ON BASIC SEQUENCES IN BANACH SPACES
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Dedicated to the memory of Professor Gottfried Köthe

SUMMARY
Let $X$ be a Banach space with $X^{**}$ separable. If $X$ has a shrinking basis and $Y$ is a closed subspace of $X^{**}$ which contains $X$, there exists a shrinking basis $(x_n)$ in $X$ with two complementary subsequences $(x_{m_j})$ and $(x_{n_j})$ so that $[x_{m_j}]$ is a reflexive space and $X + [x_{n_j}] = Y$, where we are denoting by $[x_{n_j}]$ the weak-star closure of $[x_{n_j}]$ in $X^{**}$. If $(y_n)$ is a sequence in $X$ that converges to a point in $X^{**} \sim X$ for the weak-star topology, there is a basic sequence $(y_{n_j})$ in $(y_n)$ such that $[y_{n_j}]$ is a quasi-reflexive Banach space of order one. Given a Banach space $Z$ with basis it is also proved that every basic sequence $(z_n)$ in $Z$ has a subsequence extending to a basis of $Z$.

The vector spaces we use here are defined on the field of the real or complex numbers. If $X$ is a Banach space write $|| \cdot ||$ to denote its norm; $X^*$ denotes the Banach space conjugate of $X$, $X^{**}$ and $X^{***}$ are the conjugate of $X^*$ and $X^{**}$ respectively. As it is usual, we shall consider $X$ and $X^*$ as subspaces of $X^{**}$ and $X^{***}$ respectively. Some times we write $(x,u)$ instead of $u(x)$ for $u \in X^{**}$ and $x \in X^*$. If $(x_n)$ is a sequence in $X$, we write $[x_n]$ to denote its closed linear hull. The sequences $(x_{m_j})$ and $(x_{n_j})$ are said to be complementary if

$$\{m_j : j = 1, 2, \ldots \} \quad \text{and} \quad \{n_j : j = 1, 2, \ldots \}$$

are disjoint complementary subsets of the set of positive integers. We say that the sequence $(x_n)$ is normalized when $||x_n|| = 1$, $n = 1, 2, \ldots$ if $A$ is a subset $X$, lin $A$ is the linear hull of $A$.

If $(x_n)$ is a basic sequence in a Banach space $X$ we put $x_n^*$, $n = 1, 2, \ldots$, to denote the functional coefficients of $(x_n)$ in $[x_n]^*$. Given a non negative integer $k$, the basic sequence $(x_n)$ is called $k$-shrinking if $[x_n^*]$ has codimension $k$ in $[x_n]^*$. It is said that $(x_n)$ is $k$-boundedly complete when $[x_n] + [x_n^*]^{\perp}$ has codimension $k$ in $[x_n]^*$, where $[x_n^*]^{\perp}$ is the subspace of $[x_n^{**}]$ orthogonal to $[x_n^*]$, [6].

If $X$ is a Banach space the weak-star topology on $X^*$ is the topology of point wise convergence on every vector of $X$; we write $X^*_\sigma$ to denote $X^*$ endowed with this topology. In the

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same way, the weak-star topology on $X^\ast(X^{***})$ is the topology of pointwise convergence on every vector of $X^\ast(X^{**})$. If $A$ is a subset of $X^\ast$, both $\overline{A}$ and $A^\sim$ denote its closure in $X^\ast_c$. If $B$ is a subset of $X$, $\overline{B}$ and $B^\sim$ denote the closure of $B$ in $X^{**}_g$.

A Banach space $X$ is called quasi-reflexive when it has finite codimension in $X^{**}$; when this codimension is $p$ we shall say that $X$ is quasi-reflexive of order $p$.

**Proposition 1.** Let $(x_n)$ and $(y_n)$ two bases in a Banach space $X$. If

$$\sum_{n=1}^{\infty} \| x_n^\ast \| \cdot \| x_n - y_n \| < \infty$$

then $[x_n^\ast] \supset [y_n^\ast]$.

