is clear that the image of $E(M,$ R $^n)$ under d coincides with the one of E_0 (M, $\mathbf{R}^n$) and it can be argued that this image is an open subset of C^∞(M, R $^n) /_{\mathbb{R}^n}$, see below, and under the bijection d of $E_0(M,$ R”) onto this image, $E_0($ M, R”) inherits a Fréchet manifold structure. Formula (5.2) then holds for the differentiable structures as well.

As a consequence,

$$TE(M,\mathbb{R}^n) = TE_0\,(M,\mathbb{R}^n) \oplus T\mathbb{R}^n \tag{5.3}$$

and we now have to determine the first summand more explicitly.

Let again A_0 denote the vector space of exact $\mathbb{R}^n$-valued one-forms on M. The differentiation operator d is a continuous linear surjection inducing the continuous isomorphism $C^\infty(M,\mathbb{R}^n)/\dot{\mathbb{R}}^n \cong A_0$ mentioned above; in the sequel, we use the Fréchet topology of the quotient on A_0. Then d remains a continuous linear surjection and hence is an open map. Accordingly, $C = d(E(M,\mathbb{R}^n)) \subset A_0$ is open in A_0 und thus is a Fréchet manifold. Moreover, $d\,|E_0(M,\mathbb{R}^n)$ is a diffeomotphism by the earlier definition of the differentiable structure of $E_0(M,\mathbb{R}^n)$.

Since d is linear, the (principal part of the) tangent map Td is d once more and, in particular, d yields an isomorphism of $T_{J_0}E_0(M,\mathbb{R}^n)$ onto the tangent space A_0; identifying the former with $C^\infty(M,\mathbb{R}^n)$, the kemel of this map is, of course, the subspace $\dot{\mathbb{R}}^n \subset C^\infty(M,\mathbb{R}^n)$. We now split $C^\infty(M,\mathbb{R}^n)$ at $J_0 \in E_0(M,\mathbb{R}^n) \subset E(M,\mathbb{R}^n)$ as follows:
For $L \in C^\infty(M,\mathbb{R}^n)$, set

$$u_L := \left(1/\operatorname{vol}_{J_0}(M)\right)\int_M L\mu\,(J_0)$$

and interpret u_L as an element of $\dot{\mathbb{R}}^n = \ker(d)$. Then $\int_M (L-u_L)\mu(J_0) = 0$ and the construction yields a continuous splitting

$$T_{J_0}E(M,\mathbb{R}^n) := \left\{L_0 \in C^\infty(M,\mathbb{R}^n) \mid \int_M L_0\mu(J_0) = 0\right\} \oplus \mathbb{R}^n.$$

Under d, the split subspace $\{L_0 \in C^\infty(M,\mathbb{R}^n) \mid \int_M L_0\mu(J_0) = 0\}$ is mapped isomorphically onto A_0 and we conclude that

$$T_{J_0}E_0(M,\mathbb{R}^n) := \left\{L_0 \in C^\infty(M,\mathbb{R}^n) \mid \int_M L_0\mu(J_0) = c\right\}$$

Note that the right-hand side is isomorphic to $C^\infty(M,\mathbb{R}^n)/\dot{\mathbb{R}}^n$. Lastly, there is a «splitting» of $E(M,\mathbb{R}^n)$ in $C^\infty(M,\mathbb{R}^n)$, analogous to (5.2), namely

$$E(M,\mathbb{R}^n) = E_0(M,\mathbb{R}^n) + \dot{\mathbb{R}}^n \subset C^\infty(M,\mathbb{R}^n)\,; \tag{5.4}$$

we also conclude that

$$TE(M,\mathbb{R}^n) = \left(E_0(M,\mathbb{R}^n) \times C^\infty(M,\mathbb{R}^n)/_{\dot{\mathbb{R}}^n}\right) \oplus (\mathbb{R}^n \times \mathbb{R}^n)\,. \tag{5.5}$$

Consequently, any smooth one-form F on $E(M, \mathbb{R}^n)$ can be written in the form

$$F(J)(L) = F(J_0 + u)(L_0) + F(J_0 + u)(z), \tag{5.6}$$

with $L = L_0 + z$, $L_0 \in T_{J_0}E_0(M, \mathbb{R}^n) \cong C^\infty(M, \mathbb{R}^n)/\dot{\mathbb{R}}^n$, $z \in \mathbb{R}^n$, $u \in \dot{\mathbb{R}}^n$. Formula (4.1) now shows that for representable forms F,

$$F(J)(L) = F(J_0)(L_0). \tag{5.7}$$

Remark 5.1. In the applications to continuum mechanics, (5.7) means that F only depends on those embeddings whose center of mass is fixed at $0 \in \mathbb{R}^n$.

