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COMPACT CONVEX SETS IN NON-LOCALLY-CONVEX LINEAR SPACES

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Dedicated to the memory of Professor Gottfried Köthe

0. INTRODUCTION

In the theorems about compact convex sets it is usually assumed that the compact convex set is contained in a locally convex linear space. Examples for such theorems are the Schauder-Tychonoff fixed point theorem and the Krein-Milman theorem. The problem whether the Krein-Milman theorem remains true also in non locally convex linear spaces, was solved by J.W. Roberts; he gave examples for absolutely convex compact sets without extreme points ([R 76], [R 77], [Ro 84]). It is, however, still an open problem, whether the Schauder-Tychonoff theorem remains true in non locally convex linear spaces [M 81, Problem 54 of Schauder]. A sufficient condition for a compact convex set to have e.g. the fixed point property or to have extreme points is that it can be affinely embedded in a Hausdorff locally convex linear space. This observation, which was the starting point for [JOT 76] and [R 76], was left out of account in various newer publications about Schauder-Tychonoff's fixed point theorem in non locally convex spaces, e.g. [H 84].

In the first section we give a survey and a precisation of results about sets affinely embeddable in locally convex linear spaces. In the second section we examine conditions for compact convex sets introduced in literature in connection with fixed point theorems in non locally convex linear spaces. In the last section we prove as consequence of Rosenthal's Lemma that in certain Orlicz sequence spaces for some $q \in [0,1]$ every q-convex bounded closed subset is compact.

We denote by $\mathbb{N}$ and $\mathbb{R}$ the set of all natural numbers and real numbers, respectively.

1. LOCALLY CONVEX SETS

In this section, let K be a non void subset of a Hausdorff topological linear space (E, τ).

We treat the question, when on E exists a Hausdorff locally convex linear topology σ, which induces on K the same topology as τ. By 1.1, for convex K an equivalent question is, when K is *affinely embeddable* in a locally convex linear space, i.e. when there is an affine map $h : K \to F$ with values in a locally convex linear space F such that $h : K \to h(K)$ is a homeomorphism; hereby h is called *affine* if $h(\alpha x + (1-\alpha)y) = \alpha h(x) + (1-\alpha)h(y)$ for all $x, y \in K$ and $\alpha \in [0,1]$.

Proposition 1.1. *If K is convex, then K is affinely embeddable in a locally convex linear space iff E admits a Hausdorff locally convex linear topology σ with $\sigma|K = \tau|K$.*

Proof of the implication $\Rightarrow$. Let $h : K \to (F, \rho)$ be an affine embedding in a locally convex linear space (F, ρ). $f(r(x-y)) := r(h(x) - h(y))$ $(x, y \in K; r \geq 0)$ defines a linear map on $E_0 := \text{span}(K_K)$, which is injective since h is so. Let E_1 be an algebraic complement space of E_0 in E, $\| \ \|$ a norm on E_1, $\sigma_0 := f^{-1}(\rho)$ the inverse image under f of ρ and σ the topology on E, for which $(x, y) \mapsto x + y$ is a homeomorphism from $(E_0, \sigma_0) \times (E_1, \| \ \|)$ on (E, σ). Then σ is a Hausdorff locally convex linear topology. Moreover, for a net (x_α) in K and $x \in K$, the following conditions are equivalent: $x_\alpha \to$ $\to x(\sigma)$, $f(x_\alpha) - f(x) \to 0(\rho)$, $h(x_\alpha) - h(x) \to 0(\rho)$, $x_\alpha \to x(\tau)$. Hence $\sigma|K = \tau|K$.

Proposition 1.2. *Let K be convex and compact. Then K is affinely embeddable in a locally convex linear space iff the space A of all real valued affine functions on K separates the points.*

Proof. $\Leftarrow$ We endow the algebraic dual A^* of A with the weak* topology $\sigma(A^*, A)$. The map $h : K \to (A^*, \sigma(A^*, A))$, which sends every $x \in K$ into the evaluation map $h(x) =$ $= \varphi_x$ defined by $\varphi_x(f) = f(x)$ $(f \in A)$, is affine and continuous. If A separates the points, then h is injective, hence an affine embedding, since K is compact.
$\Rightarrow$ Let σ be a topology according to 1.1. Since the continuous dual $(E, \sigma)'$ separates the points, the set of restrictions $\{f|K : f \in (E, \sigma)'\}$ is a point separating subset of A.

We mention a well-known consequence of 1.2.

Corollary 1.3. *If K is convex and compact and $(E, \tau)'$ separates the points, then K is affinely embeddable in a locally convex linear space and therefore has extreme points and the fixed point property.*

In particular, 1.3 applies for Orlicz sequence spaces.

K is said to be *(strongly) locally convex* if every point of K has a neighbourhood base consisting of (open) convex sets in the relative topology $\tau|K$.

Jamison, O'Brien and Taylor [JOT 76] used Proposition 1.2 to prove that a convex compact subset K of E is affinely embeddable in a locally convex linear space iff K is strongly locally convex. They posed the problem whether every compact convex locally convex subset is even strongly locally convex. J.W. Roberts solved this problem:

Theorem 1.4. *[R 78]. If K is compact and convex, then K is affinely embeddable in a locally convex linear space iff K is locally convex.*

Kalton [K 80] gave an example for an Orlicz function space with trivial continuous dual such that every compact convex subset is locally convex and therefore - by 1.4 - has extreme points and the fixed point property.

We denote by $co\ K$ and $aco\ K$ the convex hull and absolutely convex hull of K.

Proposition 1.5. *Let K be a compact convex subset of (E,τ) and σ be a Hausdorff linear topology on E with $\sigma|K = \tau|K$. Then $\sigma|aco\ K = \tau|aco\ K$.*

Proof. (i) We use the following fact: if X is a compact space and $\varphi : X \to Y$ a surjective map, then Y admits at most one Hausdorff topology, for which φ is continuous; the closed sets in Y are exactly the images of closed subsets of X under φ.
(ii) $\varphi(r,x) := rx$ defines a surjective map from $[0,1] \times K$ on $K_0 := co(K \cup \{0\})$. Therefore, by (i), the topology on K_0 is uniquely determined by the topology on K, i.e. $\sigma|K_0 = \tau|K_0$ since $\sigma|K = \tau|K$.
(iii) Applying (i) for $\varphi(x,y) := x - y$ $(x, y \in K_0)$ one gets $\sigma|K_0 - K_0 = \tau|K_0 - K_0$ since $\sigma|K_0 = \tau|K_0$. But $K_0 - K_0$ is an absolutely convex set containing K.

Corollary 1.6. *If K is convex, compact and locally convex, then $aco\ K$ is locally convex.*

Proof. By 1.4, K is affinely embeddable in a locally convex linear space. Let σ be a topology according to 1.1. Then $\sigma|aco\ K = \tau|aco\ K$ by 1.5 and $\sigma|aco\ K$ is locally convex.

The next theorem gives an easy proof of Roberts' deep embedding theorem (Theorem 1.4) under the additional assumption that K is *absolutely* convex. Therefore it would be of interest to find an easy proof of 1.6 without using 1.4. That would give - together with 1.7 - a new proof of Roberts' Theorem 1.4.

Theorem 1.7. *Let K be absolutely convex and $E =$ span K. Then the sets $\bigcup_{n=1}^{\infty} \sum_{i=1}^{n} U_i \cap$ $\cap iK$, where U_i are 0-neighbourhoods in (E,τ), form a 0-neighbourhood base of a linear topology σ on E, which is finer than τ and induces on K the same relative topology as τ. If 0 has a neighbourhood base of convex sets in $(K, \tau|K)$, then σ is locally convex.*

The proof of 1.7 is routine. If K is bounded, then σ is the mixed topology determined by the p_k-topology and τ, where p_k denotes the Minkowski functional of K, (see [W 61, p. 50]). The idea of 1.7 is essentially also contained in the proof of Theorem 1 of [K 80]. 1.7 suggests the following problem.

Problem 1. *Is every convex locally convex subset of E affinely embeddable in a locally convex linear space?*

A positive answer would imply a positive answer to the following problem posed by Krauthausen.