**Proof.** Let us fix a positive integer $m$. In the weak-star dual $X_g^\ast$ we have

$$y_m^\ast = \sum_{n=1}^{\infty} (x_n, y_m^\ast) x_n^\ast.$$

If we write

$$s_k = \sum_{n=1}^{k} (x_n, y_m^\ast) x_n^\ast$$

and we take $x$ in $X$ with $\| x \| < 1$, we have for $k > m$ the following:

$$|\langle x, y_m^\ast - s_k \rangle| = |\sum_{n=k+1}^{\infty} (x_n, y_m^\ast) \langle x, x_n^\ast \rangle| =$$

$$= |\sum_{n=k+1}^{\infty} (x_n - y_n, y_m^\ast) \langle x, x_n^\ast \rangle| \leq \| y_m^\ast \| \sum_{n=k+1}^{\infty} \| x_n^\ast \| \cdot \| x_n - y_n \|$$

so

$$\lim_{k} \| y_m^\ast - s_k \| = 0$$

and therefore $y_m^\ast$ belongs to $[x_n^\ast]$ and $[y_n^\ast] \subset [x_n^\ast]$. Q.E.D.

**Proposition 2.** Let $(x_n)$ be a basis in a Banach space $X$. If $(y_n)$ is a sequence in $X$ such that

$$(1) \sum_{n=1}^{\infty} \| x_n^\ast \| \cdot \| x_n - y_n \| < 1$$
then \((y_n)\) is a basis in \(X\) with \([y_n^*] = [x_n^*]\).

Proof. It is well known that \((y_n)\) is a basis in \(X\) and there is an isomorphism \(T\) from \(X\) onto \(X\) that transforms \(x_n\) in \(y_n\), \(n = 1, 2, \ldots, [1]\). From (1) it follows that \([y_n^*]\) is contained in \([x_n^*]\) because of the former proposition. On the other hand, if \(T^*\) is the isomorphism from \(X^*\) onto \(X^*\), the adjoint of \(T\), we have

\[
T^*y_n^* = x_n^*, \quad n = 1, 2, \ldots,
\]

and so

\[
\sum_{n=1}^{\infty} ||y_n^*|| \cdot ||y_n - x_n|| \leq \\
\leq ||(T^*)^{-1}|| \sum_{n=1}^{\infty} ||x_n^*|| \cdot ||y_n - x_n|| \leq ||(T^*)^{-1}||.
\]

We can apply again the former proposition to obtain that \([x_n^*]\) is contained in \([y_n^*]\). Q.E.D.

The following proposition is a straightforward consequence of Proposition 2:

Proposition 3. Let \((x_n)\) be a basis in a Banach space \(X\). If \((u_n)\) is a sequence in \([x_n^*]\) such that

\[
\sum_{n=1}^{\infty} ||x_n|| \cdot ||x_n^* - u_n|| < 1
\]

there exists a basis \((y_n)\) in \(X\) such that \(y_n^* = u_n\), \(n = 1, 2, \ldots\).

In the proof of Theorem 1 we shall need the following result that we have proved in [8]:

a) Let \(X\) be an infinite dimensional Banach space with \(X^{**}\) separable. If \(Z\) is a closed subspace of \(X^{**}\) that contains \(X\), there is an infinite dimensional closed subspace \(Y\) of \(X\) such that \(X + \overline{Y} = Z\).

Theorem 1. Let \(X\) be a Banach space with a shrinking basis. Let \(Z\) be a closed subspace of \(X^{**}\) that contains \(X\). If \(X^{**}\) is separable, there exists a shrinking basis \((x_n)\) in \(X\) with two complementary subsequence \((x_{m_j})\) and \((x_{n_j})\) such that \([x_{m_j}]\) is reflexive and \(X + [x_{n_j}] = Z\).

