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NONCLOSED SEQUENTIALLY CLOSED SUBSETS OF LOCALLY CONVEX SPACES AND APPLICATIONS

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Dedicated to the memory of Professor Gottfried Köthe

This article is a review of some methods of constructing nonclosed sequentially closed subsets in locally convex spaces (l.c.s.) as well as some applications of such subsets to problems in the theory of l.c.s. These subsets are collections of elements having two or more indexes being natural numbers, as well as convex or linear envelopes of such countable sets. As to the above-mentioned problems we regard several ones connected with the Ptak and Krein-Smulian spaces (we recall the definitions of these spaces below), problems connected with the tehory of differentiable functions on l.c.s. and some problems posed by Dieudonné and L. Schwartz and solved by Grothendieck (in the latter case we give solutions which differ from the solutions of Grothendieck). In some cases we prefer not to give the most general constructions replacing them by typical examples.

1. SOME HISTORICAL REMARKS

In 1953 V. Ptak introduced the notion of a B-completeness (independently this notion under the name «fully completeness» was introduced by H.S. Collins) and proved the open mapping theorem for linear mappings from B-complete spaces. In 1958 he introduced B_r-complete spaces and proved the closed graph theorem for linear mapping into such spaces (the latter theorem was proved by Robertsons for B-complete spaces two years earlier). Recall some definitions. Let E be a l.c.s. A subset A of E' is called nearly closed iff whenever V is neighbourhood of zero in E the set $A \cap V^0$ is closed in the space E equipped with the weak topology. The space E is called B-complete (B_r-complete, hypercomplete, Krein-Smulian) space iff every nearly closed vector subspace (dense vector subspace, absolutely convex subset, convex subset) of the space E' equipped with the weak topology is closed. Just after the first result of V. Ptak specialists supposed that almost each «ordinary» l.c.s. of functional analysis is B-complete. Only in the seventies it was proved that the situation was completely different (see [5] -[12] and the bibliography in [4]). Namely, all «ordinary» l.c.s., which are not metrizable or dual-metrizable, do not belong to the class of B-complete spaces. In particular, even the space D has a noncomplete metrizable quotient space [6]. By the way the first example of a complete l.c.s. having a noncomplete quotient space is due to G. Köthe [1]. There were some other problems concerning B-complete, B_r-complete, hypercomplete and Krein-Smulian spaces. For instance, it was unknown whether a product of two spaces belonging to one of these classes belongs to the same class, whether these classes coincide, whether some concrete l.c.s. belong to one of these classes. In the seventies almost all these problems were solved (now it is unknown only whether the classes of B-complete spaces and hypercomplete spaces coincide); moreover, most of them were solved ([5]-[10]) by the methods we describe in this paper. This is explained by the fact that in many important cases the classes of nearly closed subsets and sequentially closed subsets coincide. It is necessary to mention one exception: the strong result due to M. Valdivia [14], giving a description of B_r-complete non-B-complete l.c.s. (the results in [11], [12] can be obtained by our method - see below). It is interesting that the same method can be used for constructing real functions on spaces D and D' which are infinitely Fréchet differentiable but not continuous. It is also worth mentioning that an analogous construction can be used to prove that the so-called Pontryagin's duality does not hold for the pair (D, D'), see [9]. Finally, it is curious to note that the recently nearly the same method has been applied by E.T. Shavgulidze for constructing some selfadjoint operators connected with some models in quantum field theory [15]. In this paper we describe only the main ideas and sketch the proofs, leaving the reconstruction of the details to the reader.