Problem 2. *Let K be convex and locally convex and M a compact subset of K. Is then $co\ M$ totally bounded? (see [Kr 76, p. 10], [H 82, p. 122]; cf. [H 84, p. 31]).*

Under the additional assumption that K is absolutely convex the answer to Problem 2 is yes:

Proposition 1.8. *Let K be absolutely convex and locally convex. If M is a totally bounded subset of K, then co M is totally bounded.*

This follows from 1.7 and the fact that in a locally convex linear space the convex hull of a totally bounded set is totally bounded.

Let K be compact; then K is called *admissible* if for every 0-neighbourhood U there is a continuous map $h : K \to K$ such that $h(K)$ is contained in a finite dimensional subspace of E and $x - h(x) \in U$ for every $x \in K$. It is well-known that every compact convex admissible set has the fixed point property. Klee posed the following problem:

Problem 3. *[Kl 60]. Is every compact convex subset of E admissible?*

Nagumo proved that every compact convex subset of a Hausdorff locally convex linear space is admissible [N 51, Theorem 2]. Nagumo's theorem and Roberts' embedding Theorem 1. 4 yields:

Theorem 1.9. *Every compact convex locally convex subset of E is admissible.*

Under the additional assumption that E is metrizable Krauthausen proved 1.9 without using an embedding theorem, (see [Kr 76, Theorem 1.14] or [H 84, Theorem 3, p. 27]). It is possible to modify Krauthausen's proof such that his metrizable assumption becomes superfluous.

The problem posed by Peck and Waelbroeck [PW 70], whether every compact convex subset of E is locally convex, was answered by Roberts [R 76] negatively. Roberts' construction ([R 76], [Ro 84, section 5.6]) of a counterexample is based on the notion of a needle point. $x_0 \in E$ is called a *needle point* if $x_0 \neq 0$ and every 0-neighbourhood U contains a finite set F such that $co\ F \subset co\{0, x_0\} + U$ and $x_0 \in co\ F + U$.

Suppose that x_0 is a needle point of E and E is a complete F-normed space. Choose finite sets $F_n \subset U_n := \{x \in E :\| x \| \leq 2^{-n}\}$ with $x_0 \in co\ F_n \subset co\{0, x_0\} + U_n$ and $x_0 \in co\ F_n + U_n$, and $F_0 := \{0, x_0\}$. Then $K := \overline{co} \bigcup_{n=0}^{\infty} F_n$ and $acoK$ are compact convex sets, which are not locally convex (at 0). The difficult part of Roberts' construction is the proof of the existence of needle points. He proved: every nonzero point of an Orlicz function space $L_\varphi[0,1]$ (with respect to the Lebesgue measure on $[0,1]$) is a needle point if the Orlicz function φ is concave and $\varphi(t)/t \to 0 \ (t \to \infty)$.

We will show that Roberts' construction yields sets which are admissible and have therefore the fixed point property.

Proposition 1.10. *[J 81, 2.8.2]. For every F-semi- norm $\| \|_1$ on E there is an F-seminorm $\| \|_2$, which induces the same topology as $\| \|_1$, with the following property: for every $x \in E$ with $\| x \|_1 \neq 0$ the map $[0, +\infty[\ni t \mapsto \| tx \|_2$ is strictly increasing.*

Lemma 1.11. *Let $x_0 \in E$ and U be a 0-neighbourhood in (E, τ).*

(a) There is a uniformly continuous map $f : E \to co\{-x_0, x_0\}$ such that $f(x) = x$ for all $x \in co\{-x_0, x_0\}$.

(b) There is a uniformly continuous map $\psi_1 : E \to [0,1]$ and a 0-neighbourhood V such that for all $x \in co\{0, x_0\} + V$ and $t \in [0,1]$ one has $0 \leq \psi_1(tx) \leq t$ and $\psi_1(x)x_0 - x \in U$.

(c) There is a uniformly continuous map $\psi_2 : E \to [-1,1]$ and a 0-neighbourhood V such that for all $x \in co\{-x_0, x_0\} + V$ and $t \in [-1,1]$ one has $|\psi_2(tx)| \leq |t|$ and $\psi_2(x)x_0 - x \in U$.

Proof. We may assume that $x_0 \neq 0$. Choose a continuous F-seminorm $\| \ \|$ on (E, τ) such that $\{x \in E : \| x \| \leq 3\} \subset U \setminus \{x_0\}$. On grounds of 1.10 we may assume that $[0, +\infty[\ni$ $\ni t \mapsto \| tx_0 \|$ is strictly increasing. Define $\beta(t) := \alpha^{-1}(t)$ for $0 \leq t \leq \| x_0 \|$, $\beta(t) := 1$ for $t \geq \| x_0 \|$, $\psi(x) := \beta(\| x \|)$ and $g(x) := \psi(x) \cdot x_0$ for $x \in E$. Then ψ and g are uniformly continuous and $g(x) = x$ for $x \in co\{0, x_0\}$. Put $h(x) = 2x - x_0$ for $x \in E$. Then $f := h \circ g \circ h^{-1}$ is a function as required in (a).

Choose $\gamma > 0$ with $\| \gamma x_0 \| \leq 1$. Since ψ is uniformly continuous with respect to $\| \ \|$, there is a $\delta \in]0,1]$ such that $x_i \in E$ and $|x_1 - x_2| \leq \delta$ imply $|\psi(x_1) - \psi(x_2)| \leq \gamma$. Define $\rho : \mathbb{R} \to \mathbb{R}$ by $\rho(t) = t - \gamma$ if $t \geq \gamma$, $\rho(t) = 0$ if $-\gamma \leq t \leq \gamma$, $\rho(t) = t + \gamma$ if $t \leq -\gamma$. Then $\psi_1 := \rho \circ \psi$ is a uniformly continuous map from E into $[0,1]$. Put $V := \{z \in E : \| z \| \leq \delta\}$. Let $t \in [0,1]$ and $x = sx_0 + z$ with $0 \leq s \leq 1$ and $z \in V$. Then $\| tx - tsx_0 \| = \| tz \| \leq \delta$, hence $|\psi(t,x) - ts| = |\psi(tx) - \psi(tsx_0)| \leq \gamma$ and $\psi_1(tx) \leq ts \leq t$, since $|u - v| \leq \gamma$ implies $s(u)| \leq |v|$. We now prove that $\psi_1(x)x_0 - x \in$ $\in U$. Since $|\psi(x) - s| \leq \gamma$ and $|\psi_1(x) - \psi(x)| \leq \gamma$, we have $\| \psi(x)x_0 - sx_0 \| \leq 1$ and $\| \psi_1(x)x_0 - \psi(x)x_0 \| \leq 1$, hence $\| \psi_1(x)x_0 - sx_0 \| \leq 2$ and $\| \psi_1(x)x_0 - x \| \leq$ $\leq \| \psi_1(x)x_0 - sx_0 \| + \| z \| \leq 3$. Consequently $\psi_1(x)x_0 - x \in U$.

The proof of (c) is similar to that of (b). Write $f(x) = \psi_0(x)x_0$ for $x \in E$. Choose $\delta \in]0,1]$ such that $x_i \in E$ and $|x_1 - x_2| \leq \delta$ imply $|\psi_0(x_1) - \psi_0(x_2)| \leq \gamma$. Then $\psi_2 := \rho \circ \psi_0$ and $V := \{z \in E : \| z \| \leq \delta\}$ have the required properties.

Proposition 1.12. *Let $(E, \| \ \|)$ be a complete F-normed linear space, $x_0 \in E \setminus \{0\}$, $F_0 :=$ $= \{0, x_0\}$ and F_n finite subsets of $U_n := \{x \in E : \| x \| \leq 2^{-n}\}$ such that $co\, F_n \subset$ $\subset co\, F_0 + U_n$ and $x_0 \in co\, F_n + U_n$ for $n \in \mathbb{N}$. Assume further that M_n are closed linear subspaces of E such that $\bigcup_{i>n} \cup F_0 \subset M_n$ and $M_n \cap span \bigcup_{i=1}^{n} F_i = (0)$. Then $\overline{co} \bigcup_{n=0}^{\infty} F_n$ and $\overline{aco} \bigcup_{n=0}^{\infty} F_n$ are compact admissible sets, which are not locally convex.*

Proof. (i) Before Proposition 1.10, we have just observed that $K := \overline{aco} \bigcup_{i=0}^{\infty} F_i$ and $K_1 :=$ $= \overline{co} \bigcup_{i=0}^{\infty} F_i$ are compact sets, which are not locally convex.