Proof. We write \(X^\perp\) and \(Z^\perp\) for the subspace of \(X^{***}\) which are orthogonal to \(X\) and \(Z\) respectively. Let \((u_{2m-1})_{m=1}^{\infty}\) be a sequence in \(X^\perp\) such that its linear hull is weak-star dense in \(X^\perp\) and \(||u_{2m-1}|| \leq 1\), \(m = 1, 2, \ldots\). Let \((u_{2m})_{m=1}^{\infty}\) be a sequence in \(Z^\perp\) so that its
linear hull is a weak-star dense subspace of $Z^\perp$ and $\| u_{2, m} \| \leq 1$, $m = 1, 2, \ldots$. If we apply result a) we obtain an infinite dimensional closed subspace $Y$ of $X$ such that $X + Y = Z$. Let $G$ be the subspace of $X^*$ which is orthogonal to $Y$. The closure of $G$ in $X^{**}$ contains $Z^\perp$ and so, for every positive integer $m$, a sequence $(u_{mn})_{n=1}^{\infty}$ in $X^*$ can be determined so that it converges to $u_{m}$ in $X^{***}$ and $u_{(2m)n}$ belongs to $G$, $n = 1, 2, \ldots$. Selecting $(u_{m})_{m=1}^{\infty}$ we can suppose that $(u_{mn})_{n=1}^{\infty}$ is normalized, $m = 1, 2, \ldots$. Let $(y_n)$ be a shrinking basis in $X$ and $K$ the basis constant of $(y_n)$. We would have

$$\lim_{n} u_{mn} (y_j) = 0, \quad m, j = 1, 2, \ldots.$$  

We order the pairs $mn$ in the sequence:

$$(2) \quad 11, 12, 21, \ldots, 1n, 2(n-1), \ldots,$$

that means $pq$ is less than $mn$ if and only if either $p + q < m + n$ or $p + q = m + n$ and $p < m$. Suppose we take $\epsilon_{mn} > 0$ by asking

$$72 K (K + 1) \sum_{m, n=1}^{\infty} \epsilon_{mn} < 1.$$  

We put $m_{11} = 1$ and $n_{11} = 0$. We find a positive integer $n_{12}$ such that

$$\| u_{1, m_{11}} - z_{11} \| < \epsilon_{11}$$

where

$$z_{11} = \sum_{j=n_{11}+1}^{n_{12}} u_{1, m_{11}} (y_j) y_j^*.$$  

We proceed by recurrence and suppose for a given index $pq$ such that $rs$ is the next index in (2), we would have obtained

$$z_{pq} = \sum_{j=n_{pq}+1}^{n_{rs}} u_{pm_{pq}} (y_j) y_j^*.$$  

We could determine a positive integer $m_{rs}$ such that

$$\| \sum_{j=1}^{n_{rs}} u_{pm_{rs}} (y_j) y_j^* \| < \frac{1}{2} \epsilon_{rs}.$$
If \( h_k \) is the index coming after \( r_s \) in (2), a positive integer \( n_{h_k} \) could be found so that

\[
\| \sum_{j=n_{h_k}+1}^{\infty} u_{r_{m,s}} \left( y_j \right) y_j^* \| \leq \frac{1}{2} \varepsilon_{r_s}.
\]

Therefore

\[
\| u_{r_{m,s}} - z_{r,s} \| \leq \varepsilon_{r_s}
\]

where

\[
z_{r,s} = \sum_{j=n_{r,s}+1}^{n_{h_k}} u_{r_{m,s}} \left( y_j \right) y_j^*.
\]

The sequence

\[(3)\]

\[z_{11}, z_{12}, z_{21}, \ldots, z_{1n}, z_{2(n-1)2}, z_{n1}, \ldots\]

is a block basic sequence of \((y_n^*)\) and so, applying a result of Zippin [9] (see also [7, pp. 67-68]), there is a basis \((v_n)\) in \(X^*\) extending (3) and such that \([v_n^*] = X\) with basis constant no greater than \(18K(K+1)\). Hence

\[
\| v_n^* \| \cdot \| v_n \| \leq 36K(K+1)
\]

and we would have

\[
\| v_n^* \| \leq 72K(K+1)
\]

for the values of \(n\) corresponding with some \(z_{pq}\) because

\[
\| z_{pq} \| \geq \| u_{p_{m,w}} \| - \| u_{p_{m,w}} - z_{pq} \| \geq 1 - \varepsilon_{pq} > \frac{1}{2}.
\]

Suppose we define a sequence \((w_n)\) in \(X^*\) by letting \(w_n\) be \(v_n\) when this element does not belong to the sequence (3), and \(w_n\) be \(u_{p_{m,w}}\) when \(v_n = z_{pq}\). Then would have

\[
\sum_{n=1}^{\infty} \| v_n^* \| \cdot \| v_n - w_n \| \leq 72K(K+1) \sum_{n=1}^{\infty} \| v_n - w_n \| =
\]

\[
= 72K(K+1) \sum_{p,q=1}^{\infty} \| u_{p_{m,w}} - z_{pq} \| \leq 72K(K+1) \sum_{p,q=1}^{\infty} \varepsilon_{pq} < 1
\]
and we could apply Proposition 3 to obtain a basis \((x_n)\) in \(X\) such that \(x_n = w_n\), \(n = 1, 2, \ldots\).

