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ON BASIC SEQUENCES IN BANACH SPACES

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Dedicated to the memory of Professor Gottfried Köthe

SUMMARY

Let X be a Banach space with X^{**} separable. If X has a shrinking basis and Y is a closed subspace of X^{**} which contains X, there exists a shrinking basis (x_n) in X with two complementary subsequences (x_{m_j}) and (x_{n_j}) so that $[x_{m_j}]$ is a reflexive space and $X + \widetilde{[x_{n_j}]} = Y$, where we are denoting by $\widetilde{[x_{n_j}]}$ the weak-star closure of $[x_{n_j}]$ in X^{**}. If (y_n) is a sequence in X that converges to a point in $X^{**} \sim X$ for the weak-star topology, there is a basic sequence (y_{n_j}) in (y_n) such that $[y_{n_j}]$ is a quasi-reflexive Banach space of order one. Given a Banach space Z with basis it is also proved that every basic sequence (z_n) in Z has a subsequence extending to a basis of Z.

The vector spaces we use here are defined on the field of the real or complex numbers. If X is a Banach space write $\|\cdot\|$ to denote its norm; X^* denotes the Banach space conjugate of X, X^{**} and X^{***} are the conjugate of X^* and X^{**} respectively. As it is usual, we shall consider X and X^* as subspaces of X^{**} and X^{***} respectively. Some times we write $\langle x, u\rangle$ instead of $u(x)$ for $u \in X^{**}$ and $x \in X^*$. If (x_n) is a sequence in X, we write $[x_n]$ to denote its closed linear hull. The sequences (x_{m_j}) and (x_{n_j}) are said to be complementary if

$$\{m_j : j = 1, 2, \ldots\} \quad \text{and} \quad \{n_j : j = 1, 2, \ldots\}$$

are disjoint complementary subsets of the set of positive integers. We say that the sequence (x_n) is normalized when $\| x_n \| = 1$, $n = 1, 2, \ldots$ if A is a subset X, lin A is the linear hull of A.

If (x_n) is a basic sequence in a Banach space X we put x_n^*, $n = 1, 2, \ldots$, to denote the functional coefficients of (x_n) in $[x_n]^*$. Given a non negative integer k, the basic sequence (x_n) is called k-shrinking if $[x_n^*]$ has codimension k in $[x_n]^*$. It is said that (x_n) is k-boundedly complete when $[x_n] + [x_n^*]^{\perp}$ has codimension k in $[x_n]^*$, where $[x_n^*]^{\perp}$ is the subspace of $[x_n]^{**}$ orthogonal to $[x_n^*]$, [6].

If X is a Banach space the weak-star topology on X^* is the topology of point wise convergence on every vector of X; we write X_σ^* to denote X^* endowed with this topology. In the

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same way, the weak-star topology on $X^{**}(X^{***})$ is the topology of pointwise convergence on every vector of $X^{*}(X^{**})$. If A is a subset of X^{*}, both $\widetilde{A}$ and $A^{\sim}$ denote its closure in X^{*}_{σ}. If B is a subset of X, $\widetilde{B}$ and $B^{\sim}$ denote the closure of B in X^{**}_{σ}.

A Banach space X is called quasi-reflexive when it has finite codimension in X^{**}; when this codimension is p we shall say that X is quasi-reflexive of order p.

Proposition 1. *Let (x_n) and (y_n) two bases in a Banach space X. If*

$$\sum_{n=1}^{\infty} \| x_n^* \| \cdot \| x_n - y_n \| < \infty$$

then $[x_n^*] \supset [y_n^*]$.