(ii) We now prove that K is admissible. Let U a 0-neighbourhood in E. Choose ψ_2 and V as in 1.11 (c) and $n \in \mathbb{N}$ with $U_{n-i} \subset V$. Then $A := \overline{aco}\, F_0 \cup \bigcup_{i>n} F_i$ and $B := \overline{aco} \bigcup_{i=1}^{n} F_i$ are compact subsets of K. Furthermore, $A \subset co\{-x_0, x_0\} + V$. By the assumption on M_n

the map $A \times B \ni (x,y) \mapsto x+y$ is injective, hence an homeomorphis onto $A+B$. Therefore $h(x+y) = \psi_2(x)x_0 + y$ (if $x \in A$ and $y \in B$) defines a continuous map from $A+B$ in $E_n := \operatorname{span}\bigcup_{i=0}^{n} F_j$. By 1.11 (c) we have $h(x+y) - (x+y) = \psi_2(x)x_0 - x \in U$ for $x \in A$ and $y \in B$, hence $h(z) - z \in U$ for all $z \in K$, since $K \subset A+B$. We show that $h(K) \subset K$: let $z \in K$. Then there are $x \in A$, $y \in B$ and $s,t \in \mathbb{R}$ with $|s|+|t| \leq 1$ and $z = sx + ty$. Since $|\psi_2(sx)| \leq |s|$ by 1.11 (c), one obtains $h(z) = \psi_2(sx)x_0 + ty \in K$.
(iii) To prove the admissibility of K_1, we choose ψ_1 and V according to 1.11 (b) and define $h_1(x+y) := \psi_1(x)x_0 + y$ for $x \in A$ and $y \in B$. Then h_1 is a continuous map with values in E_n, $h_1(K_1) \subset K_1$ and $h_1(z) - z \in U$ for $z \in K_1$.

From Robert's construction sketched before Proposition 1.10 (cf. [Ro 84, section 5.6]) and from Proposition 1.12 one obtains:

Corollary 1.13. *If φ is a concave Orlicz function and $\varphi(t)/t \to 0$ $(t \to \infty)$, then the Orlicz space $L_\varphi[0,1]$ contains absolutely convex compact sets, which are not locally convex, but admissible and have therefore the fixed point property.*

2. CONVEXLY TOTALLY BOUNDED AND STRONGLY CONVEXLY TOTALLY BOUNDED SETS

In this section let (E,τ) be a Hausdorff topological linear space and K a non void subset of E.

K is said to be *convexly totally bounded* (*ctb* for short) if for every 0-neighbourhood U there are a finite number of convex subsets $C_1,\dots,C_n$ of U and points $x_1,\dots,x_n \in E$ such that $K \subset \bigcup_{i=1}^{n}(x_i + C_i)$. This notion was introduced by Idzik [I 87] to establish fixed point theorems in non locally convex linear spaces.

Theorem 2.1. *[I 88, Theorem 4.3]. If K is convex, compact and ctb, then K has the fixed point property.*

Since every totatlly bounded subset of a locally convex linear space is *ctb*, 2.1 generalizes the Schauber-Tychonoff fixed point theorem. Idzik [I 88, Problem 4.7] posed the problem, whether every compact convex subset of a (non locally convex) topological linear space is *ctb*. A positive answer would solve Schauder's problem [M 81, Problem 54]. In [DTW 93] we answer Idzik's problem negatively specializing Roberts' example for an absolutely convex compact non locally convex set (see section 1). Combining this result of [DTW 93] with 1.12 one obtains:

Theorem 2.2. *If φ is a concave Orlicz function such that $\varphi(t)/t \to 0$ $(t \to \infty)$, then the Orlicz function spaces $L_\varphi[0,1]$ contains absolutely convex compact sets with the fixed point property, which fail to be ctb.*

Easy examples show that the convex hull of a *ctb* set doesn't need to be *ctb* [DTW 93]. We therefore introduced the following notion in [DTW 93]: K is said to be *strongly convexly totally bounded* (*ctb* for short) if for every 0-neighbourhood U there is a convex subset C of U and a finte subset F of E such that $K \subset F + C$.

Proposition 2.3. *[DTW 93]. The closed absolutely convex hull of a sctb set is sctb.*

Every *sctb* set is *ctb* and every *ctb* set is totally bounded; in locally convex linear spaces these three notions are equivalent.

Proposition 2.4. *If E is metrizable, then every totally bounded set is sctb iff E is locally convex.*

Proof of the implication $\Rightarrow$. If E is not locally convex, then E contains by [MO 48, statement 1.642] a compact set K such that *co* K is not bounded. By 2.3, K is not *sctb*.

In 2.4, the metrizability assumption is not superfluous: if τ is the finest linear topology on E, then every bounded subset of E is contained in a finite dimensional subspace of E and is therefore *sctb*, but τ is not locally convex if the dimension of E is uncountable, (see [J 81, p. 80 and p. 123]).

The example [DTW 93, 5.3(b)] shows that a *ctb* set is not necessarily *sctb*. But we don't know, whether there are also *convex ctb* sets, which are not *sctb*. On grounds of 2.8 an equivalent question is:

Problem 4. *Are there compact convex ctb sets, which are not affinely embeddable in a locally convex linear space?*

The main aim of this section is the proof of Theorem 2.7. There we compare the notion of *sctb* sets also with the following notion introduced by Hadžić [H 82, Definition 6] in connection with fixed point theorems. Hadžić calls K of Z-type if for every 0-neighbourhood U there is a 0-neighbourhood V in (E,τ) such that *co* $V \cap (K-K) \subset U$. A convex set K is of Z-type iff 0 has a neighbourhood base of convex sets in the relative topology $\tau|K-K$.

Proposition 2.5. *(cf. [H 84, Proposition 3, p. 30] and [JOT, p. 204]). Let K be absolutely convex. Then K is of Z-type iff K is locally convex.*

Proof. $\Leftarrow$ This follows from the fact that $x \mapsto 2x$ defines a homeomorphism from K onto $K-K$.

$\Rightarrow$ Let $x \in K$. If U is a 0-neighbourhood in (E,τ) such that $U_0 := U \cap (K-K)$ is convex, then $(x+U_0)\cap K$ is a convex neighbourhood of x in $(K,\tau|K)$ since $(x+U)\cap K \subset x+U_0$.

Proposition 2.6. *If K is sctb, then for every 0-neighbourhood U there is an absolutely convex subset C of U and a finite subset F of K such that $K \subset F + C$.*

Proof. Let U and V be 0-neighbourhoods with $V - V \subset U$, C_0 a convex subset of V and $x_1, \dots, x_n \in E$ with $K \subset \bigcup_{i=1}^n (x_i + C_0)$ and $(x_i + C_0) \cap K \neq \emptyset$ for $i = 1, \dots, n$. Choose $y_i \in (x_i + C_0) \cap K$; then $x_i + C_0 \subset y_i - C_0 + C_0$ for $i = 1, \dots, n$. Therefore $K \subset F + C$ with $F := \{y_1, \dots, y_n\}$ and $C := C_0 - C_0$. Furthermore, C is an absolutely convex subset of U.

Theorem 2.7. *Let K be a totally bounded subset of K and $F =$ span K. Then the following conditions are equivalent.*

(1) K is sctb.

(2) K is of Z-type.

(3) aco K is locally convex.

(4) E admits a Hausdorff locally convex linear topology σ, which induces on F a finer topology than τ, such that $\sigma|K$ is totally bounded.

(5) E admits a Hausdorff locally convex linear topology σ, which induces on F a finer topology than τ, such that $\sigma|K = \tau|K$.

(6) E admits a Hausdorff locally convex linear topology σ, which induces on F a finer topology than τ, such that $\sigma|aco\ K = \tau|aco\ K$.

Proof. (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (1) and (6) $\Rightarrow$ (3) is obvious.

(3) $\Rightarrow$ (2) By 2.5, *aco* K is of Z-type. Therefore K is of Z-type, too.

(2) $\Rightarrow$ (1) Let U be a 0-neighbourhood. By (2) there is a 0-neighbourhood V such that $C := co\, V \cap (K - K) \subset U$. Since K is totally bounded, there is a finite number of points $x_1, \dots, x_n \in E$ such that $K \subset \bigcup_{i=1}^n (x_i + V)$. Since $(x_i + V) \cap K \subset x_i + C$, we get $K \subset \bigcup_{i=1}^n (x_i + C)$, where C is a convex subset of U.