The following theorem (cf. [Bi 4]) describes in full generality the structure of $\mathfrak{g}$-representable one-forms for $N \equiv \mathbb{R}^n$ and $<,>$ a fixed scalar product.

Theorem 5.2. ***Every*** $F \in A^1_{\mathfrak{g}}(E(M, \mathbb{R}^n), \mathbb{R})$ ***admits a smooth constitutive map***

$$\mathfrak{h} : E(M, \mathbb{R}^n) \to C^\infty(M, \mathbb{R}^n), \tag{5.8}$$

such that F can be expressed as

$$F(J)(L) = \int_M \langle \Delta(J)\mathfrak{h}(J), L\rangle \mu(J) + \int_{\partial M} \langle \mathrm{d}\mathfrak{h}(J)(\mathfrak{n}), L\rangle i_{\mathfrak{n}}\, \mu(J) \tag{5.9}$$

for each $J \in E(M, \mathbb{R}^n)$ ***and each*** $L \in C^\infty(M, \mathbb{R}^n)$. ***For all*** $J \in E(M, \mathbb{R}^n)$, ***the map*** $\mathfrak{h}$ ***defines*** $\Phi \in C^\infty(E(M, \mathbb{R}^n), C^\infty(\partial M, \mathbb{R}^n))$ ***respectively by***

$$\Phi(J) := \Delta(J)\mathfrak{h}(J) \tag{5.10}$$

and

$$\varphi(J) := \mathrm{d}\mathfrak{h}(J)(\mathfrak{n}), \tag{5.11}$$

which satisfy, due to the first jet dependence of F, the equation

$$0 = \int_M \Phi(J)\mu(J) + \int_{\partial M} \varphi(J) i_{\mathfrak{n}}\, \mu(J). \tag{5.12}$$

Conversely, given two smooth maps $\Phi \in C^\infty(E(M, \mathbb{R}^n), C^\infty(\partial M, \mathbb{R}^n))$ ***and*** $\varphi \in C^\infty(E(M, \mathbb{R}^n), C^\infty(\partial M, \mathbb{R}^n))$, ***for which (5.12) holds as an integrability condition there exists***

a smooth map $\mathfrak{h} \in C^\infty(E(M,\mathbb{R}^n), C^\infty(\partial M,\mathbb{R}^n))$ satisfying (5.10) and (5.11) which is uniquely determined up to a constant for each $J \in E(M,\mathbb{R}^n)$. ∎

Remark5.3. a) If $\Phi' \in C^\infty(E(M,\mathbb{R}^n), C^\infty(\partial M,\mathbb{R}^n))$ and $\varphi' \in C^\infty(E(M,\mathbb{R}^n), C^\infty(\partial M,\mathbb{R}^n))$ are given arbitrarily, we may split off a constant and components Φ and φ satisfying (5.12). Then Φ and φ can be expressed as in (5.10) and (5.11).
b) As a brief comment on the interplay between linearity and non-linearity, we point out the following:
Even if $\mathfrak{h}$ is of the form

$$\mathfrak{h}(J+K) = \mathfrak{h}(J) + D\mathfrak{h}(J)(K),$$

for any $K \in C^\infty(M,\mathbb{R}^n)$ such that $J+K$ still lies in $E(M,\mathbb{R}^n)$, the two maps in (5.10) respectively (5.11) do not vary in a similary simple manner since the Laplacian on J is considerably more subtle (cf. (3.11)).
c) Introducing the $\wedge$-product and the Hodge-star operator as done in [A,M,R] we may write

$$\int_M \mathrm{d}L_1 \cdot \mathrm{d}L_2\,\mu(J) = \int_M \mathrm{d}L_1 \wedge *\mathrm{d}L_2 \tag{5.13}$$

for any pair $L_1, L_2 \in C^\infty(M,\mathbb{R}^n)$. This is easily seen by converting the right-hand side of (5.13) into the right hand side of (5.9). In fact, equality already holds at the level of the integrands (cf. [A]).
d) A theorem analogous to theorem 5.2 holds in the general case as well, cf. [Ho 2].