Proof. Let us fix a positive integer m. In the weak-star dual X^*_σ we have

$$y_m^* = \sum_{n=1}^{\infty} \langle x_n, y_m^* \rangle x_n^*.$$

If we write

$$s_k = \sum_{n=1}^{k} \langle x_n, y_m^* \rangle x_n^*$$

and we take x in X with $\| x \| < 1$, we have for $k > m$ the following:

$$|\langle x, y_m^* - s_k \rangle| = |\sum_{n=k+1}^{\infty} \langle x_n, y_m^* \rangle \langle x, x_n^* \rangle| =$$

$$= |\sum_{n=k+1}^{\infty} \langle x_n - y_n, y_m^* \rangle \langle x, x_n^* \rangle| \leq \| y_m^* \| \sum_{n=k+1}^{\infty} \| x_n^* \| \cdot \| x_n - y_n \|$$

so

$$\lim_k \| y_m^* - s_k \| = 0$$

and therefore y_m^* belongs to $[x_n^*]$ and $[y_n^*] \subset [x_n^*]$. Q.E.D.

Proposition 2. *Let (x_n) be a basis in a Banach space X. If (y_n) is a sequence in X such that*

$$\sum_{n=1}^{\infty} \| x_n^* \| \cdot \| x_n - y_n \| < 1 \tag{1}$$

then (y_n) *is a basis in* X *with* $[y_n^*] = [x_n^*]$.

Proof. It is well known that (y_n) is a basis in X and there is an isomorphism T from X onto X that transforms x_n in y_n, $n = 1, 2, \ldots$, [1]. From (1) it follows that $[y_n^*]$ is contained in $[x_n^*]$ because of the former proposition. On the other hand, if T^* is the isomorphism from X^* onto X^*, the adjoint of T, we have

$$T^* y_n^* = x_n^*, \qquad n = 1, 2, \ldots,$$

and so

$$\sum_{n=1}^{\infty} \| y_n^* \| \cdot \| y_n - x_n \| \leq$$

$$\leq \| (T^*)^{-1} \| \sum_{n=1}^{\infty} \| x_n^* \| \cdot \| y_n - x_n \| \leq \| (T^*)^{-1} \| .$$

We can apply again the former proposition to obtain that $[x_n^*]$ is contained in $[y_n^*]$ .Q.E.D.

The following proposition is a straightforward consequence of Proposition 2:

Proposition 3. *Let* (x_n) *be a basis in a Banach space* X. *If* (u_n) *is a sequence in* $[x_n^*]$ *such that*

$$\sum_{n=1}^{\infty} \| x_n \| \cdot \| x_n^* - u_n \| < 1$$

there exists a basis (y_n) *in* X *such that* $y_n^* = u_n$, $n = 1, 2, \ldots$.

In the proof of Theorem 1 we shall need the following result that we have proved in [8]:

a) Let X *be an infinite dimensional Banach space with* X^{**} *separable. If* Z *is a closed subspace of* X^{**} *that contains* X, *there is an infinite dimensional closed subspace* Y *of* X *such that* $X + \widetilde{Y} = Z$.

Theorem 1. *Let* X *be a Banach space with a shrinking basis. Let* Z *be a closed subspace of* X^{**} *that contains* X. *If* X^{**} *is separable, there exists a shrinking basis* (x_n) *in* X *with two complementary subsequence* (x_{m_j}) *and* (x_{n_j}) *such that* $[x_{m_j}]$ *is reflexive and* $X + \widetilde{[x_{n_j}]} = Z$.

Proof. We write $X^{\perp}$ and $Z^{\perp}$ for the subspace of X^{***} which are orthogonal to X and Z respectively. Let $(u_{2m-1})_{m=1}^{\infty}$ be a sequence in $X^{\perp}$ such that its linear hull is weak-star dense in $X^{\perp}$ and $\| u_{2m-1} \| \leq 1$, $m = 1, 2, \ldots$. Let $(u_{2m})_{m=1}^{\infty}$ be a sequence in $Z^{\perp}$ so that its linear bull is a weak-star dense subspace of $Z^{\perp}$ and $\| u_{2m} \| \leq 1$, $m = 1, 2, \ldots$. If we apply result a) we obtain an infinite dimensional closed subspace Y of X such that $X + \widetilde{Y} = Z$. Let G be the subspace of X^* which is orthogonal to Y. The closure of G in X^{***}_{σ} contains $Z^{\perp}$ and so, for every positive integer m, a sequence $(u_{mn})_{n=1}^{\infty}$ in X^* can be determined so that it converges to u_m in X^{***}_{σ} and $u_{(2m)n}$ belongs to G, $n = 1, 2, \ldots$. Selecting $(u_m)_{m=1}^{\infty}$ we can suppose that $(u_{mn})_{n=1}^{\infty}$ is normalized, $m = 1, 2, \ldots$. Let (y_n) be a shrinking basis in X and K the basis constant of (y_n). We would have