(1) $\Rightarrow$ (6) (i) Let $\widetilde{E}$ be the completion of (E, τ). Then $\overline{aco} K$ (taken in $\widetilde{E}$) is *sctb* by 2.3, hence compact. Therefore we may assume that K is absolutely convex, compact and *sctb*. Furthermore we may assume that $E = \text{span}\, K$, since every Hausdorff locally convex linear topology on F can be extended to a Hausdorff locally convex linear topology on E (cf. the proof of 1.1).

(ii) Let $(U_\alpha)_{\alpha \in A}$ a 0-neighbourhood base in (E, τ). We will define σ by a family $(p_\alpha)_{\alpha \in A}$ of seminorms, where each p_α is the Minkowski functional of a closed absolutely convex subset B_α of U_α.

Let $\alpha \in A$. Choose 0-neighbourhoods V_n and W_n such that U_α contains the closure of $V_0 + V_0$ and $V_n + V_n \subset V_{n-1}$ and $nW_n \subset V_n$ for $n \in \mathbb{N}$. By 2.6, there are absolutely convex subsets C_n of W_n and finite sets $F_n \subset E$ such that $K \subset F_n + C_n$ for $n \in \mathbb{N}$. Let $\varepsilon > 0$ with $\varepsilon K \subset V_0$. Define B_α as the closure of $\varepsilon K + \bigcup_{n=1}^\infty \sum_{i=1}^n iC_1$. Then B_α is a closed absolutely convex subset of U_α. Since $\varepsilon K \subset B_\alpha$, we have span $B_\alpha = E$ and therefore the Minkowski functional p_α of B_α is a seminorm on E. Since $B_\alpha \subset U_\alpha$ for all $\alpha \in A$, the topology σ induced by the family $(p_\alpha)_{\alpha \in A}$ is finer than τ.

K is totally bounded in (E,σ): let $\alpha \in A$. We have only to prove that K is totally bounded in (E,p_α). Let $n \in \mathbb{N}$ and $B := \{x \in E : p_\alpha(x) \leq 1/n\}$. Then we have, with the notation above, $C_n \subset B$ since $nC_n \subset B_\alpha \subset nB$. It follows that $K \subset F_n + B$.

K is compact in (E,σ): let $(x_\beta)_{\beta\in B}$ be a net in K. Since $(K,\sigma|K)$ is totally bounded and $(K,\tau|K)$ is compact, (x_β) has a subnet $(y_\gamma)_{\gamma\in\Gamma}$, which is Cauchy in (E,σ) and converges to some point $y \in K$ in (E,τ). It follows that (y_γ) converges to y in (E,σ), since (E,σ) has a 0-neighbourhood base of sets, which are closed in (E,τ).

From the facts that τ is Hausdorff and coarser than σ and that K is compact in (E,τ) and in (E,σ) follow that $\sigma|K = \tau|K$.

Corollary 2.8. *Let K be a compact convex subset of (E,τ) and $F =$ span K. Then the following conditions are equivalent.*

(1) K is sctb.

(2) K is of Z-type.

(3) K is locally convex.

(4) K is affinely embeddable in a locally convex linear space.

(5) E admits a Hausdorff locally convex linear topology σ, which induces on F a finer topology than τ, such that $\sigma|K = \tau|K$.

Proof. (1) $\Longleftrightarrow$ (2) $\Longleftrightarrow$ (5) and (3) $\Longleftrightarrow$ (4) hold by 2.7 and 1.4. (5) $\Longrightarrow$ (4) is obvious. (4) $\Longrightarrow$ (2) Let σ be a locally convex topology according to 1.1 and $K_0 := aco\, K$. Then $\sigma|K_0 = \tau|K_0$ by 1.5 and so $\sigma|2K_0 = \tau|2K_0$. Therefore $\tau|2K_0$ is locally convex and consequently K is of Z-type, since $K - K \subset 2K_0$.

Let $p > 0$; then every *convex ctb* subset of ℓ_p is *sctb* by 2.8 (4) $\Longrightarrow$ (1). We will see in 3.8, that *every ctb* subset of ℓ_p is *sctb*.

Krauthhausen [Kr 76, Satz 2.18] proved that every compact convex order-bounded subset of an Orlicz function space is locally convex. By 2.8, these sets are *sctb*.

3. COMPACT CONVEX SETS IN CERTAIN ORLICZ SEQUENCE SPACES

In this section let $\varphi : [0,\infty[\to [0,\infty[$ be an increasing function continuous in 0 such that $\varphi(t) = 0$ iff $t = 0$.

We put $\| x \|_\varphi = \sum_{n=1}^{\infty} \varphi(|x_n|)$ for every real sequence $x = (x_n)$ and $\ell_\varphi = \{x \in \mathbb{R}^{\mathbb{N}} : \| x \|_\varphi < \infty\}$. For $n \in \mathbb{N}$, we denote by e_n the sequence $(\delta_{in})_{i\in\mathbb{N}}$, where $\delta_{nn} = 1$ and $\delta_{in} = 0$ if $i \neq n$.

φ is said to satisfy the Δ_2*-condition* at 0 if $\limsup_{t\to 0} \varphi(2t)/\varphi(t) < \infty$. The following fact is well-known.

Proposition 3.1. *(a) The following conditions are equivalent:*

(1) φ satisfies the Δ_2-condition at 0.

(2) $\sup_{0<t\leq\alpha} \varphi(st)/\varphi(t) < \infty$ *for all* $\alpha, s > 0$.

(3) For every $\alpha > 0$ *there is a* $\beta > 0$ *such that* $x, y \in \mathbb{R}^{\mathbb{N}}$ *and* $\| x \|_\varphi, \| y \|_\varphi \leq \alpha$ *imply* $\| x + y \|_\varphi \leq \beta(\| x \|_\varphi + \| y \|_\varphi)$.

(4) $x \in \ell_\varphi$ *implies* $2x \in \ell_\varphi$.

(5) The sets $\{x \in \mathbb{R}^{\mathbb{N}} : \| x \|_\varphi < \varepsilon\}$ $(\varepsilon > 0)$ *form a* 0*-neighbourhood base for a group topology on* $\mathbb{R}^{\mathbb{N}}$.

(6) ℓ_φ *is a linear subspace of* $\mathbb{R}^{\mathbb{N}}$ *and the sets* $\{x \in \ell_\varphi : \| x \|_\varphi < \varepsilon\}$ $(\varepsilon > 0)$ *form a* 0*-neighbourhood base for a linear topology* τ_φ *on* ℓ_φ .

(b) If φ *satisfies the* Δ_2*-condition at 0, then* $(\ell_\varphi, \tau_\varphi)$ *is a complete metrizable topological linear space and* $(e_n)_{n\in\mathbb{N}}$ *is an equicontinuous basis in* ℓ_φ *(in the sense of [J, p. 296]).*

The main aim of this section is to prove that in certain Orlicz sequence spaces ℓ_φ, in particular in the spaces ℓ_p for $0 < p < 1$, every closed bounded convex subset is compact (see Theorem 3.7). More generally, we treat q-convex sets; [J 81, p. 101] for the definition. Hereby the set Q_φ of all $q > 0$, for which

$$\lim_{0<s\to 0} \psi(q, s) = \infty, \quad \text{where} \quad \psi(q, s) = \inf_{0<t\leq 1} \varphi(st)/s^q\varphi(t),$$

plays an important role. We examine this set more in detail at the end of this section. Here we only note:

Lemma 3.2. *If* $Q_\varphi \neq \emptyset$ *, then* φ *satisfies the* Δ_2*-condition at 0.*

Proof. Let $q \in Q_\varphi$. Then there is a positive number $s \leq 1/2$ such that $\varphi(st)/s^q\varphi(t) \geq 1$ for all $t \in [0, 1]$. Therefore

$$\varphi(t/2)/\varphi(t) \geq \varphi(st)/\varphi(t) = (\varphi(st)/s^q\varphi(t)) \cdot s^q \geq s^q \quad \text{for} \quad 0 < t \leq 1,$$

hence $\limsup_{t\to 0} \varphi(t/2)/\varphi(t) \leq s^{-q}$.

In the proof of Theorem 3.7 we use one implication $(\Leftarrow)$ of the following known fact.

