Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **34** (2014) no. 1, 23–33. doi:10.1285/i15900932v34n1p23

Products of groups which contain abelian subgroups of finite index

Bernhard Amberg
Fachbereich Physik, Mathematik und Informatik,
Institut für Mathematik der Universität Mainz,
D-55099 Mainz, Germany
amberg@uni-mainz.de

Abstract. It is unknown whether every group $G = AB$ which is the product of two abelian-by-finite subgroups A and B must always have a soluble or even metabelian subgroup of finite index. Here we deal with the special case of this problem when A and B contain abelian subgroups of "small" index, notably of index at most 2. Some recent results on the solubility of such groups are discussed which depend on special calculations involving involutions.

Keywords: factorized groups, abelian subgroups, soluble groups, dihedral groups.

MSC 2010 classification: 20D40, 20F16, 20F50

1 Introduction

A group G is called **factorized**, if $G = AB = \{ab \mid a \in A, b \in B\}$ is the product of two subgroups A and B of G.

If a factorized group $G = AB$ with two subgroups A and B is given, the question arises in which way the factorization as a product of two subgroups has influence on the structure of the factorized group. What can be said about G if the structures of the two subgroups A and B are known? Statements of this type are of special interest when they are valid for arbitrary groups G without additional assumptions for G. In the following we discuss some recent results about groups which are the product of two abelian-by-finite subgroups A and B.

2 Itô's Theorem

The following celebrated theorem of N.Itô (1955) was very influential in the investigation of factorized groups. It is the basis for almost all known results about products of two abelian subgroups.

Theorem 2.1. *If the group $G = AB$ is the product of two abelian subgroups A and B, then G is metabelian.*

The proof is by a surprisingly short commutator calculation (see for instance [1], Theorem 2.1.1). It seems almost impossible to generalize this argument to more general situations, for instance for products of two nilpotent groups (even of class two).

3 Products of abelian-by-finite groups

In view of Itô's theorem the following conjecture has been made (see also Question 3 in [1]).

Conjecture. Let the group $G = AB$ be the product of two abelian-by-finite subgroups A and B (i.e. A and B have abelian subgroups of finite index). Then G is soluble-by-finite and perhaps even metabelian-by-finite.

The validity of this conjecture could only be varified so far under additional requirements. Ya.Sysak proved it in 1986 for linear groups (see [19] and [20]), and J.Wilson in 1990 for residually finite groups (see [8] or [1], Theorem 2.3.4).

The importance of the following theorem of N.Chernikov (1981) lies in the fact that it requires no additional assumptions on the factorized group G and uses only properties from the factorization of G (see [1], Theorem 2.2.5).

Theorem 3.1. *If the group $G = AB$ is the product of two central-by-finite subgroups A and B, then G is soluble-by-finite.*

It is unknown whether G must be metabelian-by-finite in this case.

If N is a normal subgroup of a group G and A and B are abelian subgroups of G, then subgroup $NA = NA \cap AB = A(NA \cap B)$ is metabelian by Itô's theorem.

This useful extension of Itô's theorem may be generalized to any subgroup H as follows (see [20], Lemma 9).

Lemma 3.2. *Let G be a group and A and B two abelian subgroups of G. If the subgroup H of G is contained in the set AB, then H is metabelian.*

Consider now a group $G = AB$ which is the product of two subgroups A and B, which have (abelian) subgroup A_0 resp. B_0 of finite index $n =| A : A_0 |$ and $m =| B : B_0 |$. Then by Lemma 1.2.5 of [1] the subgroup $< A_0, B_0 >$ has finite index at most nm in G.

Clearly, if also we should have that $A_0B_0 = B_0A_0$ is a subgroup of G, then G has a metabelian subgroup of finite index (by Itô).

This shows that if additional permutability conditions are imposed, some factorization problems become much easier and sometimes trivial.