Let us consider now the increasing sequence \((m_j)\) of all positive integers such that \(x_{m_j}\) is an element of the form \(u_{(2p-1)m_{(2p-1)},p}\). Let \(x_{n_j}\) be the subsequence of \((x_n)\) complementary to \((x_{m_j})\). Every element of the form \(u_{(2p)m_{(2p)},pq}\) belongs to \((x_{n_j}^*)\) and so the closure of \([x_{n_j}^*]\) in \(X_{\sigma}^{**}\) contains \(X^\perp\) from where it follows that \([x_{m_j}^*]\) is reflexive. The closure of \([x_{m_j}^*]\) in \(X_{\sigma}^{***}\) contains \(Z^\perp\) from where it follows that \([x_{n_j}^*]\) is contained in \(Z\). On the other hand, \(x_{m_j}^*\) belongs to \(G, j = 1, 2, \ldots\), and so \([x_{n_j}^*]\) contains \(Y\). Finally we have \(X + [x_{n_j}] = Z\).

Q.E.D.

Corollary 1. Let \(X\) be a Banach space with shrinking basis. Let \(p\) be a non negative integer less or equal than the dimension of \(X^{**}/X\). If \(X^{**}\) is separable there is a shrinking basis \((x_n)\) of \(X\) and two complementary subsequences \((x_{m_j})\) and \((x_{n_j})\) of \((x_n)\) such that \([x_{m_j}^*]\) is reflexive and \([x_{n_j}^*]\) is quasi-reflexive of order \(p\).

Proof. It is enough to take \(Z\) in the former theorem such that \(X\) has codimension \(p\) in \(Z\). Q.E.D.

Corollary 2. Let \(X\) be a Banach space with shrinking basis. Let \(p\) be a non negative integer less or equal than the dimension of \(X^{**}/X\). If \(X^{**}\) is separable there is a shrinking basis \((x_n)\) of \(X\) and two complementary subsequences \((x_{m_j})\) and \((x_{n_j})\) of \((x_n)\) such that \([x_{m_j}^*]\) is reflexive and \(X/[x_{n_j}^*]\) is quasi-reflexive of order \(p\).

Proof. It is enough to take \(Z\) in the former theorem so that the codimension of \(Z\) in \(X^{**}\) be equal to \(p\). Q.E.D.

Proposition 4. Let \((x_n)\) be a normalized sequence in a Banach space \(X\). If \(X^*\) is separable the following two conditions are equivalent:

1) The set \(\{x_n : n = 1, 2, \ldots\}\) is not weakly relatively compact.

2) There is a basic subsequence \((z_n)\) of \((x_n)\) so that every subsequence of \((z_n)\) is \(1\)-shrinking.

Proof. Firstly we suppose that condition 1) holds. Since \(X^*\) is separable there is a point \(x_0\) in \(X^{**} \sim X\) and a subsequence \((y_n)\) of \((x_n)\) weak-star converging to \(x_0\). Indeed, the sequence \((y_n)\) could be also found being a basic sequence [3, pp. 41-42]. Let \(H\) be the closed hyperplane of \(X^*\) defined by

\[H = \{u \in X^* : (x_0, u) = 0\},\]
and let us take a sequence \((u_n)\) in \(H\) with \([u_n] = H\). We obviously have

\[
\lim_{n} \langle y_n, u_m \rangle = 0, \quad m = 1, 2, \ldots.
\]

Let \(n_1\) be a positive integer such that

\[
|\langle y_n, u_1 \rangle| < \frac{1}{2}.
\]

Proceeding by recurrence, let us suppose we have found the positive integer \(n_p\). We could choose another integer \(n_{p+1} > n_p\) so that

\[
|\langle y_{n_p}, u_m \rangle| < \frac{1}{2^{p+1}}, \quad m = 1, 2, \ldots, p + 1.
\]