Proposition 3.3. *Let* $(x_n)_{n\in\mathbb{N}}$ *be an equicontinuous basis in a Hausdorff topological linear space* E *and* $P_k : E \to E$ *the expansion operators defined by* $P_k(x) = \sum_{n=1}^{k} t_n x_n$ *if* $t_n \in \mathbb{R}$ *and* $x = \sum_{n=1}^{\infty} t_n x_n$. *Then a subset* K *of* E *is totally bounded iff* $P_k(K)$ *is bounded for all* $k \in \mathbb{N}$ *and* $P_k(x) \to x$ $(k \to \infty)$ *uniformly in* $x \in K$.

Proof. Define $R_k : E \to E$ by $R_k(x) = x - P_k(x)$.

(i) Suppose that $P_k(K)$ is bounded for all $k \in \mathbb{N}$ and $P_k(x) \to x$ $(k \to \infty)$ uniformly in $x \in K$. Let U be a 0-neighbourhood in E. Choose $k \in \mathbb{N}$ with $R_k(K) \subset U$.

Since $P_k(K)$ is a bounded subset of the finite dimensional space span$\{x_1, \ldots, x_k\}$, the set $P_k(K)$ is totally bounded and therefore $P_k(K) \subset F + U$ for some finite set $F \subset E$. Hence $K \subset P_k(K) + R_k(K) \subset F + U + U$. It follows that K is totally bounded.
(ii) Let K be a totally bounded subset of E. Then K and therefore $P_k(K)$ is bounded for each $k \in \mathbb{N}$. To prove that $R_k(x) \to 0$ $(k \to \infty)$ uniformly in $x \in K$, we need that $\{R_k : k \in \mathbb{N}\}$ is equicontinuous. Let U and V be 0-neighbourhood in E such that $R_k(V) \subset U$ for all $k \in \mathbb{N}$. Choose a finite set $F \subset E$ such that $K \subset F + V$ and $k_0 \in \mathbb{N}$ such that $R_k(x) \in U$ for all $x \in F$ and $k \geq k_0$. Then we have $R_k(K) \subset R_k(F) + R_k(V) \subset U + U$ for $k \geq k_0$.

An essential tool in the proof of 3.7 is the following special version of Rosenthal's Lemma [DU 77, Lemma 1, p. 18].

Rosenthal's Lemma 3.4. *Let $\alpha_{ij} \in [0, +\infty[$ for $i, j \in \mathbb{N}$ and $\sup_{i\in\mathbb{N}} \sum_{j=1}^{\infty} \alpha_{ij} < \infty$. Then for every $\varepsilon > 0$ there is a sequence of natural numbers $n_1 < n_2 < n_3 < \ldots$ such that $\sum_{j=1, j\neq 1} \alpha_{n_i n_j} \leq \varepsilon$ for every $i \in \mathbb{N}$, in particular $\alpha_{n_i n_j} \leq \varepsilon$ if $i \neq j$.*

Lemma 3.5. $\| sz \|_\varphi \geq s^q \cdot \| z \|_\varphi \cdot \psi(q,s) \cdot \min\{1, \varphi(1)/\alpha\}$ *if $s, q, \alpha > 0$ and $z = (z_n) \in \ell_\varphi$ with $\sup_{n\in\mathbb{N}} (|z_n|) \leq \alpha$.*

Proof. Let $s, q, \alpha > 0$. Then $\varphi(st) \geq \psi(q,s) s^q \varphi(t)$ if $0 < t \leq 1$, and

$$\varphi(st) \geq \varphi(s \cdot 1) \geq \psi(q,s) s^q \varphi(1) \geq \psi(q,s) s^q \varphi(t) \varphi(1)/\alpha$$

if $t \geq 1$ and $\varphi(t) \leq \alpha$; hence

$$\varphi(st) \geq \psi(q,s) s^q \varphi(t) \cdot \min\{1, \varphi(1)/\alpha\} \quad \text{for all} \quad t > 0 \quad \text{with} \quad \varphi(t) \leq \alpha.$$

It follows that for $z = (z_n) \in \ell_\varphi$ with $\sup_{n\in\mathbb{N}} \varphi(|z_n|) \leq \alpha$

$$\| sz \|_\varphi = \sum_{n=1}^{\infty} \varphi\left(s|z_n|\right) \geq \sum_{n=1}^{\infty} \psi(q,s) \cdot s^q \cdot \varphi\left(s|z_n|\right) \cdot \min\{1, \varphi(1)/\alpha\} =$$

$$= \psi(q,s) s^q \cdot \| z \|_\varphi \cdot \min\{1, \varphi(1)/\alpha\}.$$

A subset K of ℓ_φ is called $\| \, \|_\varphi$ *-bounded* if $\sup_{x\in K} \| x \|_\varphi < \infty$. $\| \, \|_\varphi$ -boundedness implies pointwise boundedness in ℓ_φ iff $\sup_{t>0} \varphi(t) = \infty$.

Proposition 3.6. *Assume that φ satisfies the Δ_2-condition at 0. Then every bounded subset of $(\ell_\varphi, \tau_\varphi)$ is $\| \, \|_\varphi$ -bounded and $\| \, \|_\infty$ -bounded, hence pointwise bounded.*

Proof. (i) The $\| \, \|_\infty$ -topology is weaker than the $\| \, \|_\varphi$ -topology τ_φ, since $\{x \in \ell_\varphi : \| x \|_\varphi \leq \delta\} \subset \{x \in \ell_\varphi : \| x \|_\infty \leq \varepsilon\}$ for $\varepsilon > 0$ and $0 < \delta < \varphi(\varepsilon)$. Therefore τ_φ-boundedness implies $\| \, \|_\infty$ -boundedness, hence pointwise boundedness.

(ii) Let K be a bounded subset of $(\ell_\varphi, \tau_\varphi)$. Then $\alpha := \sup\{\| x \|_\infty : x \in K\} < \infty$ by (i) and $\beta := \sup_{0<t\leq\alpha} \varphi(st)/\varphi(t) < \infty$ by 3.1(a) (2). Let $0 < r < 1$ such that $x \in K$ implies $\| rx \|_\varphi \leq 1$ and $s := 1/r$. Then, for $x = (x_n) \in K$, we have

$$\| x \|_\varphi = \sum_{n=1}^{\infty} \varphi\left(r|x_n|\right) \cdot \varphi\left(s \cdot r|x_n|\right) / \varphi(r|x_n|) \leq \| rx \|_\varphi \cdot \beta \leq \beta.$$

Theorem 3.7. *Let $q \in Q_\varphi \cap [0,1]$ and K be a q-convex subset of ℓ_φ. Then the following conditions are equivalent:*

(1) K is $\| \, \|_\varphi$ -bounded and pointwise bounded.

(2) K is τ_φ-bounded.

(3) K is totally bounded in (E, τ_φ).

Proof. By 3.2, φ satisfies the Δ_2-condition at 0.

(3) $\Rightarrow$ (2) is obvious, (2) $\Rightarrow$ (1) holds by 3.6.

(1) $\Rightarrow$ (3) Put $P_k(x) := \sum_{n=1}^{k} x_n e_n$ and $R_k(x) := x - P_k(x)$ if $x = (x_n) \in \ell_\varphi$. Then $P_k(K)$ is bounded for all $k \in \mathbb{N}$, since K is pointwise bounded. Suppose that K is not totally bounded. Then, by 3.3, $(R_k(x))$ does not convergence uniformly in $x \in K$ to 0. Therefore there are a positive number $\eta > 0$, a sequence of integers $0 = k_1 < k_2 < k_3 < \ldots$ and $x_i = (x_{in})_{n\in K}$ such that

$$(*) \qquad \sum_{k_i < n \leq k_{i+1}} \varphi\left(|x_{in}|\right) > \eta$$

for each $i \in \mathbb{N}$. Put $f_j := P_{k_{j+1}} - P_{k_j}$ and $y_{ij} := f_j(x_i)$ $(i, j \in \mathbb{N})$.

(*) means that $\| y_{ii} \|_\varphi > \eta$ for each $i \in \mathbb{N}$. We will apply Rosenthal's Lemma 3.4 to $\alpha_{ij} := \| y_{ij} \|_\varphi$ $(i, j \in \mathbb{N})$. The assumption of 3.4 is satisfied, since $\sum_{j=1}^{\infty} \alpha_{ij} = \| x_i \|_\varphi \leq$ $\leq \sup_{x\in K} \| x \|_\varphi := \alpha < \infty$.