2. NOTATION

Everywhere below E, G are l.c.s., $T = E + G$ (topological direct sum), $A = (a_{k,n} : k, n \in N) \subset E, B = (b_{n,k} : k, n \in N) \subset G, C = (a_{k,n} + b_{n,k} : k, n \in N) \subset T$, and for every $m \in N$.

$$C_m = (a_{k,n} + b_{n,k} : k \in N; n = 1, 2, \dots, m);$$

$$A_m = (a_{k,n} : k \in N; n = 1, 2, \dots, m).$$

If V is a subset of a topological space F then $clos(V)$ denotes the closure of $V. clos(V)_S$ denotes the sequential closure of V (i.e. the intersection of all sequentially closed subsets of F containing V) and $clos(V)_{SS}$ denotes the collection of limits of sequences $(x_n) \subset V$. If X is a subset of l.c.s. Z then $M(X)$ and $L(X)$ denote the linear manifold (= the plane) and the linear subspace generated by X; correspondingly, $M_n = M(C_n)$, $M = \cup clos(M_n)$.

3. SEQUENTIALLY CLOSED SETS

Theorem 1. *Let the following conditions be fulfilled:*

a) $\forall n \in N b_{n,k} \to 0 (k \to \infty)$*:*

b) if $b_{n(l),k(l)} \to 0 (l \to \infty)$*, then* $\sup n(l) \neq \infty$*.*

Then there exists a countable set V *such that* $clos(V) \neq clos(V)_{SS}$*.*

Proof. Let $a \in G, a \neq 0$ be such that $0 \notin (n^{-1} a + b_{n,k} : n, k \in N)$; then $0 \in clos(V), 0 \notin clos(V)_{SS}$.

Example 1. $G = D[-1,1], b_{n,k} = k^{-1}\delta^{(n)}$.

Example 2. $G = (l_2, \sigma(l_2, l_2)), (e_k) \subset G; \forall i,j \in N, (e_i, e_j) = \delta_{i,j}, b_{nk} = ne_k (n,k \in N)$ *(this example is similar to an example of von Neumann).*

Theorem 2. *Let the following conditions be fulfilled: (0)* $0 \notin A \cap B$ *: (1)* $a_{k,n} \to 0 (n \to \infty)$ *uniformly with respect to* $k (\in N)$ *; (2) Whenever* $n \in N$ *there does not exist a converging subsequence of the sequence* $(a_{kn} : k \in N)$ *; (3) Whenever* $n \in N b_{nk} \to 0 (k \to \infty)$ *; (4) If a sequence* $(b_{n(l)k(l)} : l \in N)$ *converges then* $\sup n(l) \neq \infty$*. Then* $C = clos(C)_S \neq clos(C) (0 \in clos(C), 0 \notin clos(C)_s)$.

Example 3. *E and G are the following topological vector subspaces of* $D'(R^1)$: $E = (f \in D'(R^1); \sup(f) \subset N); G = (f \in D'(R^1); \sup(f) \subset (0)); a_{kn} = \delta^{(k)}(x-n); b_{nk} = k^{-1}\delta^{(n)}$ *. Then the conditions of Theorem 2 are fulfilled.*

Example 3a. *Let G and* b_{nk} *be as in the preceding example,* $E = D[0,1]$ *and* $a_{kn} (\in E)$ *are defined as follows:* $a_{kn}(t) = g(t)(k+1)^{-n} \sin kt, g \neq 0$ *. Then the conditions of Theorem 2 are fulfilled.*

Example 3b. *Let E and* a_{kn} *be as in the preceding example.* $G = D[1,\infty)$ *and* $b_{nk} = k^{-1} p(t-n)$ *where* $p \in G, p \neq 0$ *. Then the conditions of Theorem 2 are fulfilled.*

Theorem 3. *Let the conditions 1), 3) (Theorem 2) be fulfilled and let also the following conditions be true: 2a) For all* $m \in N 0 \notin clos(M(A_m)_{ss}$*. 4a) If the sequence of sums* $\sum_{k,n} l^j_{kn} b_{nk} (j \in N)$ *converges to 0 then there exists* $m \in N$ *such that* $l^j_{kn} = 0$ *for* $n \geq m$*. Then* $0 \in clos(\cup M_n)$. $0 \notin clos(\cup M_n)_{ss}$.