$$\lim_n u_{mn}\left(y_j\right) = 0, \qquad m, j = 1, 2, \ldots.$$

We order the pairs mn in the sequence:

(2) $$11, 12, 21, \ldots, 1n, 2(n-1), \ldots,$$

that means pq is less than mn if and only if either $p + q < m + n$ or $p + q = m + n$ and $p < m$. Suppose we take $\varepsilon_{mn} > 0$ by asking

$$72\, K(K+1) \sum_{m,n=1}^{\infty} \varepsilon_{mn} < 1.$$

We put $m_{11} = 1$ and $n_{11} = 0$. We find a positive integer n_{12} such that

$$\| u_{1m_{11}} - z_{11} \| < \varepsilon_{11}$$

where

$$z_{11} = \sum_{j=n_{11}+1}^{n_{12}} u_{1m_{11}}\left(y_j\right) y_j^*.$$

We proceed by recurrence and suppose for a given index pq such that rs is the next index in (2), we would have obtained

$$z_{pq} = \sum_{j=n_{pq}+1}^{n_{rs}} u_{pm_{pq}}\left(y_j\right) y_j^*.$$

We could determine a positive integer m_{rs} such that

$$\| \sum_{j=1}^{n_{rs}} u_{rm_{rs}}\left(y_j\right) y_j^* \| < \frac{1}{2} \varepsilon_{rs}.$$

If hk is the indew coming after rs in (2), a positive integer n_{hk} could be found so that

$$\| \sum_{j=n_{hk}+1}^{\infty} u_{rm_{rs}}\left(y_j\right) y_j^* \| < \frac{1}{2}\varepsilon_{rs}.$$

Therefore

$$\| u_{rm_{rs}} - z_{rs} \| < \varepsilon_{rs}$$

where

$$z_{rs} = \sum_{j=n_{rs}+1}^{n_{hk}} u_{rm_{rs}}\left(y_j\right) y_j^*.$$

The sequence

(3) $$z_{11}, z_{12}, z_{21}, \ldots, z_{1n}, z_{2(n-1)2}, z_{n1}, \ldots.$$

is a block basic sequence of (y_n^*) and so, applying a result of Zippin [9] (see also [7, pp. 67-68]), there is a basis (v_n) in X^* extending (3) and such that $[v_n^*] = X$ with basis constant no greater than $18K(K+1)$. Hence

$$\| v_n^* \| \cdot \| v_n \| \leq 36K(K+1)$$

and we would have

$$\| v_n^* \| \leq 72K(K+1)$$

for the values of n corresponding with some z_{pq} because

$$\| z_{pq} \| \geq \| u_{pm_{pq}} \| - \| u_{pm_{pq}} - z_{pq} \| \geq 1 - \varepsilon_{pq} > \frac{1}{2}.$$

Suppose we define a sequence (w_n) in X^* by letting w_n be v_n when this element does not belongs to the sequence (3), and w_n be $u_{pm_{pq}}$ when $v_n = z_{pq}$. Then would have

$$\sum_{n=1}^{\infty} \| v_n^* \| \cdot \| v_n - w_n \| \leq 72K(K+1) \sum_{n=1}^{\infty} \| v_n - w_n \| =$$

$$= 72K(K+1) \sum_{p,q=1}^{\infty} \| u_{pm_{pq}} - z_{pq} \| \leq 72K(K+1) \sum_{p,q=1}^{\infty} \varepsilon_{pq} < 1$$

and we could apply Proposition 3 to obtain a basis (x_n) in X such that $x_n^* = w_n$, $n = 1, 2, \ldots$.