We define the sequence \((z_m)\) by letting \(z_m = y_{n_m}, m = 1, 2, \ldots\). The sequence \((z_m)\) is the basic subsequence of \((x_n)\) that we are looking for condition 2) to be held. Indeed, let \((z_m)\) be any subsequences of \((z_m)\) and let \(L\) be equal to the orthogonal subspace to \([z_m]\) in \(X^*\). We shall denote by \(\Psi\) the canonical mapping from \(X^*\) onto \(X^*/L\). It is obvious that \(L\) is contained in \(H\), so \(\Psi(H)\) is a closed hyperplane of \(\Psi(X^*) = [z_m]^*\). It is easily shown that \([z_m]^*\) is contained in \(\Psi(H)\). On the other hand,

\[
\sum_{j=1}^{\infty} |\langle z_m, u_p \rangle| < \infty, \quad p = 1, 2, \ldots.
\]

Therefore the series

\[
\sum_{j=1}^{\infty} \langle z_m, \Psi(u_p) \rangle z_m^*
\]

converges in \([z_m]^*\) to \(\Psi(u_p), p = 1, 2, \ldots\). Consequently, \([z_m]^*\) coincides with the hyperplane \(\Psi(H)\) and \((z_m)\) is 1-shrinking.

Conversely, let us now suppose that condition 2) holds. Since \(X^*\) is separable there is a subsequence \((t_n)\) of \((z_m)\) that converges to a point \(t_0\) of \(X^{**}\) for the weak-star topology. We always have \(t_0 \neq 0\) because in case \(t_0 = 0\) [5, Proposition 2.3] could be applied to obtain a shrinking subsequence of \((t_n)\), so a contradiction with condition 2). Let us write

\[
M = \{ u \in [t_n]^*: \langle t_0, u \rangle = 0 \}.
\]
Obviously
\[ \lim_{n} \langle t_n, t_m^* \rangle = 0 = \langle t_0, t_m^* \rangle, \quad m = 1, 2, \ldots \]
and \( \{ t_n^* \} \subset M \), from where it follows that \( M \) is weak-star dense in \( \{ t_n \}^* \), so \( t_0 \) can not belong to \( X \) and 1) is verified. Q.E.D.

For the proof of Proposition 5 we need the following result [8]:

b) Let \( B \) be the closed unit ball of a Banach space \( X \). Let \( F \) be a subspace of finite codimension and weak-star closed in \( X^{**} \). Then \( (F \cap B)\) = \( F \cap \overline{B} \).

**Proposition 5.** Let \( X \) be a Banach space with \( X^* \) separable. Let \( (u_n) \) be a sequence in \( X^* \) that converges to \( u \) for the weak-star topology. If

\[ \bigcap_{n=1}^{\infty} [u_n, u_{n+1}, \ldots] = \text{lin}\{u\} \]

there exists a subsequence \( (u_{n_k}) \) of \( (u_n) \) such that

\[ \overline{[u_{n_k}]} = [u_{n_k}] + \text{lin}\{u\}. \]

**Proof.** Let \( G \) be the subspace of \( X^{**} \) which is orthogonal to \( \{ u_n \} \cup \{ u \} \). Since \( X^* \) is separable, we could take a sequence \( (v_n) \) in \( G \) with \( \| v_n \| \leq 1, \ n = 1, 2, \ldots \), so that its linear hull would be weak-star dense in \( G \). Let \( B \) be the closed unit ball of \( X \). For every positive integer \( n \), we denote by

(3) \[ \{ V_{mn} : m = 1, 2, \ldots \} \]

a fundamental system of neighbourhoods of \( v_n \) in \( B \) for the weak-star topology. We order the neighbourhoods of (4) in a sequence

\[ \{ V_m; m = 1, 3, \ldots \} . \]