Theorem 4. *Let the conditions 1) and 3) (Theorem 2) be fulfilled and let also the following conditions be true: 2)* $\forall m 0 \notin clos(M(A_n))$ *; 4b) if a sequence* $(m_j) \subset M$ *converges then* $\exists n \in N$ *such that* $(m_j) \subset clos(M_n)$ *. Then* $M = clos(M)_s \neq clos(M) (0 \in clos(M), 0 \notin clos(M)_s)$.

Example 4. *Let E and A be the same as in example 3.* $G = D'[-1/2, 1/2], b_{nk} = (k+1)^{-1}\delta^{(n)}(x - k^{-1})$ *then the conditions of Theorem 4 are fulfilled. To verify the hypotheses of Theorem 4 the following lemmas are useful.*

Lemma 1. *Let there exist a collection* $K = (K_n : n = 1,2,\ldots)$ *of vector subspaces of G such that I)* $\bigcup_{n=1}^{\infty} K_n = G$ *; II) if a generalized sequence* $\left(\sum_{k,n} l^s_{kn} (a_{kn} + b_{nk}) : s \in S \right)$

converges to a point $a \in E + K_m$, *then* $l^s_{kn} \to 0$ *for* $n \geq m+1$ *and* $\sum_{k,n} l'^s_{kn} b_{nk} \to 0$, *where* $l'^s_{kn} = l^s_{kn}$ *for* $n \geq m+1$ *and* $l'^s_{kn} = 0$ *for* $n \leq m$. *III) if a generalized sequence* $(\sum r^s_{kn} a_{kn})$ *converges to a point* $d(\in E)$, *then* $\forall m$ *the generalized sequence* $(\sum r'^s_{kn} a_{kn})$ *(where* $r'^s_{kn} = r^s_{kn}$ *for* $n \leq m$ *and* $r'^s_{kn} = 0$ *for* $n \geq m+1$*) converges in* E; *if also* $r^s_{kn} \to 0 \forall k, n$ *then* $d = 0$. *Then the space* $L(M)$ *is closed in* T.

Lemma 2. *Let the conditions of Lemma 1 and the condition 2b) of Theorem 4 be fulfilled and whenever we have a converging sequence* $(b_n) \subset G$ *there exist such* $m \in N$ *that* $(b_n) \subset K_m$. *Then the condition 4) of Theorem 4 is fulfilled.*

Theorem 5. *Let the conditions of Theorem 4 and Lemma 2 be fulfilled. Then the vector space* $L(M)$ *is closed in* T *and the linear functional* $f : L(M) \to R^1$ *such that* $f(M) = 1$ *is sequentially continuous but is not continuous. The conditions of Theorem 5 are fulfilled in examples 4-6.*

Example 5. *Let* G *and* B *be the same as in example 4,* $E = D[0,1]$, $a_{kn} \in E$ *are defined as follows:* $a_{kn}(t) = (k+1)^{-1}(\sin l_{kn} t) f(t)$ *where* $f \in D[0,1], f \neq 0$ *and* $(l_{kn} : k, n \in N)$ *is a collection of natural numbers such that* $l_{k_1 n_1} = l_{k_2 n_2}$ *iff* $k_1 = k_2$ *and* $n_1 = n_2$. *Then* $M = clos(M)_s \neq clos(M) (0 \in clos(M), 0 \notin clos(M)_s)$.

Example 6. *Let* E *and* a_{kn} *be as in the preceding example,* $G = D[1,\infty)$ *and* $b_{nk} \in G$ *be defined as follows:* $b_{nk}(t) = 2^{-(2^k)} g(2^k(t-n))$, *where* $g \in D[1,\infty), g \neq 0$ *and* $g(t) = 0$ *for* $t \geq 2$. *We can conclude from the preceding results that there exist nonclosed sequentially closed vector subspaces of the spaces* $D(R^1)$ *and* $D'(R^1)$. *In fact, the space* $E + G$ *from example 3 is a closed vector subspace of* $D'(R^1)$, *the space* $D[0,1] + D[1,\infty)$ *(considered in example 6) is a closed vector subspace of* $D(R^1)$ *and these spaces (*$E + G$ *and* $D[0,1] + D[1,\infty)$*) possess unclosed sequentially closed vector subspaces. Using the method which is similar to the above-described method of constructing nonclosed sequentially closed vector subspaces of* $E + G$, *it is possible to construct such subspaces of* $D(R)$ *and* $D'(R)$, *which are even dense in the corresponding spaces. Now we prefer not to formulate some general theorems and restrict ourselves to examples.*