To deal with the above conjecture it is natural to consider first the case when the indices of the abelian subgroups A_0 in A and B_0 in B are small. We will consider the following

Problem. Let the group $G = AB$ be the product of two subgroups A and B, where A contains an abelian subgroup A_0 and B contains an abelian subgroup B_0 such that the indices $n =| A : A_0 |$ and $m =| B : B_0 |$ are at most 2. Is then G soluble and/or metabelian-by-finite?

4 Finite products of subgroups with soluble subgroups of index at most 2

Finite products of two subgroups that contain cyclic subgroups of index at most 2 were for instance considered by B.Huppert [11], W.Scott [17] and V.Monakhov [14] and [15].

V. Monakhov showed in [14] that a finite group $G = AB$ is soluble if A and B have cyclic subgroups of index at most 2.

The following result by L.Kazarin [13] generalizes the well-known theorem of O.Kegel and H.Wielandt on the solubility of every finite product of two nilpotent subgroups (see for instance [1], Theorem 2.4.3).

Theorem 4.1. *If the finite group $G = AB$ is the product of two subgroups A and B, each of which possesses a nilpotent subgroup of index at most 2, then G is soluble.*

5 Products of cyclic-by-finite groups

Products of finite cyclic groups were for instance studied in [10] (see also [12]). P.Cohn proved in [9] that every group $G = AB$ which is the product of two infinite cyclic subgroups A and B has a non-trivial normal subgroup of G which is contained in A or B. This implies that G has a subgroup H of finite index whose derived subgroup H' and the factor group H/H' are both cyclic (see [18]), so that $G = AB$ is metacyclic-by-finite. The question arises whether every group which is the product of two cyclic-by-finite subgroups, must be metacyclic-by-finite.

The following theorem of B.Amberg and Ya.Sysak [5] generalizes the result of P.Cohn.

Theorem 5.1. *If the group $G = AB$ is the product of two subgroups A and B, each of which has a cyclic subgroup of index at most 2, then G is metacyclic-by-finite.*

Note that a non-abelian infinite group which has a cyclic group of index 2 must be the infinite dihedral group. This ensures the existence of involutions in A and B which we may use for computations inside the factorized group G.

An important idea in the proof is to show that the normalizer in G of an infinite cyclic subgroup of one of the factors A or B has a non-trivial intersection with the other factor.

Recall that a group G is **dihedral** if it can be generated by two distinct involutions. The structure of such groups is well-known. The main properties are collected in the following lemma.

Lemma 5.2. *Let the dihedral group G be generated by the two involutions x and y. Let $c = xy$ and $C = < c >$. Then we have*

a) *The cyclic subgroup C is normal in G with index 2, the group $G = C \rtimes < i >$ is the semidirect product of C and a subgroup $< i >$ of order 2,*

b) *If G is non-abelian, then C is characteristic in G.*

c) *Every element of $G \setminus C$ is an involution which inverts every element of C, i.e. if $g \in G \setminus C$, then $c^g = c^{-1}$ for $c \in C$,*

d) *The set $G \setminus C$ is a single conjugacy class if and only if the order of C is finite and odd; it is the union of two conjugacy classes otherwise.*

6 Finite Products of dihedral groups

By Theorem 4.1 finite products of dihedral subgroups are soluble. The following more precice statement about their derived length is proved in [2].

Theorem 6.1. *Let $G = AB$ be a finite group, which is a product of subgroups A and B, where A is dihedral and B is either cyclic or a dihedral group. Then $G^{(7)} = 1$.*

It is also shown in [2] that if G is a finite 2-group then we even have $G^{(5)} = 1$ in this case, but there exist finite 2-groups of derived lenth 3 which are the product of two dihedral subgroups.

The proof of Theorem 6.1 is based on methods for finite factorized groups. We mention here only two well-known lemmas that play a role in our investigations (see for instance [1], Lemma 1.3.2 and [7], Lemma 1.1.20).