We write \( n_1 = 1 \) and we suppose that for a positive integer \( p \) we have obtained the positive integers \( n_1, n_2, \ldots, n_p \). We put

\[ L_p = \text{lin}\{u_{n_1}, u_{n_2}, \ldots, u_{n_p}, u\} . \]
Let $H_p$ be the subspace orthogonal to $L_p$ in $X^{**}$. We write $A_{p_n}$ to denote the subspace of $X$ which is orthogonal to $L_p$ in $X^{**}$. We write $A_{p_n}$ to denote the subspace of $X$ which is orthogonal to

$$L_p \cup [u_n, u_{n+1}, \ldots]^{\sim}.$$ 

Since

$$\bigcap_{n=1}^{\infty} \left( L_p \cup [u_n, u_{n+1}, \ldots]^{\sim} \right) = L_p \cup \text{lin}(u) = L_p$$

it follows that $\bigcup_{n=1}^{\infty} A_{p_n}$ is a dense subset of $H_p \cap X$. We claim that there is a positive integer $n_{p+1} > n_p$ so that

$$A_{p_{n_{p+1}}} \bigcap V_p \neq \emptyset.$$ 

Indeed,

$$v_n \in H_p, \; n = 1, 2, \ldots,$$

and result b) assures us that $(B \cap H_p)^{\sim} = \overline{B} \cap H_p$, from where it follows that

$$V_p \bigcap B \bigcap H_p \neq \emptyset$$

and so

$$\left( \bigcup_{n=1}^{\infty} A_{p_n} \right) \bigcap V_p \neq \emptyset,$$

therefore, the positive integer $n_{p+1} > n_p$ such that $A_{p_{n_{p+1}}} \bigcap V_p \neq \emptyset$ can be found. Let $x_p$ be a point in this non-void subset. Obviously,

$$\langle x_p, u_{n_j} \rangle = 0, \; j = 1, 2, \ldots, p, \; \langle x_p, u_n \rangle = 0, \; n = n_{p+1}, n_{p+1} + 1, \ldots$$

We are going to see now how the sequence $(u_{n_j})_{j=1}^{\infty}$ is the subsequence we are looking for. Let $M$ be the subspace of $X$ which is orthogonal to $[u_{n_j}]$. Since

$$x_j \in M, \; j = 1, 2, \ldots$$

and $v_n$ is a weak-star cluster point in $X^{**}$ of the sequence $(x_j)$, $n = 1, 2, \ldots$, it follows that $G \subset \overline{M}$, therefore

$$[u_{n_j}] \subset [u_{n_j}] + \text{lin}(u),$$

from where the conclusion follows.

Q.E.D.
Theorem 2. Let \((x_n)\) be a normalized sequence in a Banach space \(X\). If \(X^{**}\) is separable, the following conditions are equivalent:

1) The set \(\{x_n : n = 1, 2, \ldots\}\) is not weakly relatively compact.

2) There is a subsequence \((z_n)\) of \((x_n)\) such that if \((y_n)\) is any subsequence of \((z_n)\), then \([y_n]\) is a quasi-reflexive Banach space of order one.

Proof. Let us suppose firstly that 1) holds. We could apply Proposition 4 to find a basic subsequence \((t_n)\) of \((x_n)\) converging to a point \(t_0\) in \(X^{**} \sim X\) in the weak-star topology and such that every subsequence of \((t_n)\) is 1-shrinking; \([t_n^*]\) is an hyperplane of \([t_n]\) orthogonal to \(t_0\) and consequently

\[
\bigcap_{n=1}^{\infty} [t_n, t_{n+1}, \ldots] = \text{lin}\{t_0\}.
\]

Applying now the former proposition we obtain a subsequence \((z_n)\) of \((t_n)\) so that

\[
[z_n] = [z_n] + \text{lin}\{t_0\}.
\]

It results obvious that if \((y_n)\) is any subsequence of \((z_n)\), \([y_n]\) is a quasi-reflexive Banach space of order one.

Conversely, when suppose that 2) is true, 1) also follows bearing in mind Proposition 4.
Q.E.D.

Proposition 6. Let \((x_n)\) be a basis in a Banach space \(X\). Let \(F\) be a subspace of \(X^*\) that contains \([x_n^*]\). If \([x_n^*]\) has finite codimension in \(F\) there is a basis \((y_n)\) in \(X\) such that \([y_n^*]\) = \(F^*\).