Example 7. *Let* (r_j) *be the sequence of all rational numbers and* $(p, j, s,) \mapsto n(p, j, s)$ *a one to one mapping from* N^3 *to* $(2, 3, \ldots)$. *Then the linear manifold* F, *generated in* $D'(R^1)$ *by the set* $(r_p \delta(x - r_j) + k^{-1}\delta^{(n(p,j,s))}(x - k^{-1}) + \delta^{(k+1)}(x - n(p, j, s)) : p, k, j, s \in N)$ *is sequentially closed and dense in* $D'(R^1)$, *but is not closed (as* $0 \in clos(F), 0 \notin F$*); consequently* $F - a$ *(where* $a \in F$*) is a dense sequentially closed but nonclosed vector subspace in* $D'(R^1)$.

Example 8. *Let $(k,n) \to l(k,n)$ and $(j,s) \to n(j,s)$ be bijections from N^2 onto N and (f_j) be a sequence of elements of $D(R^1)$. The following conditions have to be fulfilled: (1) If $\min_s n(j,s) = a(j)$ and $\min_{k,s} l(k,n(j,s)) = b(j)$, then $f_j^{(a(j))}(x) = 0$ for $x \in (0, 2(b(j))^{-1}\pi)$; (2) supp $f_j \subset (-\infty, a(j))$; (3) clos $((f_j)) = D(R^1)$ (the existence of such (f_j) is a consequence of the fact that $a(j) \to \infty$ and $b(j) \to \infty$). Let F_D be the sequentially closed linear manifold generated in $D(R^1)$ by the set $(f_j(t) + f(t)(k+1)^{n(j,s)} \sin l(k,n(j,s)t + 2^{-2^k} g(2^k(t-n(j,s)) : k,j,s \in N)$ (here $g \in D(R^1)$, supp$g \subset [1,\infty)$, $f \in D(R^1)$, supp$f \subset (-\infty,1)$, $f(t) = 1 \forall t \in (0,1/2)$). Then $F_D - a(a \in F_D)$ is a dense sequentially closed nonclosed vector subspace of $D(R^1)$. It is possible to use in essence the same method to construct a sequentially closed nonclosed dense vector subspace in the space $D[0,1] + D'[0,1]$ as well as in the space $(l_\infty, \sigma(l_\infty, l_1) \oplus \prod_{j=1}^{\infty} E_j$ where for every $j E_j$ is a copy of the space $(l_2, \sigma(l_2, l_2))$. We shall do it in the following example.*