Lemma 6.2. *Let the finite group $G = AB$ be the product of two subgroups A and B, then for every prime p there exists a Sylow-p-subgroup of G which is a product of a Sylow-p-subgroup of A and a Sylow-p-subgroup of B.*

Lemma 6.3. *Let the finite group $G = AB$ be the product of subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^xB_0 = B_0A_0^x$ for all $x \in G$. Assume in addition that A_0 and B_0 are π-groups for a set of primes π. If $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.*

7 Products of periodic locally dihedral groups

A group G is **locally dihedral** if it has a local system of dihedral subgroups, i.e. every finite subset of G is contained in some dihedral subgroup of G.

Every **periodic locally dihedral group** is locally finite and every finite subgroup of such a group is contained in a finite dihedral subgroup.

Lemma 7.1. *Every periodic locally dihedral group G has a locally cylic normal subgroup C of index 2, and every element of $G \setminus C$ is an involution that inverts every element of C;*
$G = C \rtimes < i >$ is the semidirect product of C and a subgroup $< i >$ of order 2.

The following well-known lemma is useful for the study of locally finite factorized groups (see for instance [1], Lemma 1.2.3).

Lemma 7.2. *Let the locally finite group $G = AB$ be the product of two subgroups A and B, and let A_0 and B_0 be finite normal subgroups of A and B, respectively. Then there exists a finite subgroup E of G such that*
$A_0, B_0 \subseteq E \subseteq N_G(A_0, B_0)$ and $E = (A \cap E)(B \cap E)$.

The following solubility criterion is proved in [2]). The special case when the group G is periodic was already dealt with in [4].

Theorem 7.3. *Let the group $G = AB$ be the product of two periodic locally dihedral subgroups A and B. Then G is soluble.*

For a complete proof we refer the reader to [2]. Here we sketch only the argument that if the result is false, there will be a counterexample with no nontrivial soluble normal subgroup.

Assume that the theorem is false and there exists a nonsoluble group $G = AB$ with periodic locally dihedral subgroups A and B. Then $A = A_0\langle c\rangle$, $B = B_0\langle d\rangle$ for two involutions $c \in A \setminus A_0$ and $d \in B \setminus B_0$, with $cac = a^{-1}$ for each $a \in A_0$

and $dbd = b^{-1}$ for each $b \in B_0$; A_0 and B_0 are locally cyclic normal subgroups of A resp. B.

By Lemma 3.2 it is easy to see that both subgroups A and B must be non-abelian. The case that $A \cap B = < c >$ is easier and considered first. Thus we may assume that $A \cap B = 1$.

Assume that $N \neq 1$ is a soluble normal subgroup of G. Clearly $R = NA = A(NA \cap B)$ is soluble by the modular law, and so locally finite (as a soluble product of two periodic groups).

If L is a finite normal subgroup of A and S is a finite normal subgroup of $NA \cap B$, then the subgroup $H = < L, S >$ is finite and so the normalizer $K = N_R(< L, S >) = (K \cap A)(K \cap B)$ is factorized by Lemma 7.2.

Since the finite group $K/C_K(H)$ is the product of two subgroups of dihedral groups, its derived length is at most 7 by Theorem 6.1. Since $H \cap C_K(H) = Z(H)$, this implies that $H^{(8)} = 1$.

If $R_0 = < A_0, R \cap B_0 >$, then $R_0^{(8)} = 1$ and since $| R : R_0 | \leq 4$, the subgroup R and so N have derived length at most 9. Then also the product T of all soluble normal subgroups of G is a soluble normal subgroup of G of derived length at most 9. Thus G/T is a counterexample with no soluble normal subgroup $N \neq 1$. The claim is proved.

8 Products of generalized dihedral groups

It turns out that groups with the following property can be handled by our methods.

Definition 8.1. A group G is **generalized dihedral** if it is **of dihedral type**, i.e. G contains an abelian subgroup X of index 2 and an involution τ which inverts every element in X.

It is easy to see that $A = X \rtimes <a>$ is the semi-direct product of an abelian subgroup X and an involution a, so that $x^a = x^{-1}$ for each $x \in X$.