APPENDIX 5.1.

Let us motivate (3.1) in the context of the present section: given two $I, J \in E(M,\mathbb{R}^n)$ which lie in the same connected component, we may write

$$\mathrm{d}J = Q(J)\cdot \mathrm{d}I$$

with $Q(J) \in C^\infty(M, L(\mathbb{R}^n,\mathbb{R}^n))$. Using the classical polardecomposition (cf. [Bi,Sn,Fi]) the map $Q(J)$ *can* be expressed in the form

$$Q(J) = g(J)\cdot \bar{f}(J),$$

where $g(J) \in C^\infty(M, SO(n))$ and $\bar{f}(J) \in C^\infty(M, L_s(\mathbb{R}^n,\mathbb{R}^n))$, the index s meaning «self-adjoint» with respect to $\langle,\rangle$. Moreover, for all $X, Y \in \Gamma(TM)$

$$\begin{aligned} m(J)(X,Y) &= \langle \bar{f}(J)\mathrm{d}IX, \bar{f}(J)\mathrm{d}IY\rangle = \\ &= m(I)(f(J)X, f(J)Y), \end{aligned} \tag{5.14}$$

where f(J) is the (positive self-adjoint) square root of the strong bundle isomorphism $A'(J) \in L(TM, TM)$, defined by

$$m(J)(X,Y) = m(I)(A'(J)X,Y), \qquad \forall X,Y \in \Gamma(TM).$$

Defining $f'(J) \in C^\infty(M, L_s(\mathbb{R}^n, \mathbb{R}^n))$ by

$$f'(J) \cdot \mathrm{d}I = \mathrm{d}I \cdot f(J),$$

with $f'(J) \,|(T(JTM))^\perp = 0$, we conclude by (5.14)

$$d(J) = g \cdot \mathrm{d}I \cdot f.$$

Letting J depend on a smooth real parameter t with $J(0) = I$, we find

$$\mathrm{d}\dot{J}(0) = \dot{g}(0)\mathrm{d}I + \mathrm{d}I\dot{f}(0). \tag{5.15}$$

Thus there is a unique $C \in C^\infty(M, L_a(\mathbb{R}^n, \mathbb{R}^n))$, the index a meaning skew-adjoint, such that

$$\dot{g}(0)\mathrm{d}I = c\mathrm{d}I + \mathrm{d}I \cdot C,$$

with c as in (3.1). Collecting C and $\dot{f}(0)$ into $A(\mathrm{d}J, \mathrm{d}I)$, yields

$$\mathrm{d}\dot{J}(0) = c \cdot \mathrm{d}I + \mathrm{d}I \cdot A(\mathrm{d}J, \mathrm{d}I) \tag{5.16}$$

the decomposition (3.1) in case of $a = \mathrm{d}\dot{J}(0)$. Equation (5.16) then motivates the general decomposition (3.1). The meaning of the coefticients c, C and $\dot{f}$ is discussed e.g. in [Bi,Sc,So].

6. 5-REPRESENTABLE ONE-FORMS ON $E(M, \mathbb{R}^n)$ AS CONSTITUTIVE LAWS

In this part of the paper we link the formalism developed earlier to classica1 elasticity as presented e.g. in [L,L]. In doing so, we work in a C^∞-setting. First of all we introduce the work caused by deforming a body, the body being identitied with the manifold M with boundary enjoying the properties of the previous sections. To this end we consider the derivative of the map $m : E(M, \mathbb{R}^n) \to \mathfrak{M}(M)$, at any $J \in E(M, \mathbb{R}^n)$ in the direction of any $L \in C^\infty(M, \mathbb{R}^n)$. It is determined by

$$Dm(J)(L)(X,Y) = \langle \mathrm{d}JX, \mathrm{d}LY\rangle + \langle \mathrm{d}LX, \mathrm{d}JY\rangle, \qquad \forall X,Y \in \Gamma(TM). \tag{6.1}$$