Example 9. *Let $G = (l_\infty, \sigma(l_\infty, l_1))$; $E = \prod_{j=1}^{\infty} E_j$; $\forall n, ke_n = (\delta_{nk} k \in N) \in l_2, l_\infty \ni g_{nk} : N \to (-1,1), \forall n_i, k_i \in N, \alpha_i \in R^1 (i = 1,\dots,r)$ if $\sum \alpha_i = 1$ then card$(n \in N : \sum \alpha_i g_{n_i k_i}(n) = 1) = \infty$; $a_{kn} = (0,0,\dots,ke_k,0,0,\dots)(\in E)$ the number of the place occupied by an element ke_k being n; $b_{nk} = n(0,\dots,g_{nk}(k+1), g_{nk}(k+2),\dots)(\in G)$ the first k places being occupied by 0. Then the set C is sequentially closed but nonclosed in $T = E + G$ and the vector subspace clos$(M(C))_s - a(a \in M(C))$ has similar properties. It is not difficult to construct a dense vector subspace with the same properties: namely let $(j,s) \mapsto n(j,s)$ be a bijection from N^2 onto N and let (c_j) be a dense subset of T such that the following conditions are fulfilled: $\forall j \in N c_j = c_j^1 + c_j^2, c_j^1 \in G, c_j^1 = (h_1^j,\dots,h_p^j,0,0,\dots)$ (h_n^j are rational numbers; of course p depends on j); $c_j^2 = (d_1^j,\dots,d_n^j,\dots), (d_n^j \in E_n, d_n^j = (w_{n_1}^j, w_{n_2}^j,\dots,w_{nt}^j,0,0,\dots))$ (here $t = t(j,n)$). Let us assume that $K = (c_j + a_{k+t(j,n(j,s)),n(j,s)} + b_{n(j,s)k} : j,k,s \in N)$. Then the plane clos$(M(K))_s$ is dense sequentially close nonclosed; consequently clos$(M(K))_s - a(a \in M(K))$ is a vector subspace having the same properties.*

4. APPLICATIONS

If for every neighbourhood V of zero in a l.c.s. F, its polar V^0 is metrizable then every sequentially closed subset of $(F', \sigma(F',F))$ is nearly closed. So if a l.c.s. F is separable (in this case the polars in F' of neighbourhoods of zero of F are metrizable) and the space

$(F', \sigma(F', F))$ satisfies the conditions of Theorem 4, then the space F is not a B-complete space. Besides it follows from some of the preceding examples that the space $R^\infty \times R^{(\infty)}$ is not a Krein-Smulian space (because the convex envelope of the set G from example 3 is sequentially closed but nonclosed in the space $E + G$ from the example, the latter space being isomorphic to $R^\infty \times R^{(\infty)}$ (though the latter space is hypercomplete [7])), and that the spaces $D(R^1), D'(R^1), D[0,1] \oplus D'[0,1], l_1 \oplus \sum_{j=1}^{\infty} E_j$ (where for every $j \in N$ the space E_j is a copy of l_2) are not B_r-complete. The existence of discontinuous sequentially continuous linear functionals is connected with the problem of hereditary completeness of l.c.s. In fact, if a l.c.s. F is semi-reflexive and if F_1 is a (closed) vector subspace, then the space $(F_1', \beta(F_1', F_1))$ is canonically isomorphic to the quotient space $(F', \beta(F', F))/F_1^0$ ([2]); on the other hand, a theorem of Grothendieck shows that the space $(F_1', \beta(F_1', F_1))$ is noncomplete iff there exists a linear discontinuous functional on F_1 whose restrictions to every bounded subset of F_1 is continuous. Consequently we can conclude from Theorem 5 that if a l.c.s. F is reflexive and separable and its strong dual satisfies the conditions of Theorem 5, then there exists a noncomplete factor-space of F. So the spaces $D(R), D'(R), D[0,1] \times D'[0,1]$ have such factor-spaces. It is probably worth mentioning that the examples of section 2 contain the answers to some questions posed long ago in [2]. We list these answers (the figures in brackets correspond to the numbers of questions in [2]). There exists a sequentially closed non-closed vector subspace in $D \equiv D(R^1)$ (this is a consequence of example 6); (2) There exists a closed vector subspace of D which is not a strict inductive limit of a sequence of Fréchet spaces; in fact, in accordance with Theorem 5, the vector subspace $L(M)$ in example 6 has such a property because the linear functional f defined in this theorem is sequentially continuous but not continuous: (3) D has a non-complete factor space (we mentioned this above): (4) If $F_1 = L(M) \subset D$ where M is constructed using the families a_{kn} and b_{nk} from example 6, then there exists a compact subset in D/F_1 which is not the image of any bounded set in D (for the detailed proof see [9]); (9) The standard topology in D' is coarser than the finest of all topologies in D' which coincide with $\sigma(D, D')$ on every bounded set (indeed, the set G given by example 3 is nonclosed, but sequentially closed). In the paper [3] Grothendieck has used special spaces to solve these problems; for the same purpose we used here the standard spaces D and D'. Using the sets described in examples 3-3b it is possibile to construct real functions on $D(R), D'(R)$, which are infinitely differentiable everywhere but are not everywhere continuous.