Let us consider now the increasing sequence (m_j) of all positive integers such that $x_{m_j}^*$ is an element of the form $u_{(2p-1)m_{(2p-1)q}}$. Let x_{n_j} be the subsequence of (x_n) complementary to (x_{m_j}). Every element of the form $u_{(2p)m_{(2p)q}}$ belongs to $(x_{n_j}^*)$ and so the closure of $[x_{n_j}^*]$ in X_σ^{***} contains $X^\perp$ from where it follows that $[x_{m_j}]$ is reflexive. The closure of $[x_{m_j}^*]$ in X_σ^{***} contains $Z^\perp$ from where it follows that $\widetilde{[x_{n_j}]}$ is contained in Z. On the other hand, $x_{m_j}^*$ belongs to G, $j = 1, 2, \ldots$, and so $\widetilde{[x_{n_j}]}$ contains Y. Finally we have $X + [x_{n_j}] = Z$. Q.E.D.

Corollary 1. *Let X be a Banach space with shrinking basis. Let p be a non negative integer less or equal than the dimension of X^{**}/X. If X^{**} is separable there is a shrinking basis (x_n) of X and two complementary subsequences (x_{m_j}) and (x_{n_j}) of (x_n) such that $[x_{m_j}]$ is reflexive and $[x_{n_j}]$ is quasi-reflexive of order p.*

Proof. It is enough to take Z in the former theorem such that X has codimension p in Z. Q.E.D.

Corollary 2. *Let X be a Banach space with shrinking basis. Let p be a non negative integer less or equal than the dimension of X^{**}/X. If X^{**} is separable there is a shrinking basis (x_n) of X and two complementary subsequences (x_{m_j}) and (x_{n_j}) of (x_n) such that $[x_{m_j}]$ is reflexive and $X/[x_{n_j}]$ is quasi-reflexive of order p.*

Proof. It is enough to take Z in the former theorem so that the codimension of Z in X^{**} be equal to p. Q.E.D.

Proposition 4. *Let (x_n) be a normalized sequence in a Banach space X. If X^* is separable the following two conditions are equivalent:*

1) The set $\{x_n : n = 1, 2, \ldots\}$ is not weakly relatively compact.

2) There is a basic subsequence (z_n) of (x_n) so that every subsequence of (z_n) is 1-shrinking.

Proof. Firstly we suppose that condition 1) holds. Since X^* is separable there is a point x_0 in $X^{**} \sim X$ and a subsequence (y_n) of (x_n) weak-star converging to x_0. Indeed, the sequence (y_n) could be also found being a basic sequence [3, pp. 41-42]. Let H be the closed hypperplane of X^* defined by

$$H = \{u \varepsilon X^* : \langle x_0, u \rangle = 0\},$$

and let us take a sequence (u_n) in H with $[u_n] = H$. We obviously have

$$\lim_n \langle y_n, u_m \rangle = 0, \qquad m = 1, 2, \ldots.$$

Let n_1 be a positive integer such that

$$|\langle y_{n_1}, u_1 \rangle| < \frac{1}{2}.$$

Proceeding by recurrence, let us suppose we have found the positive integer n_p. We could choose another integer $n_{p+1} > n_p$ so that

$$|\langle y_{n_{p+1}}, u_m \rangle| < \frac{1}{2^{p+1}}, \qquad m = 1, 2, \ldots, p+1.$$

We define the sequence (z_m) by letting $z_m = y_{n_m}$, $m = 1, 2, \ldots$. The sequence (z_n) is the basic subsequence of (x_n) that we are looking for condition 2) to be held. Indeed, let (z_{m_j}) be any subsequences of (z_m) and let L be equal to the orthogonal subspace to $[z_m]$ in X^*. We shall denote by Ψ the canonical mapping from X^* onto X^*/L. It is obvious that L is contained in H, so $\Psi(H)$ is a closed hyperplane of $\Psi(X^*) = [z_{m_j}]^*$. It is easily shown that $[z^*_{m_j}]$ is contained in $\Psi(H)$. On the other hand,

$$\sum_{j=1}^{\infty} |\langle z_{m_j}, u_p \rangle| < \infty, \qquad p = 1, 2, \ldots.$$

Therefore the series

$$\sum_{j=1}^{\infty} \langle z_{m_j}, \Psi(u_p) \rangle z^*_{m_j}$$

converges in $[z_{m_j}]^*$ to $\Psi(u_p)$, $p = 1, 2, \ldots$. Consequently, $[z^*_{m_j}]$ coincides with the hyperplane $\Psi(H)$ and (z_{m_j}) is 1-shrinking.