Let $m \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$z_i \in \ell_\varphi \quad \text{and} \quad \| z_i \|_\varphi \leq \varepsilon \quad (i = 1, \ldots, m) \quad \text{imply} \quad \| \sum_{i=1}^{m} z_i \|_\varphi \leq \alpha/m.$$

By 3.4 there is a sequence $n_1 < n_2 < n_3 < \ldots$ of natural numbers such that $\alpha_{n_i n_j} \leq \varepsilon$ if $i \neq j$. Put $z_{ij} := y_{n_i n_j}$. Then $\| \sum_{\substack{i=1\\ i\neq j}}^{m} z_{ij} \|_\varphi \leq \alpha/m$. We will estimate $\| sz \|_\varphi$, where $s := m^{-1/q}$ and $z := \sum_{j=1}^{m} z_{jj}$. We have

$$\| \sum_{\substack{1\leq i,j\leq m\\ i\neq j}} sz_{ij} \|_\varphi = \sum_{j=1}^{m} \| \sum_{\substack{i=1\\ i\neq j}}^{m} sz_{ij} \|_\varphi \leq \sum_{j=1}^{m} \| \sum_{\substack{i=1\\ i\neq j}}^{m} z_{ij} \|_\varphi \leq \alpha.$$

Since K is q-convex, we have $x := s\sum_{i=1}^{m} x_{n_i} \in K$ and therefore

$$\| \sum_{1\leq i,j\leq m} sz_{ij} \|_\varphi \leq \| sx \|_\varphi \leq \alpha.$$

Choose a number $\beta > 0$ for α according to 3.1 (a) (3). Then

$$\| sz \|_\varphi \leq \beta \left(\| \sum_{1\leq i,j\leq m} sz_{ij} \|_\varphi + \| \sum_{\substack{1\leq i,j\leq m \\ i\neq j}} sz_{ij} \|_\varphi \right) \leq 2\alpha\beta.$$

On the other hand, $\| z \|_\varphi = \sum_{j=1}^{m} \| z_{jj} \|_\varphi > m \cdot \eta = s^{-q}\eta$ hence, by 3.5 with $\gamma :=$ $= \min\{1, \varphi(1)/\alpha\}$,

$$\| sz \|_\varphi \geq s^q \| z \|_\varphi \cdot\psi(q,s) \cdot \gamma \geq \eta \cdot \psi(q,s) \cdot \gamma$$

since each member of the sequence $z =: (z_n)$ is a member of some sequence of K and therefore $\varphi(|z_n|) \leq \alpha$ for all $n \in \mathbb{N}$.

We have proved that

$$\eta \cdot \psi(q,s) \cdot \gamma \leq \| sz \|_\varphi \leq 2\alpha\beta.$$

It follows that for every $m \in \mathbb{N}$

$$\psi\left(q, m^{-1/q}\right) \leq 2\alpha\beta/\gamma\eta,$$

a contradiction to the assumption $\psi(q,r) \to \infty \ (0 < r \to 0)$.

Corollary 3.8. *If* $\inf_{0<t\leq 1} \varphi(st)/s\varphi(t) \to \infty \ (0 < s \to 0)$, *then every ctb subset of* ℓ_φ *is sctb.*

Proof. Modifying $\varphi(t)$ for $t \geq 1$ we may assume that $\sup_{t>0} \varphi(t) = \infty$. Let K be a *ctb* subset of ℓ_φ and $U := \{x \in \ell_\varphi : \| x \|_\varphi \leq 1\}$. Choose a finite set $F = \{x_1, \dots, x_n\} \subset \ell_\varphi$ and convex subsets C_i of U such that $K \subset \bigcup_{i=1}^{n}(x_i + C_i)$. We may assume that $0 \in C_i$. Then $co\, K \subset co\, F + \sum_{i=1}^{n} C_i$. Therefore $co\, K$ is $\| \, \|_\varphi$ -bounded, since by 3.1 (a) (3) the sum of $\| \, \|_\varphi$ -bounded sets is $\| \, \|_\varphi$ -bounded. Consequently $co\, K$ is totally bounded by 3.7 and $\overline{co}\, K$ is compact. Since the continuous dual of ℓ_φ separates the points, $\overline{co}\, K$ is *sctb* by 2.8 (4) $\Rightarrow$ (1). Therefore K is *sctb*, too.

If $p > 0$ and $0 < q \leq 1$, then by 3.7 every bounded q-convex subset of ℓ_p is totally bounded iff $p < q$. An analogous statement does not hold for the function spaces $L_p[0,1]$: since the set of all characteristic functions of $L_p[0,1]$ is not totally bounded, the unit ball of $L_\infty[0,1]$ is an absolutely convex bounded subset of $L_p[0,1]$, which is not totally bounded.

Recently G. Metafune has drawn my attention to Pitt's Theorem [LT 77, 2.c.3], which says that every continuous linear operator $T : \ell_q \to \ell_p$ is compact if $1 \leq p < q < \infty$. Turpin [T 76, Théorème 3.5.3] proved a theorem of the same type, which is strictly related to the following corollary of 3.7. Corollary 3.9 is actually equivalent to 3.7 (2) $\Rightarrow$ (3).

Corollary 3.9. *Assume that $q \in Q_\varphi \cap [0,1]$ and E is a locally q-convex linear space. Then every continuous linear operator $T : E \to \ell_\varphi$ is compact.*

Proof. Let U be a bounded 0-neighbourhood in ℓ_φ and V a q-convex 0-neighbourhood in E such that $T(V) \subset U$. Then $T(V)$ is q-convex and bounded, hence relatively compact by 3.7 (2) $\Rightarrow$ (3).

We now show that, viceversa, 3.7 (2) $\Rightarrow$ (3) is a consequence of 3.9: let $q \in Q_\varphi \cap [0,1]$ and K be a q-convex bounded subset of ℓ_φ. Then $B := K - K$ is an absolutely q-convex bounded subset of E. Therefore the sets εB, $\varepsilon > 0$, form a 0-neighbourhood base for a locally q-convex linear topology σ on $E :=$ span B, which is finer than $\tau_\varphi | E$. Therefore the identity map $(E, \sigma) \ni x \to x \in (\ell_\varphi, \tau_\varphi)$ is continuous, hence compact by 3.9. It follows that B and therefore K are relatively compact in $(\ell_\varphi, \tau_\varphi)$.

We briefly examine boundedness for non q-convex sets.

Proposition 3.10. *Assume that φ satisfies the Δ_2-condition at 0. The boundedness and $\| \, \|_\varphi$ -boundedness are equivalent in $(\ell_\varphi, \tau_\varphi)$ iff*

(i) $\sup_{t>0} \varphi(t) = \infty$ *and*

(ii) $\sup_{0<t\leq 1} \varphi(st)/\varphi(t) \to 0 \quad (0 < s \to 0)$.

Proof. $\Rightarrow$ (i) If $\sup_{t>0} \varphi(t) < \infty$, then $\{te_1 : t \in \mathbb{R}\}$ is $\| \, \|_\varphi$ -bounded, but not bounded in $(\ell_\varphi, \tau_\varphi)$.

(ii) Suppose that $\sup_{0<t\leq 1} \varphi(st)/\varphi(t)$ does not converge to 0 for $0 < s \to 0$. Put $B_\varepsilon :=$ $= \{x \in \ell_\varphi : \| x \|_\varphi \leq \varepsilon\}$ for $\varepsilon > 0$. We show that for no $\varepsilon > 0$ the set B_ε is bounded.