Let us recall the definition of Fréchet differentiability. A mapping f form an open set V of a l.c.s. E into a l.c.s. G is said to be (one time) Fréchet differentiable at a point $a \in V$ if there exist a linear continuous map $f'(a) : E \to G$ (called the Fréchet derivative of f at the point a) and an «infinitesimal mapping» $r : E \to G$ with the following properties: (1) $f(a+h) - f(a) = f'(a)h + r(h)$ whenever $h \in E, a+h \in V$, (2) for each bounded subset

B in E the relation $r(th)/t \to 0$ holds uniformly for h in B. The map f is said to be twice Fréchet differentiable at a point $a \in V$ if it is Fréchet differentiable in some neighbourhood W of a and the map $f' : W \to L_b(E,G), x \mapsto f'(x)$, is Fréchet differentiable at a point a. It is easy to construct a (real) function on D' which is infinitely differentiable at a given point, but is not continuous at this point. For example, it suffices to define f as the indicator of a set A possessing the following property: the origin belongs to the closure of A, but not to the sequential closure of A. Then f is infinitely Fréchet differentiable at the origin but not continuous at this point. Nevertheless there exist points whre f has not a Fréchet derivative. The construction of a discontinuous everywhere (infinitely) differentiable function is more complicated (by the way, it is very easy to construct such a function on the space $D \times D'$: the mapping of evaluation has the necessary properties).

Let, for every $k, n \in N, \Phi_{kn_1}$ and Φ_{kn_2} be functions from D possessing the following properties:

(1) $\forall k, n,\ \Phi^{(n)}_{kn_1}(-k^{-1}) = k;$

(2) $supp\Phi_{kn_1} \subset [-2,0];$

(3) $\forall n \in N \forall r \in N\backslash\{n\} \forall k \in N, \Phi^{(r)}_{kn_1}(-k^{-1}) = 0;$

(4) $\forall r \in N \forall n, k \in N$, if $r < n$ then $\sup_t |\Phi^{(r)}_{kn_1}(t)| \leq 1/(n^2 k^2);$

(5) $\forall n, k \in N,\ \Phi^{(k)}_{kn_2}(n) = 1;$

(6) $\forall k, n \in N,\ supp\Phi_{kn_2} \subset [n - 1/2, n + 1/2];$

(7) $\forall k \in N \forall r \in N\backslash\{k\} \forall n \in N, \Phi^{(r)}_{kn_2}(n) = 0;$

(8) $\forall r \in N \forall n, k \in N$, if $r < k$ then $\sup_t |\Phi^{(r)}(t)| < 1/(n^2 k^2)$.

Let also $h \in D, 0 \leq h(t) \leq 1, h(t) = 0$ if $|t| \geq \frac{1}{4}, h(t) = 1$ if $|t| \leq 1/8$. Then the function $F : D' \to R^1, F(g) = \sum_{k,n=1}^{\infty} h((g, \Phi_{kn_1}) - 1) h((g, \Phi_{kn_2}) - 1)$ is everywhere infintely Fréchet differentiable, all its derivatives (of order ≥ 1) being (everywhere) continuous (as mappings from D into the corresponding spaces of linear mappings, equipped with the topology of bounded convergence), but this function is discontinuous at the origin. The detailed proof can be found in [16]; here we remark only that F assigns the value 1 to all elements $\delta^{(n)}(x + 1/k)/k + \delta^{(k)}(x - n)$. So the function F is the result of «smoothing» the above mentioned functions. I do not know whether there exists a function with similar properties on the space D. But an infinitely Fréchet differentiable discontinuous function on the space D exists, but in the examples which I mean, the derivatives are discontinous also.

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