Conversely, let us now suppose thay condition 2) holds. Since X^* is separable there is a subsequence (t_n) of (z_n) that converges to a point t_0 of X^{**} for the weak-star topology. We always have $t_0 \neq 0$ because in case $t_0 = 0$ [5, Proposition 2.3] could be applied to obtain a shrinking subsequence of (t_n), so a contradiction with condition 2). Let us write

$$M = \{u \in [t_n]^* : \langle t_0, u \rangle = 0\}.$$

Obviously

$$\lim_n \langle t_n, t_m^* \rangle = 0 = \langle t_0, t_m^* \rangle, \qquad m = 1, 2, \ldots$$

and $[t_n^*] \subset M$, from where it follows that M is weak-star dense in $[t_n]^*$, so t_0 can not belong to X and 1) is verified. Q.E.D.

For the proof of Proposition 5 we need the following result [8]:

b) Let B be the closed unit ball of a Banach space X. Let F be a subspace of finite codimension and weak-star closed in X^{**}. Then $(F \cap B)^{\sim} = F \cap \widetilde{B}$.

Proposition 5. *Let X be a Banach space with X^* separable. Let (u_n) be a sequence in X^* that converges to u for the weak-star topology. If*

$$\bigcap_{n=1}^{\infty} [u_n, u_{n+1}, \ldots]^{\sim} = lin\{u\}$$

there exists a subsequence (u_{n_j}) of (u_n) such that

$$\widetilde{\left[u_{n_j}\right]} = \left[u_{n_j}\right] + lin\{u\}.$$

Proof. Let G be the subspace of X^{**} which is orthogonal to $[u_n] \cup \{u\}$. Since X^* is separable, we could take a sequence (v_n) in G with $\| v_n \| \leq 1$, $n = 1, 2, \ldots$, so that its linear hull whould be weak-star dense in G. Let B be the closed unit ball of X. For every positive integer n, we denote by

$$(3) \qquad \{V_{mn} : m = 1, 2, \ldots\}$$

a fundamental system of neighbourhoods of v_n in B for the weak-star topology. We order the neighbourhoods of (4) in a sequence

$$\{V_m; m = 1, 3, \ldots\}.$$

We write $n_1 = 1$ and we suppose that for a positive integer p we have obtained the positive integers $n_1, n_2, \ldots, n_p$. We put

$$L_p = lin\left\{u_{n_1}, u_{n_2}, \ldots, u_{n_p}, u\right\}.$$

Let H_p be the subspace orthogonal to L_p in X^{**}. We write A_{pn} to denote the subspace of X which is orthogonal to L_p in X^{**}. We write A_{pn} to denote the subspace of X which is orthogonal to

$$L_p \bigcup [u_n, u_{n+1}, \ldots]^{\sim}.$$

Since

$$\bigcap_{n=1}^{\infty} \left(L_p \bigcup [u_n, u_{n+1}, \ldots]^{\sim} \right) = L_p \bigcup lin\{u\} = L_p$$

it follows that $\bigcup_{n=1}^{\infty} A_{pn}$ is a dense subset of $H_p \cap X$. We claim that there is a positive integer $n_{p+1} > n_p$ so that

$$A_{pn_{p+1}} \bigcap V_p \neq \emptyset.$$

Indeed,

$$v_n \in H_p, \ n = 1, 2, \ldots,$$

and result b) assures us tht $(B \cap H_p)^{\sim} = \widetilde{B} \cap H_p$, from where it follows that