Proof. It is clear the only case need to show is when \([x_n^*]\) is an hyperplane of \(F\). If we take a vector \(u\) in \(F^* \sim [x_n^*]\) with \(||u|| = 1\), we can find an increasing sequence \((n_p)_{p=1}^{\infty}\) of positive integers such that if \(n_0 = 0\) and

\[
z_p = \sum_{n=n_{p-1}+1}^{n_p} u(x_n) x_n^*
\]

it follows that

\[
\inf \left\{ ||z_p|| : p = 1, 2, \ldots \right\} > 0.
\]

Applying now a method due to Zippin [9] (see also [7, pp. 67-68]), it is possible to obtain a basis \((v_n)\) in \([x_n^*]\) such that \([v_n^*]\) coincides with the restriction of \(X\) on \([x_n^*]\) and

\[
v_{n_p} = z_p, \quad p = 1, 2, \ldots
\]
We define a sequence \((w_n)\) in the following way:

\[ w_n = v_n \quad \text{if} \quad n \neq n_p, \quad w_n = v_1 + v_2 + \ldots + v_p, \quad p = 1, 2, \ldots. \]

The sequence \((w_n)\) is a basis in \([x_n^*]\) such that

\[ w_n^* = v_n^* \quad \text{if} \quad n \neq n_p, \quad w_n^* = v_p^* - v_{p+1}^*, \quad p = 1, 2, \ldots. \]

Let \(H\) be the hyperplane of \(X\) orthogonal to \(\{u\}\). Let \(y_1\) be a vector in \(X\) such that \(\langle y_1, u \rangle = 1\). The restriction of \(H\) on \([x_n^*]\) obviously coincides with \([w_n^*]\) and therefore if \(y_{n+1}^*\) is the vector in \(X^*\) with restriction on \([x_n^*]\) equal to \(w_n^*\), \(n = 1, 2, \ldots\), we have the basis \((y_n)\) in \(X\) and

\[ y_1^* = u, \quad y_n^* = w_{n+1}^* - (y_1, w_{n-1})u, \quad n = 1, 2, \ldots \]

from where the conclusion follows. \(\text{Q.E.D.}\)

**Proposition 7.** Let \(X\) be a Banach space with basis. Let \(k\) be a non negative integer. The following conditions are equivalent:

1) \(X\) is quasi-reflexive of order \(k\).

2) \(X\) has a \(k\)-boundedly complete basis and every basis of \(X\) is \(j\)-boundedly complete with \(0 \leq j \leq k\).

**Proof.** The implication 1) \(\Rightarrow\) 2) is obvious. Let us now suppose that 2) holds. Let \((x_n)\) be a \(k\)-boundedly complete basis in \(X\). Then \(X + \lim\{x_n\}^\bot\) has codimension \(k\) in \(X^{**}\), where \(\lim\{x_n\}^\bot\) denotes the subspace of \(X^{**}\) which is orthogonal to \([x_n^*]\). Let us suppose that \(X\) is not quasi-reflexive of order \(k\). It follows that there is a vector \(v\) non equal to zero in \([x_n^*]\). Let \(u\) be a vector in \(X^*\) such that \(\langle v, u \rangle \neq 0\). If \(F\) is the linear hull of \([x_n^*] \cup \{u\}\) we can apply the former proposition to \(F\) and we obtain a basis \((y_n)\) of \(X\) such that \([y_n^*]\) = \(F\). Obviously, \((y_n)\) is \((k + 1)\)-boundedly complete and this contradiction finishes the proof. Q.E.D.

**Proposition 8.** Let \((z_n)\) be a basis in a Banach space \(Z\). Let \(G\) be a be a closed subspace of finite codimension in \([z_n^*]\). If \(G\) is weak-star dense in \(Z^*\) there exists a basis \((t_n)\) in \(Z\) such that \([t_n^*]\) = \(G\).

**Proof.** It is clear that the only case we need to show is when \(G\) is an hyperplane of \([z_n^*]\). Let \(u\) be a vector in \(Z^{**}\) with \(\|u\| = 1\) which is zero on \(G\). Let \(S\) be the canonical mapping from \(Z^{**}\) onto \(Z^{**}/[z_n^*]\), where \([z_n^*]\) is the orthogonal to \([z_n^*]\) in \(Z^{**}\). Then \(S(Z) = [z_n^{**}].\) If \(F\) is the linear hull of \([z_n^*] \cup Su\), we have the hyperplane \([z_n^{**}]\) in \(F\). Proceeding as in
the proof of Proposition 6 we could obtain a basis \((y_n)\) in the hyperplane of \([z_n^*]\) orthogonal to \(\{u\}\) in such a way that, if \((u_n)\) are the elements of \([z_n^{**}]\) verifying
\[
(u_n, y_n) = 1, \quad (u_m, y_n) = 0, \quad m \neq n, \quad m, n = 1, 2, \ldots
\]
we would have \([u_n] = [z_n^{**}]\). If \(t_n\) is the vector of \(X\) such that \(St_n = y_n\), \(n = 1, 2, \ldots, (t_n)\) is a basis in \(Z\) with \([t_n^*] = G\).