Clearly dihedral and locally dihedral groups are also generalized dihedral.

The main properties of generalized dihedral groups are collected in the following lemma.

Lemma 8.2. *Let A be a generalized dihedral group. Then the following holds*

1) *every subgroup of X is normal in A;*

2) *if A is non-abelian, then every non-abelian normal subgroup of A contains the derived subgroup A' of A;*

3) $A' = X^2$ and so the commutator factor group A/A' is an elementary abelian 2-group;

4) the center of A coincides with the set of all involutions of X;

5) the coset aX coincides with the set of all non-central involutions of A;

6) two involutions a and b in A are conjugate if and only if $ab^{-1} \in X^2$.

7) if A is non-abelian, then X is characteristic in A.

The solubility of every product of two generalized dihedral groups is proved in B.Amberg and Ya.Sysak [6].

Theorem 8.3. *Let the group $G = AB$ be the product of two subgroups A and B, each of which is either abelian or generalized dihedral. Then G is soluble.*

The proof of this theorem is elementary and uses almost only computations with involutions. Extensive use is made by the fact that every two involutions of a group generate a dihedral subgroup. A main idea of the proof is to show that the normalizer in G of a non-trivial normal subgroup of one of the factors A or B has a non-trivial intersection with the other factor.

If this is not the case we may find commuting involutions in A and B and produce a nontrivial abelian normal subgroup by other computations. The following lemma gives some conditions under which two permutable generalized dihedral subgroups A and B of a group have permutable involutions $a \in A$ and $b \in B$. A special case of this is already used in the proof of Theorem 5.1 and discussed in [20].

Lemma 8.4. *Let G be a group of the form $G = AB$ with subgroups A and B such that $A = X \rtimes <c>$ and $B = Y \rtimes <d>$ for abelian subgroups X and Y and involutions c and d. If $x^c = x^{-1}$ for each $x \in X$ and*

$$N_A(<y>) = 1 = N_B(<x>)$$

for every non-trivial cyclic normal subgroup $<x>$ of A and $<y>$ of B, then the subgroup B is non-abelian and there exist involutions $cx \in A$ and $yd \in B$ such that $(cx)(yd) = (yd)(cx)$.

The case when one of the two subgroups A and B is abelian, is considered separately and leads to stronger results. In this case we obtain a bound on the solubility length of G.

Theorem 8.5. *Let the group $G = AB$ be the product of a generalized dihedral subgroup A and an abelian subgroup B, then the derived length of G does not exceed* 5.

The following consequence of Theorem 8.3 should be compared with Theorem 5.1.

Corollary 8.6. *Let the group $G = AB$ be the product of two subgroups A and B, each of which contains a torsion-free locally cyclic subgroup of index at most 2. Then G is soluble and metabelian-by-finite.*

9 Groups saturated by dihedral subgroups

A group G is **saturated by subgroups in a set** $\mathfrak{S}$ if every finite subgroup S of G is contained in subgroup of G which is isomorphic to a subgroup in $\mathfrak{S}$.

Lemma 9.1. *A locally finite group which is saturated by dihedral subgroups is locally dihedral.*

Proof. Let x and y be two elements of G with $o(x) > 2$ and $o(y) > 2$. By hypothesis the finite group $< x, y >$ is contained in a proper finite dihedral group $D =< a > \rtimes < i >$. Since $x \in< a >$, $y \in< a >$, it follows that $xy = yx$. This shows that the elements of G with order more than 2 generate a locally cyclic normal subgroup H of G. Clearly the set $G \setminus H$ is non-empty and consists only of involutions.

Let $t \in G \setminus H$ be a fixed and $x \in G \setminus H$ an arbitrary involution. If $h \in H$ with $o(x) > 2$, then the finite subgroup $< h, x, t >$ is contained in a dihedral subgroup $D =< h_1 > \rtimes < t >$. Then $h_1 \in H$ by the definition of H. Thus $x \in D \subseteq H \rtimes < t >$ for every involution $x \in G$. It follows that $G = H \rtimes < t >$. The lemma is proved. QED

A.Shlopkin and A.Rubashkin in [16] extended Lemma 9.1 to several classes of periodic groups. Using Theorem 7.3 on products of two periodic locally dihedral subgroups we prove the following in [4].