$$V_p \bigcap B \bigcap H_p \neq \emptyset$$

and so

$$\left(\bigcup_{n=1}^{\infty} A_{pn} \right) \bigcap V_p \neq \emptyset,$$

therefore, the positive integer $n_{p+1} > n_p$ such that $A_{pn_{p+1}} \cap V_p \neq \emptyset$ can be found. Let x_p be a point in this non-void subset. Obviously,

$$\langle x_p, u_{n_j} \rangle = 0, \quad j = 1, 2, \ldots, p, \quad \langle x_p, u_n \rangle = 0, \quad n = n_{p+1}, n_{p+1} + 1, \ldots.$$

We are going to see now how the sequence $(u_{n_j})_{j=1}^{\infty}$ is the subsequence we are looking for. Let M be the subspace of X which is orthogonal to $[\widetilde{u_{n_j}}]$. Since

$$x_j \in M, \quad j = 1, 2, \ldots$$

and v_n is a weak-star cluster point in X^{**} of the sequence (x_j), $n = 1, 2, \ldots$, it follows that $G \subset \widetilde{M}$, therefore

$$\widetilde{\left[u_{n_j}\right]} \subset \left[u_{n_j}\right] + lin\{u\},$$

from where the conclusion follows. Q.E.D.

Theorem 2. *Let (x_n) be a normalized sequence in a Banach space X. If X^{**} is separable, the following conditions are equivalent;*

1) The set $\{x_n : n = 1, 2, \ldots\}$ is not weakly relatively compact.

2) There is a subsequence (z_n) of (x_n) such that if (y_n) is any subsequence of (z_n), then $[y_n]$ is a quasi-reflexive Banach space of order one.

Proof. Let us suppose firstly that 1) holds. We could apply Proposition 4 to find a basic subsequence (t_n) of (x_n) converging to a point t_0 in $X^{**} \sim X$ in the weak-star topology and such that every subsequence of (t_n) is 1-shrinking; $[t_n^*]$ is an hyperplane of $[t_n]^*$ orthogonal to t_0 and consequently

$$\bigcap_{n=1}^{\infty} [t_n, t_{n+1}, \ldots]^{\sim} = lin\{t_0\}.$$

Applying now the former proposition we obtain a subsequence (z_n) of (t_n) so that

$$[z_n]^{\sim} = [z_n] + lin\{t_0\}.$$

It results obvious tht if (y_n) is any subsequence of (z_n), $[y_n]$ is a quasi-reflexive Banach space of order one.

Conversely, when suppose that 2) is true, 1) also follows bearing in mind Proposition 4. Q.E.D.

Proposition 6. *Let (x_n) be a basis in a Banach space X. Let F be a subspace of X^* that contains $[x_n^*]$. If $[x_n^*]$ has finite codimension in F there is a basis (y_n) in X such that $[y_n^*] = F$.*

Proof. It is clear the only case need to show is when $[x_n^*]$ is an hyperplane of F. If we take a vector u in $F \sim [x_n^*]$ with $\| u \| = 1$, we can find an increasing sequence $(n_p)_{p=1}^{\infty}$ of positive integers such that if $n_0 = 0$ and

$$z_p = \sum_{n=n_{p-1}+1}^{n_p} u(x_n)\, x_n^*$$

it follows that

$$\inf \left\{ \| z_p \| : p = 1, 2, \ldots, \right\} > 0.$$

Applying now a method due to Zippin [9] (see also [7, pp. 67-68]), it is possible to obtain a basis (v_n) in $[x_n^*]$ such that $[v_n^*]$ coincides with the restriction of X on $[x_n^*]$ and

$$v_{n_p} = z_p, \ p = 1, 2, \ldots.$$

We define a sequence (w_n) in the following way:

$$w_n = v_n \quad \text{if} \quad n \neq n_p, \ w_{n_p} = v_1 + v_2 + \ldots + v_p, \ p = 1, 2, \ldots.$$