Q.E.D.

**Proposition 9.** Let \(X\) be a Banach space with a basis. Let \(k\) be a non negative integer. The following conditions are equivalent:

1) \(X\) is quasi-reflexive of order \(k\).

2) \(X\) has a \(k\)-shrinking basis and every basis of \(X\) is \(j\)-shrinking, \(0 \leq j \leq k\).

**Proof.** 1) \(\Rightarrow\) 2) is obvious. Let us suppose now that 2) holds. Let \((x_n)\) be a \(k\)-shrinking basis of \(X\). If \([x_n^*] \perp\) is the subspace of \(X^{**}\) which is orthogonal to \([x_n^*]\), the dimension of \([x_n^*] \perp\) is precisely \(k\). Let us suppose that \(X\) is not quasi-reflexive of order \(k\). We could find a vector \(u\) in \(X^{**} \sim (X + [x_n^*] \perp)\). If \(F\) is the subspace of \([x_n^*]\) which is orthogonal to \(\{u\}\), \(F\) is weakly-star dense in \(X^*\) and so, applying the former proposition, we could obtain a basis \((y_n)\) of \(X\) such that \([y_n^*] = F\). Consequently, \((y_n)\) would be a \((k + 1)\)-shrinking basis. This contradiction finishes the proof.

Q.E.D.

**Theorem 3.** Let \(X\) be a Banach space with basis. If \((y_n)\) is a basic sequence in \(X\) there exists a basis in \(X\) which extends some subsequence of \((y_n)\).

**Proof.** We can suppose without any restriction that \(\|y_n\| = 1, \ n = 1, 2, \ldots\). Let \((x_n)\) be a basis in \(X\). If
\[
\lim_{n} \langle y_n, x_m^* \rangle = 0, \quad m = 1, 2, \ldots,
\]
it would be enough to apply [2, Theorem 3]. Let us now suppose that (5) does not hold. Then there is a weak-star cluster point \(y\) in \(X^{**} \sim X\) of the sequence \((y_n)\). We put
\[
H = \{ u \in X^* : \langle y, u \rangle = 0 \}.
\]

If \(H \cap [x_n^*]\) is weak-star dense in \(X^*\) we could apply Proposition 8 and obtain a basis \((z_n)\) in \(X\) such that \([z_n^*] = H \cap [x_n^*]\). In that case, we can take a subsequence \((y_{n_j})\) of \((y_n)\) such that
\[
\lim_{j} \langle y_{n_j}, z_m^* \rangle = 0, \quad m = 1, 2, \ldots,
\]
and the proof is reduced to the former case. If \( H \cap [x^*_n] \) is not weak-star dense in \( X^* \) there is an element \( x_0 \) in \( X \) such that

\[
\langle x_0, v \rangle = 0, \quad v \in \bigcap [x^*_n].
\]

There exists an element \( u \) in \( H \) such that \( \langle x_0, u \rangle \neq 0 \). Let \( F \) be the linear hull of \([x^*_n] \cup \{u\}\). We could apply Proposition 6 and obtain a basis \((z_n)\) of \( X \) such that \([z^*_n] = F\). We would have now that \( H \cap F \) is weak-star dense in \( X^* \) and so, according to Proposition 8, there is a basis \((w_n)\) in \( X \) such that \([w^*_n] = H \cap [z^*_n]\). If \((y_n)\) is now a subsequence of \((y_n)\) such that

\[
\lim_j \langle y_{n_j}, w_m^* \rangle = 0, \quad m = 1, 2, \ldots
\]

the proof is also reduced to the first case in that situation. Q.E.D.

Note. The original proof of Proposition 2 was longer than the one presented here. We are grateful to Dr. V. Montesinos for providing the proof given here.
REFERENCES


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