Theorem 9.2. *If the infinite periodic group G is saturated by finite dihedral subgroups, then G is a locally finite dihedral group.*

Assume there exists a periodic group G saturated by dihedral subgroups which is not locally dihedral. Then G is not locally finite by Lemma 9.1. By results in [16] the centralizer $C_G(\gamma)$ of every involution γ in G is a (finite or infinite) periodic locally dihedral group, and there exist at least two involutions τ and $\mu \neq \tau$ in a Sylow-2-subgroup S of G.

Then we show that $G = AB$ where $A = C_G(\tau)$ and $B = C_G(\mu)$ are locally dihedral. By Theorem 7.3 the factorized group G is soluble and so locally finite. This contradiction proves the theorem.

10 Chernikov groups

An abelian-by-finite group with minimum condition on its subgroup is called a **Chernikov group**. Its structure is as follows.

The **finite residual** $J = J(G)$ of a group G is the intersection of all subgroups of G with finite index

$$J(G) = \bigcap G/N, N \subseteq G, |G : N| < \infty$$

A group G is a Chernikov group if and only if $G/J(G)$ is finite and $J(G)$ is the direct product of finitely many quasicyclic (Prüfer) p-groups for finitely many primes p,

Chernikov groups may be handled by considering the following induction parameter.

The **type** of a Chernikov group X is the parameter $\Theta(X) = (r, m)$ where

(1) $r = r(X)$ is the number of quasicyclic (Prüfer) subgroups in a decomposition of the radicable abelian group $J(X)$ (the **rank** of $J(X)$)

(2) $m = m(X) = |X : J(X)|$.

A linear ordering on the set of pairs (r, s) is given by $(r, s) < (r_1, s_1)$ if $r < r_1$ or $r = r_1$ and $s < s_1$.

If U is a subgroup of X, then $\Theta(U) \leqslant \Theta(X)$.

If $\Theta(U) = \Theta(X)$, then $U = X$.

N.F. Sesekin has shown in 1968 that every group, which is the product of two abelian subgroups with minimum condition, also satisfies the minimum condition on all its subgroups and is therefore a metabelian Chernikov group. The present author proved in 1973 that every soluble product $G = AB$ of two Chernikov subgroups A and B is likewise a Chernikov group and $J(G) = J(A)J(B)$ (see [1], Corollary 3.2.8 and 3.2.10).

These results were widely generalized by several authors over the years, which may be summarized as follows (see for example[1], section 3.2).

Theorem 10.1. *Let the group $G = AB$ be the product of two Chernikov subgroups A and B. If G is soluble or generalized soluble in some sense, then G is* *also a Chernikov group and we have*

$$J(G) = J(A)J(B).$$

11 Products of Chernikov subgroups "with index at most 2"

Is an arbitrary group $G = AB$ which is the product of two Chernikov subgroups A and B likewise a Chernikov group? It is natural to consider this question first in the case that the two subgroups A and B have abelian subgroups of index at most 2. If one of the two subgroups A or B is of dihedral type, a positive answer is given in B.Amberg and L.Kazarin [3]).

Theorem 11.1. *Let the group $G = AB$ be the product of two Chernikov subgroups A and B, which both have abelian subgroups A_0 and B_0 respectively with index at most* 2.
Let further one of the two subgroups, A say, be of dihedral type, i.e. A contains an involution τ which inverts every element of A_0.
Then G is a soluble Chernikov group.

Moreover, we have that $J(G) = J(A)J(B)$ and if the index of $J(A)$ in A is m and the index of $J(B)$ in B is n, then the index of $J(G)$ in G is at most mn.
The proof of Theorem 11.1 is by induction on the sum of the types $\Theta(A)$ of A and $\Theta(B)$ of B. For details the reader is referred to [3].