The sequence (w_n) is a basis in $[x_n^*]$ such that

$$w_n^* = v_n^* \quad \text{if} \quad n \neq n_p, \ w_{n_p}^* = v_p^* - v_{p+1}^*, \ p = 1, 2, \ldots.$$

Let H be the hyperplane of X orthogonal to $\{u\}$. Let y_1 be a vector in X such that $\langle y_1, u \rangle = 1$. The restriction of H on $[x_n^*]$ obviously coincides with $[w_n^*]$ and therefore if y_{n+1} is the vector in X with restriction on $[x_n^*]$ equal to u_n^*, $n = 1, 2, \ldots$, we have the basis (y_n) in X and

$$y_1^* = u, \ y_n^* = w_{n=1}^* - \langle y_1, w_{n-1} \rangle u, \ n = 1, 2, \ldots$$

from where the conclusion follows. Q.E.D.

Proposition 7. *Let X be a Banach space with basis. Let k be a non negative integer. The following conditions are equivalent:*

1) X is quasi-reflexive of order k.

2) X has a k-boundedly complete basis and every basis of X is j-boundedly complete with $0 \leq j \leq k$.

Proof. The implication 1) $\Rightarrow$ 2) is obvious. Let us now suppose that 2) holds. Let (x_n) be a k-boundedly complete basis in X. Then $X + [x_n^*]^\perp$ has codimension k in X^{**}, where $[x_n^*]^\perp$ denotes the subspace of X^{**} which is orthogonal to $[x_n^*]$. Let us suppose that X is not quasi-reflexive of order k. It follows that there is a vector v non equal to zero in $[x_n^*]^\perp$. Let u be a vector in X^* such that $\langle v, u \rangle \neq 0$. If F is the linear hull of $[x_n^*] \bigcup \{u\}$ we can apply the former proposition to F and we obtain a basis (y_n) of X such that $[y_n^*] = F$. Obviously, (y_n) is $(k+1)$-boundedly complete and this contradiction finishes the proof. Q.E.D.

Proposition 8. *Let (z_n) be a basis in a Banach space Z. Let G be a be a closed subspace of finite codimension in $[z_n^*]$. If G is weak-star dense in Z^* there exists a basis (t_n) in Z such that $[t_n^*] = G$.*

Proof. It is clear that the only case we need to show is when G is an hyperplane of $[z_n^*]$. Let u be a vector in Z^{**} with $\| u \| = 1$ which is zero on G. Let S be the canonical mapping from Z^{**} onto $Z^{**}/[z_n^*]^\perp$, where $[z_n^*]^\perp$ is the orthogonal to $[z_n^*]$ in Z^{**}. Then $S(Z) = [z_n^{**}]$. If F is the linear hull of $[z_n^{**}] \bigcup Su$, we have the hyperplane $[z_n^{**}]$ in F. Proceedin as in the proof of Proposition 6 we could obtain a basis (y_n) in the hyperplane of $[z_n^*]$ orthogonal to $\{u\}$ in such a way that, if (u_n) are the elements of $[z_n^{**}]$ verifying

$$\langle u_n, y_n\rangle = 1, \quad \langle u_m, y_n\rangle = 0, \quad m \neq n, \quad m, n = 1, 2, \ldots,$$

we would have $[u_n] = [z_n^{**}]$. If t_n is the vector of X such that $St_n = y_n$, $n = 1, 2, \ldots, (t_n)$ is a basis in Z with $[t_n^*] = G$. Q.E.D.

Proposition 9. *Let X be a Banach space with a basis. Let k be a non negative integer. The following conditions are equivalent:*

1) X is quasi-reflexive of order k.

2) X has a k-shrinking basis and every basis of X is j-shrinking, $0 \leq j \leq k$.