Writing Dm(J)(L) with respect to m(J) yields the strong smooth bundle endomorphism

$$B(\mathrm{d}L, \mathrm{d}J) : TM \to TM. \tag{6.2}$$

Hence, $B(\mathrm{d}L, \mathrm{d}J)$ is the symmetric part of $A(\mathrm{d}L, \mathrm{d}J)$ a coefticient appearing in (3.1). This is easily seen by using (3.1) and (6.1); the tensor

$$m(J)(B(\mathrm{d}L,\mathrm{d}J)\ \dots,\dots) = \frac{1}{2}Dm(J)(L)$$

is called the linearized *deformation tensor.*

Let us assume that some smooth map

$$\mathfrak{J} : m(E(M;\mathbf{R}^n)) \to S^2(M)$$

is prescribed, where the range is the collection of all symmetric two tensors on M endowed with the C^∞-topology . $\mathfrak{J}(m(J))$ is called the stress *tensor* at m(J) . $\mathfrak{J}$ (m(J)) determines a uniquely defined smooth strong bundle map of TM, such that

$$\mathfrak{J}(m(J))(X,Y) = m(J)(\widehat{\mathfrak{J}}(\mathrm{d}J)X,Y), \qquad \forall X,Y \in \Gamma(TM). \tag{6.3}$$

We define

$$F_m(m(J))\left(\frac{1}{2}Dm(J)(L)\right) := \int_M tr(\widehat{\mathfrak{J}}(m(J)) \cdot B(\mathrm{d}L)\mathrm{d}J)\mu(J), \tag{6.4}$$

for any $m(J) \in m(E(M,\mathbf{R}^n))$ and any $Dm(J)(L) \in Dm(E(M,\mathbf{R}^n))(C^\infty(M,\mathbf{R}^n))$.

It is not clear as to whether $m(E(M,\mathrm{R}^n))$ is a manifold or not. It is one if the codimension of M in $\mathbf{R}^n$ is high enough (cf.[St]). Hence the usual techniques in analysis and differential geometry cannot by applied without camion. However, $E(M,\mathrm{R}^n)$ is a Fréchet manifold and it makes sense to lift (6.4) to $E(\mathrm{M}, \mathrm{R}^n)$ by introducing the one-form

$$F: E(M,\mathbf{R}^n) \times C^\infty(M,\mathbf{R}^n) \to \mathbf{R}$$

given by

$$F(J)(L) = F_m m(J)\left(\frac{1}{2}Dm(J)(L)\right),$$

for any of the variables of F. It makes also sense to require that F is smooth even though smoothness is not detined for F_m. As shown in [Bi 4] there is a map

$$\mathfrak{h} : E(M,\mathbf{R}^n) \to C^\infty(M,\mathbf{R}^n),$$

for which

$$F(J)(L) = \int_M \mathrm{d}\mathfrak{h}(\mathrm{d}J)\cdot \mathrm{d}L\mu(J) \tag{6.5}$$

holds for all variables of F. Hence, prescribing the stress tensor at each conliguration in $m(E(M, \mathbb{R}^n))$ yields a $\mathfrak{g}$-representable one-form F. Since $\mathfrak{J}$ is a constitutive entity in elasticity, we call F a constitutive lw (cf.[E,S]). Equation (6.5) is the motivation for calling any $F \in A^1_{\mathfrak{g}}(E(M, \mathbb{R}^n), \mathbb{R})$ a constitutive law.

As shown in [S], given any $\mathfrak{g}$-representable one-form F invariant under the natura1 action of the euclidean group of $\mathbb{R}^n$ on $E(M, \mathbb{R}^n)$, satisfying an additional condition, there is a map $\mathfrak{J}$ such that (6.4) holds. The additional condition amounts to say that no rigid motion in $\mathbb{R}^n$ causes any work.

The force densities associated with any constitutive law F with $\mathfrak{g}$-kemel d $\mathfrak{h}$ are given ateach $J \in E(M, \mathbb{R}^n)$ by

$$\Delta(J)\mathfrak{h}(J) \quad \text{on} \quad M \tag{6.6}$$

and

$$d\mathfrak{h}(J)(\mathfrak{n}) \quad \text{on} \quad \partial M, \tag{6.7}$$

(cf.[Bi 4]7. Thus, the formalism presented in these nots refines the usual treatment of elasticity and carries over to any ambient manifold N (cf. Remark 5.3 d) in the previous section). If $N \subset \mathbb{R}^n$, then it may reflect constraints a deformation of a body in $\mathbb{R}^n$ has to satisfy.

If N has no non-trivial isometry group, then there is in general no natura1 symmetric stress-tensor available at each configuration. Hence the generality of the mechanism presented here, which describes all the deformable media admitting smooth force densities at each contiguration acting upon M and ∂M respectively seems to be necessary.

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