It would be interesting to know whether in Theorem 11.1 the condition that one of the two subgroups A and B is of dihedral type can be omitted.

Perhaps this problem could first be studied for trifactorized groups of the form

$$G = AB = AC = BC$$

for three subgroups A, B and C.

In Problem 13.27 of the Kourovka Note Book it was asked whether every trifactorized group with three Chernikov subgroups A, B and C is likewise a Chernikov group.
The following special case of this unsolved problem seems to be open even for a locally finite group G.

Question. Is the group $G = AB = AC = BC$ a Chernikov group, if the subgroups A, B, C are Chernikov groups with $A/J(A)$, $B/J(B)$ and $C/J(C)$ of order at most 2?

On the other hand, it seems likely that there exist examples of trifactorized groups $G = AB = AC = BC$ which are the product of three subgroups with Min, but G itself does not satisfy Min. A corresponding statement is probably true for locally finite groups.

References

[1] Amberg, B., Franciosi, S., de Giovanni, F., Products of Groups. The Clarendon Press, Oxford University Press, Oxford, 1992.

[2] Amberg, B., Fransman A., Kazarin, L., Products of locally dihedral subgroups. J. Algebra 350 (2012), 308-317.

[3] Amberg, B., Kazarin, L., On the product of two Chernikov subgroups. Israel J. Math. 175 (2010), 363-389.

[4] Amberg, B., Kazarin, L., Periodic groups saturated by dihedral groups, Proceedings "Ischia Group Theory 2010", World Scientific, Singapore (2011), 11-19.

[5] Amberg, B., Sysak, Ya., Products of two groups containing cyclic subgroups of index at most 2, Arch. Math. 90 (2008), 101-111.

[6] Amberg, B., Sysak, Ya., On products of groups which contain abelian subgroups of index at most 2, J. Group Theory, 16 (2013), 299-318

[7] Ballester-Bolinches, A., Esteban-Romero R., Assad, M., Products of finite groups, de Gruyter, 2010.

[8] Chernikov N.S., On the product of almost abelian groups, Ukr. Mat. Zh. 33 (1981), 136-138.

[9] Cohn, P. M., A remark on the general product of two infinite cyclic groups. Arch. Math. 7 (1956), 94-99.

[10] Huppert, B., Über das Produkt von paarweise vertauschbaren zyklischen Gruppen, Math. Z. 58 (1953), 243-264.

[11] Huppert, B., Über die Auflösbarkeit faktorisierbarer Gruppen, Math. Z. 59 (1953), 1-7.

[12] Huppert, B., Endliche Gruppen I, Springer (1967).

[13] Kazarin L., On the product of two groups that are close to being nilpotent, Mat. Sb. 110 (1979), 51-65.

[14] Monakhov, V. S., The product of two groups, one of which contains a cyclic subgroup of index $\leqslant 2$. (Russian) Mat. Zametki 16 (1974), 285-295.

[15] Monakhov, V. S., On the product of two groups with cyclic subgroups of index 2. (Russian) Vesti Akad. Navuk. Belarusi, Ser. Fiz-Mat. Navuk no. 3(1996), 21-24.

[16] Shlyopkin, A.K., Rubashkin, A.G., A class of periodic groups, Algebra i Logika 44 (2005), 114-125.

[17] Scott, W. R., Solvable factorizable groups. Illinois J. Math. 1 (1957), 389-394.

[18] Sysak, Ya., Products of locally cyclic torsion-free groups. Algebra i Logika 25 (1986), 672–686.

[19] Sysak, Ya., Products of almost abelian groups. In: Investigations of groups with restrictions for subgroups (In Russian), Akad. Nauk Ukrain. SSR, Inst. Mat., Kiev (1988), 81–85.

[20] Sysak, Ya., Products of groups and local nearrings, Note di Mat. 28 (2008), 181–216.