Proof. 1) $\Rightarrow$ 2) is obvious. Let us suppose now that 2) holds. Let (x_n) be a k-shrinking basis of X. If $[x_n^*]^\perp$ is the subspace of X^{**} which is orthogonal to $[x_n^*]$, the dimension of $[x_n^*]^\perp$ is precisely k. Let us suppose that X is not quasi-reflexive of order k. We could find a vector u in $X^{**} \sim (X + [x_n^*]^\perp)$. If F is the subspace of $[x_n^*]$ which is orthogonal to $\{u\}$, F is weakly-star dense in X^* and so, applying the former proposition, we could obtain a basis (y_n) of X such that $[y_n^*] = F$. Consequently, (y_n) would be a $(k+1)$-shrinking basis. This contradiction finishes the proof. Q.E.D.

Theorem 3. *Let X be a Banach space with basis. If (y_n) is a basic sequence in X there exists a basis in X which extends some subsequence of (y_n).*

Proof. We can suppose without any restriction that $\| y_n \| = 1$, $n = 1, 2, \ldots$. Let (x_n) be a basis in X. If

$$\lim_n \langle y_n, x_m^*\rangle = 0, \quad m = 1, 2, \ldots, \tag{5}$$

it would be enough to apply [2, Theorem 3]. Let us now suppose thay (5) does not hold. Then there is a weak-star cluster point y in $X^{**} \sim X$ of the sequence (y_n). We put

$$H = \{u \in X^* : \langle y, u\rangle = 0\}.$$

If $H \cap [x_n^*]$ is weak-star dense in X^* we could apply Proposition 8 and obtain a basis (z_n) in X such that $[z_n^*] = H \cap [x_n^*]$. In that case, we can take a subsequence (y_{n_j}) of (y_n) such that

$$\lim_j \langle y_{n_j}, z_m^*\rangle = 0, \quad m = 1, 2, \ldots,$$

and the proof is reduced to the former case. If $H \cap [x_n^*]$ is not weak-star dense in X^* there is an element x_0 in X such that

$$\langle x_0, v\rangle = 0, \quad v \in \bigcap [x_n^*] .$$

There exists an element u in H such that $\langle x_0, u\rangle \neq 0$. Let F be the linear hull of $[x_n^*] \bigcup \{u\}$. We could apply Proposition 6 and obtain a basis (z_n) of X such that $[z_n^*] = F$. We would have now that $H \cap F$ is weak-star dense in X^* and so, according to Proposition 8, there is a basis (w_n) in X such that $[w_n^*] = H \cap [z_n^*]$. If (y_{n_j}) is now a subsequence of (y_n) such that

$$\lim_j \langle y_{n_j}, w_m^*\rangle = 0, \quad m = 1, 2, \ldots$$

the proof is also reduced to the first case in that situation. Q.E.D.

Note. The original proof of Proposition 2 was longer than the one presented here. We are grateful to Dr. V. Montesinos for providing the proof given here.

REFERENCES

[1] C. BESSAGA, A. PELCZYNSKI, *On bases and unconditional convergence of series in Banach spaces*, Studia Math., **17** (1958), pp. 151-164.

[2] D.W. DEAN, I. SINGER, L. STERNBACH, *On shrinking basic sequences in Banach spaces*, Studia Math., **40** (1971), pp. 23-33.

[3] J. DIESTEL, *Sequences and series in Banach spaces*, Springer-Verlag, New York-Berlin-Heidelberg-Tokyo, 1984.

[4] R.C. JAMES, *Bases and reflexivity of Banach spaces*, Ann. Math., (2) **52** (1950), pp. 518-527.

[5] W.B. JOHNSON, H.P. ROSENTHAL, *On w^* basic sequences and their applications to the study of Banach spaces*, Studia Math., **43** (1972), pp. 77-92.

[6] I. SINGER, *Bases and quasi-reflexivity of Banach spaces*, Ann. Math., **153** (1964), pp. 199-209.

[7] I. SINGER, *Bases in Banach spaces I*, Springer-Verlag, New York-Berlin-Heidelberg, 1970.

[8] M. VALDIVIA, *Banach spaces X with X^{**} separable*, Israel J. Math., **59** (1987), pp. 107-111.

[9] M. ZIPPIN, *A remark on bases and reflexivity in Banach spaces*, Israel J. Math., **6** (1968), pp. 74-79.

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