Let $\varepsilon, \delta \in]0,1]$ with $\varphi(\delta) \leq \varepsilon$. Since

$$\sup_{\delta \leq t \leq 1} \varphi(st)/\varphi(t) \leq \varphi(s)/\varphi(\delta) \to 0 \quad (0 < s \to 0),$$

$\sup_{0<t\leq\delta} \varphi(st)/\varphi(t)$ does not converge to 0 for $0 < s \to 0$. Therefore there are sequences $(s_n), (t_n)$ in $]0, \delta]$ and a number $\eta > 0$ such that $s_n \to 0$ $(n \to \infty)$ and

$\varphi(s_n t_n)/\varphi(t_n) \geq 2\eta$ for all $n \in \mathbb{N}$. Let $s > 0$. We show that $sB_\varepsilon \not\subset B_{\eta\varepsilon}$. Choose $n \in \mathbb{N}$ with $s_n \leq s$. Since $\varphi(t_n) \leq \varphi(\delta) \leq \varepsilon$, there is a natural number k with $k\varphi(t_n) \leq \varepsilon < 2k\varphi(t_n)$. Then $x := t_n \sum_{i=1}^{k} e_i \in B_\varepsilon$, but $sx \notin B_{\eta\varepsilon}$ since

$$\| sx \|_\varphi = k\varphi\left(st_n\right) \geq k\varphi\left(s_n t_n\right) \geq 2\eta k\varphi\left(t_n\right) > \eta\varepsilon.$$

$\Leftarrow$ Suppose that (i) and (ii) hold. In view of 3.6 it is enough to show that B_α is bounded for every $\alpha > 0$. Let $\alpha, \varepsilon > 0$. By (i) there is a number $u \geq 1$ with $\varphi(u) > \alpha$. Since by (ii)

$$\psi(s) := \sup_{0<t\leq u} \varphi(st)/\varphi(t) \leq \max\left\{\sup_{0<t\leq u} \varphi(st)/\varphi(t), \varphi(su)/\varphi(1)\right\} \to 0$$

for $0 < s \to 0$, there is an $s > 0$ with $\alpha\psi(s) \leq \varepsilon$.

We show that $sB_\alpha \subset B_\varepsilon$: let $x = (x_n) \in B_\alpha$. Then $|x_n| \leq u$ for $n \in \mathbb{N}$ and $\| sx \|_\varphi = \sum_{n=1}^{\infty} \varphi(s|x_n|) \leq \sum_{n=1}^{\infty} \varphi(|x_n|)\psi(s) = \| x \|_\varphi \ \psi(s) \leq \varepsilon$.

As in [LT 77, p. 143] we define

$$\beta_\varphi := \inf \ B_\varphi, \quad \text{where}$$

$$B_\varphi := \left\{q > 0 : \inf_{0<s,t\leq 1} \varphi(st)/s^q\varphi(t) > 0\right\} = \left\{q > 0 : \inf_{0<s\leq 1} \psi(q,s) > 0\right\}.$$

In [LT 77], φ is always assumed to be a *convex* Orlicz function; in this case $\varphi(st)/s\varphi(t) = \varphi(st + (1-s)\cdot 0)/(s\varphi(t) + (1-s)\varphi(0)) \leq 1$ for $s,t \in]0,1]$, hence $\beta_\varphi \geq 1$. Analogously one sees that $\beta_\varphi \leq 1$ if φ is concave. In our context the condition $\beta_\varphi < 1$ is important (cf. 3.12).

Lemma 3.11. *If* $\beta_\varphi < \infty$, *then* $\sup_{0<s\leq 1} \psi(\beta_\varphi, s) = 1$.

Proof. Let $\beta_\varphi < \infty$. Since $\psi(q,1) = 1$ and $\psi(0,s) \leq 1$ for $q > 0$ and $0 < s \leq 1$, we have only to prove that $\psi(\beta_\varphi, s) \leq 1$ if $\beta_\varphi > 0$ and $0 < s < 1$.

Let $p := \beta_\varphi > 0$, $0 < s < 1$ and $a > 1$. We prove that $\psi(p,s) \leq a$. Choose $q \in]0,p[$ with $s^{q-p} \leq a$. Since $\inf_{0<r\leq 1} \psi(q,r) = 0$, there are numbers $r, t \in]0,1]$ with $\varphi(rt)/r^q\varphi(t) \leq s^{2q}$. Let k and n be integers ≥ 0 such that

$$s^{k+1} \leq r \leq s^k \quad \text{and} \quad s^{n+1} \leq t \leq s^n.$$

Then

$$s^{2q} > \varphi(rt)/r^q\varphi(t) \geq \varphi\left(s^{n+k+2}\right)/s^{kq}\varphi(s^n) = y_{n+k+2}/s^{kq}y_n,$$

where $y_i := \varphi(s^i)$, hence

$$\prod_{i=1}^{k+2} y_{n+i}/y_{n+i-1} = y_{n+k+2}/y_n \leq s^{q(k+2)}.$$

Therefore one of the $k+2$ factors y_{n+i}/y_{n+i-1} is $\leq s^q$. Let m be an integer ≥ 0 with $y_{m+1}/y_m \leq s^q$. Then

$$\varphi(s \cdot s^m)/s^p\varphi(s^m) = y_{m+1}/s^p y_m \leq s^{q-p} \leq a,$$

hence $\psi(p,s) \leq a$.

Obviously, $B_\varphi = [\beta_\varphi, \infty[$ or $B_\varphi =]\beta_\varphi, \infty[$. There are examples for $\beta_\varphi \in B\varphi$ and for $\beta_\varphi \notin B\varphi$, (see 3.16).

Proposition 3.12. $Q_\varphi =]\beta_\varphi, \infty[$.

Proof. (i) Obviously, $p > q \in Q_\varphi$ implies $p \in Q_\varphi$.

(ii) $\beta_\varphi \notin Q_\varphi$ by 3.11.

(iii) $q > \beta_\varphi$ implies $q \in Q_\varphi$: let $p \in B_\varphi$ with $p < q$. Then $\inf_{0<s\leq 1} \psi(p,s) > 0$, hence $\psi(q,s) = \psi(p,s) \cdot s^{p-q} \to \infty$ $(0 < s \to 0)$.

Proposition 3.13. *Let* $\alpha, \beta \geq 0$ *with* $\liminf_{0<t\to 0} \varphi(2t)/\varphi(t) = 2^\alpha$ *and* $\limsup_{0<t\to 0} \varphi(2t)/\varphi(t) = 2^\beta$. *Then* $\alpha \leq \beta_\varphi \leq \beta$.

Proof. (a) We first prove that $q > \beta$ implies $q \in B_\varphi$. Choose a number a with $2^\beta < a < 2^q$ and $\varepsilon \in]0,1[$ such that $\varphi(2t)/\varphi(t) \leq a$ for all $t \in]0, \varepsilon[$. Then $\varphi(2^{-n}t) \geq \varphi(t) \cdot a^{-n}$ for $t \in]0, \varepsilon]$ and $n \in \mathbb{N}$. Put $b := \min\{1, 2^{-q}\} \cdot \varphi(\varepsilon)/\varphi(1)$. Let $s, t \in]0, 1]$. We prove that $\varphi(st)/s^q\varphi(t) \geq b$. From that follows that $q \in B_\varphi$.

Case 1, $0 < t \leq \varepsilon$. Choose $k \in \mathbb{N}$ with $2^{-k} < s \leq 2^{-k+1}$. Then

$$\varphi(st)/s^q\varphi(t) \geq \varphi\left(2^{-k}t\right)/2^{-(k-1)q}\varphi(t) \geq a^{-k}/2^{-(k-1)q} \geq 2^{-q} \geq b.$$

Case 2, $t \geq \varepsilon$ and $st \leq \varepsilon$. Then $s \leq r := st/\varepsilon \leq 1$. Using case 1 one obtains

$$\varphi(st)/s^q\varphi(t) \geq \left(\varphi(r\varepsilon)/r^q\varphi(\varepsilon)\right) \cdot (\varphi(\varepsilon)/\varphi(1)) \geq 2^{-q} \cdot \varphi(\varepsilon)/\varphi(1) \geq b.$$

Case 3, $st \geq \varepsilon$. Then $\varphi(st)/s^q\varphi(t) \geq \varphi(\varepsilon)/\varphi(1) \geq b$.
(b) Let $0 < q < \alpha$. We prove that $q \notin B_\varphi$. Choose a number a with $2^q < a < 2^\alpha$ and $\varepsilon \in]0,1[$ such that $\varphi(2t)/\varphi(t) \geq a$ for all $t \in]0,\varepsilon]$. Then $\varphi(2^{-n}t) \leq \varphi(t) \cdot a^{-n}$ for $t \in]0,\varepsilon]$ and $n \in \mathbb{N}$. It follows that

$$\varphi\left(2^{-n}t\right) / \left(2^{-n}\right)^q \varphi(t) \leq \left(2^q/a\right)^n \to 0 \quad (n \to \infty),$$

hence $\inf_{0<s,t\leq 1} \varphi(st)/s^q\varphi(t) = 0$ and $q \notin B_\varphi$.

From 3.2, 3.12, 3.13 follows:

Corollary 3.14. *φ satisfies the Δ_2-condition at 0 iff $\beta_\varphi < \infty$ iff $Q_\varphi \neq \emptyset$.*

If $\lim_{0<t\to 0} \varphi(2t)/\varphi(t)$ exists and is equal to 2^β, then $\beta = \beta_\varphi$ by 3.13. The following fact shows in particular that there are functions φ with $\beta_\varphi < 1 < \beta$, where $\limsup_{0<t\to 0} \varphi(2t)/\varphi(t) = 2^\beta$.

Proposition 3.15. *For every β and q with $0 < q \leq \beta$ there is a continuous increasing function $\varphi : [0,\infty[\to [0,\infty[$ such that $\varphi(t) = 0$ iff $t = 0$, $\sup_{t>0} \varphi(t) = \infty$, $\limsup_{0<t\to 0} \varphi(2t)/\varphi(t) = 2^\beta$ and $\beta_\varphi = q$.*

Proof. Define a function $\varphi : [0,\infty[\to [0,\infty[$ in the following way: put $d := 2^\beta$, $a := = 2^{-\beta/q} = d^{-1/q} \in]0,1/2]$; $\varphi(a^n) := d^{-n}$ and $\varphi(a^n/2) := d^{-n-1}$ for $n \in \mathbb{N} \cup \{0\}$; $\varphi(0) = 0$, $\varphi(t) = t$ if $t \geq 1$, define φ linear on the intervals $[a^{n+1}, a^n/2]$ and $[a^n/2, a^n]$.

We show $\limsup_{0<t\to 0} \varphi(2t)/\varphi(t) = 2^\beta$: let $n \in \mathbb{N}$ and $a^{n+1} \leq 2t \leq a^n$. If $t \in [a^{n+1}/2, a^{n+1}]$ and $2t \in [a^n/2, a^n]$, then

$$\begin{aligned}\varphi(t) &= d^{-n-2}\left[2(d-1)a^{-n-1} \cdot t + 2 - d\right] \geq \\ &\geq d^{-n-2}\left[2(d-1)a^{-n} \cdot 2t + 2 - d\right] = d^{-1}\varphi(2t),\end{aligned}$$

hence $\varphi(2t)/\varphi(t) \leq d$. If $t \in [a^{n+1}/2, a^{n+1}]$ and $2t \in [a^{n+1}, a^n/2]$, then $\varphi(2t)/ /\varphi(t) \leq \varphi(a^{n+1})/\varphi(a^{n+1}/2) = d$. If $t \in [a^{n+1}, a^n]$, then $\varphi(2t)/\varphi(t) \leq \varphi(a^n)/ /\varphi(a^n/2) = d$.

$q \geq \beta_\varphi$: let $s,t \in]0,1]$ and $k,n \in \mathbb{N}$ such that $a^k \leq s \leq a^{k-1}$ and $a^n \leq s \leq a^{n-1}$. Then

$$\varphi(st)/s^q\varphi(t) \geq \varphi\left(a^{k+n}\right)/a^{(k-1)q}\varphi\left(a^{n-1}\right) = \left(a^q d\right)^{-k} a^q/d = d^{-2}.$$

Therefore $q \in B_\varphi$ and $q \geq \beta_\varphi$.

$q \leq \beta_\varphi$: if $0 < p < q$, then for $s = a^k$ and $t = 1$ we have

$$\varphi(st)/s^q\varphi(t) = d^{-k}/a^{kp} = a^{(q-p)k} \to 0 \quad (k \to \infty),$$

hence $p \notin B_\varphi$ and $p \leq \beta_\varphi$. It follows that $q \leq \beta_\varphi$.

Example 3.16. *(a) Let* $p > 0$ *and* $\varphi(t) = t^p$ *for* $t \geq 0$. *Then* $\lim_{t\to 0} \varphi(2t)/\varphi(t) = 2^p$ *and* $B_\varphi = [p, \infty[$.

(b) Let $p \geq 0$. *Define* φ *by* $\varphi(t) = -t^p/ln\ t$ *if* $0 < t < 1/e$ *and* $\varphi(t) = e^{1-p} \cdot t$ *if* $t \in [0, \infty[\setminus]0, 1/e[$. *Then* $\lim_{t\to 0} \varphi(2t)/\varphi(t) = 2^p$ *and* $B_\varphi =]p, \infty[$.

(c) Define φ *as in (b) with* $p = 0$. *In particular we have* $\sup_{t>0} \varphi(t) = \infty$. *Since* $\beta_\varphi = 0$, *every* $\| \|_\varphi$ *-bounded subset of* ℓ_φ, *which is* q*-convex for some* $q \in]0, 1]$, *is totally bounded by 3.7. On the other hand,* ℓ_φ *contains by 3.10* $\| \|_\varphi$ *-bounded sets, which are not bounded, since for every* $s \in]0, 1]$ $\sup_{0<t\leq 1} \varphi(st)/\varphi(t) \geq \varphi(s^2)/\varphi(s) = 1/2$.

REFERENCES

[DU 77] J. Diestel, J.J. Uhl Jr, *Vector measures*, American Mathematical Society, Mathematical Surveys, **15** (1977).

[DTW 93] E. De Pascale, G. Trombetta, H. Weber, *Convexly totally bounded and strongly convexly totally bounded sets. Solution of a problem of Idzik*, Ann. Scuola Norm. Sup. Pisa, Cl, Sci. (4) (to appear).

[H 82] O. Hadžić, *Fixed point theorems in not necessarily locally convex topological vector spaces*, Lecture Notes in Mathematics, **948**, Springer-Verlag (1982), pp. 118-130.

[H 84] O. Hadžić, *Fixed point theory in topological vector spaces*, University of Novi Sad, Novi Sad, 1984.

[I 87] A. Idzik, *On γ-almost fixed point theorems. The single-valued case*, Bull. Pol. Acad. Sci. Math., **35** (1987), pp. 461-464.

[I 88] A. Idzik, *Almost fixed point theorems*, Proc. Amer. Math. Soc., **104** (1988), pp. 779-784.

[JOT 76] R.F. Jamison, R.C. O'Brien, P.D. Taylor, *On embedding a compact convex set into a locally convex topological vector space*, Pacific J. Math., **64** (1976), pp. 193-205.

[J 81] H. Jarchow, *Locally convex spaces*, B. G. Teubner, Stuttgart, 1981.

[K 80] N.J. Kalton, *An Fspace with trivial dual where the Krein-Milman theorem holds*, Israel J. Math., **36** (1980), pp. 41-49.

[Kl 60] V. Klee, *Leray-Schauder theory without local convexity*, Math. Ann., **141** (1960), pp. 286-296.

[Kr 76] C. Krauthausen, *Der Fixpunktsatz von Schauder in nicht notwendig konvexen Räumen sowie Anwendungen auf Hammerstein'sche Gleichungen*, Dissertation, Aachen, 1976.

[M 81] R.D. Mauldin, *The Scottish Book*, Birkhäuser, 1981.

[LT 77] J. Lindenstrauss, L. Tzafriri, *Classical Banach spaces I*, Springer-Verlag, 1977.

[MO 48] S. Mazur, W. Orlicz, *Sur les espaces métrique linéaires (I)*, Stud. Math., **10** (1948), pp. 184-208.

[N 51] M. Nagumo, *Degree of mapping in convex linear topological spaces*, Amer. J. Math., **73** (1951), pp. 497-511.

[PW 70] N.T. Peck, L. Waelbroeck, *Problem 58*, Stud. Math., 38 (1970), p. 482.

[R 76] J.W. Roberts, *Phatological compact convex sets in the spaces $L^p([0,1]), 0 \leq p < 1$. The Altgeld Book*, University of Illinois, Functional Analysis Seminar, 1976.

[R 77] J.W. Roberts, *A compact convex set with no extreme points*, Studia Math., **60** (1977), pp. 255-266.

[R 78] J.W. Roberts, *The embedding of compact convex sets in locally convex spaces*, Can. J. Math., **30** (1978), pp. 449-454.

[Ro 84] S. Rolewicz, *Metric linear spaces*, PWN - Polish Scientific Publishers, Warszawa, 1984.

[T 76] P. Turpin, *Convexités dans les espaces vectoriels topologiques généraux*, Dissertationes Mathematicae, **131**, Warszawa, 1976.

[W 61] A. Wiweger, *Linear spaces with mixed topology*, Studia Math., **20** (1961), pp. 47-68.

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