Note di Matematica Vol. XII, 167-202 (1992)

ON AN ABSTRACT FORM OF WEIL'S INTEGRALITY THEOREM

A. MALLIOS

Dedicated to the memory of Professor Gottfried Kothe

0. INTRODUCTION

The purpose of the following discussion is to obtain the classical theorem in the title of this paper as an application of our previous considerations in [29: (i), (ii), (iv), (v)] (an early announcement, under the same title, has been given in [29: (iii)] as well). These, including of course the present study, concern in effect an abstract (axiomatic) approach to the standard *differential geometry of C^∞-manifolds and/or of complex* (analytic) *ones without employing differential calculus at all.* **So** here again one realizes, and essentially in a strengthened way, that «certain [fundamental] quantities which *a priori depend on the local differential geometry are actually global topological invariants* » (see e.g. [8: Introduction]). Indeed, our treatment is quite *topological-algebraic in nature,* to the extent that this is accomplished via *sheaf thwry* and, in particular, through *sheaf cohomology.* Thus, our study might also be viewed as *algebraically* (viz. *operator-theoretically*) *oriented.* Yet, to make the exposition more comprehensible, we do develop, more or less, the necessary framework for the treatment of the theorem in question, material which, otherwise, is fully discussed in [31].

On the other hand, the connection of the classical *Weil's theorem* [45] with the theory of *geometric quantization* is standard (see e.g. [19]). *So* as a consequence of our study, we also exhibit, in brief (in the final section 9), the result of a similar application of our formulation of the latter theorem (see Theorem 7.1 in the sequel), in conjunction with an interpretation of *elementary* (free) *particles* through (sections of) *vector sheaves* ; the latter point of view has been essentially advocated by S.A. Selesnick (cf., for instance, [38]). Finally, we also give in section 8 an outline of particular concrete cases, apart of course from that of the classical differential geometry (real and/or complex), where the present point of view can (in part, see e.g. (8.4) below) be applied. In this respect, it is probably worth noting too that these specific applications come from abstract (commutative) harmonic analysis (cf., for instance, [35], [36] as well as [41], [42]).

1. $\mathcal{A}$-CONNECTIONS. PRELIMINARIES

To start with we first consider a given (fixed) $\mathbb{C}$*-algebraized space*

$$(X, \mathcal{A}). \tag{1.1}$$

So X stands here for an arbitrary *topological space* and $\mathcal{A}$ for a *sheaf of* $\mathbb{C}$*-algebras* over X ; the algebras involved are, in particular, commutative associative linear algebras over the complexes $\mathbb{C}$, having also identity elements (concerning the sheaf theory applied hereafter, we refer for instance to [9], [5] and/or [31]).

Our second basic assumption is that, apart from (1.1), we are also given a triplet

$$(\mathcal{A}, \partial, \Omega^1) \tag{1.2}$$

(deliberately) called a *differential triad*; here Ω^1 stands *for* an $\mathcal{A}$*-module* on X, viz. a sheaf of $\mathcal{A}$-modules (the upper index «1» of Ω will presently be justified below; thus, we are going to consider a suitable finite sequence of relevant $\mathcal{A}$-modules on X - cf. sections 3, 5 in the sequel). Furthermore,

$$\partial : \mathcal{A} \to \Omega^1 \tag{1.3}$$

is a $\mathbb{C}$*-linear morphism* of the corresponding sheaves of $\mathbb{C}$-vector spaces on X, satisfying the following *(Leibniz) condition*

$$\partial(s \cdot t) = s \cdot \partial t + t \cdot \partial s, \tag{1.4}$$

for any s, t in $\mathcal{A}(U)$ (viz. local sections of $\mathcal{A}$ over an arbitrary open set $U \subseteq X$). In other words, ∂ thus defines a *derivation* (in fact, a $\mathbb{C}$*-derivation*) of $\mathcal{A}$ into Ω^1.

We can now establish the following fundamental notion, for all that follows. So we have.

Definition 1.1. *Let $(\mathcal{A}, \partial, \Omega^1)$ be a given differential triad on a topological space X, and $\mathcal{E}$ an $\mathcal{A}$-module on X. Then an $\mathcal{A}$-connection of $\mathcal{E}$ (in fact, we should call it an $(\mathcal{A}, \partial, \Omega^1)$-connection) is a $\mathbb{C}$-linear morphism*

$$D : \mathcal{E} \to \mathcal{E} \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{E}) \tag{1.5}$$

such that (Leibniz condition)

$$D(\alpha \cdot s) = \alpha \cdot Ds + s \otimes \partial\alpha, \tag{1.6}$$

for any $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$, with U open in X

The tensor product appeared in the previous relation (1.6) is meant, of course, with respect to $\mathcal{A}$; this actually will always be the case in the sequel, even for tensor products not adorned.

So according to our hypothesis for (1.3), and since $\Omega^1(\mathcal{A}) \equiv \mathcal{A} \otimes_{\mathcal{A}} \Omega^1 \cong \Omega^1$, we see that $\partial : \mathcal{A} \to \Omega^1$ *is*, in fact, *an $\mathcal{A}$-connection of $\mathcal{A}$*. For reasons that will become clear in the sequel we cail it the *standard flat* $\mathcal{A}$*-connection of* $\mathcal{A}$. On the other hand, by considering the *canonical* $\mathbb{C}$*-linear isomorphism (into)*

$$(1.7)\qquad \mathbb{C} \overset{\subset}{\to}_{\varepsilon} \mathcal{A}$$

($\mathbb{C}$ is viewed here as a *constant sheaf* on X), one gets, by virtue of (1.4),

$$(1.8)\qquad \mathbb{C} \overset{\subset}{\to} \ker \partial$$

(i.e., ∂ vanishes on every *constant section* of $\mathcal{A}$).

Furthermore, by looking at the *free* $\mathcal{A}$*-module* $\mathcal{A}^n$, one gets the following $n-th$ (or yet n*-dimensional*) *extension of* ∂ (see also (1.5))

$$(1.9)\qquad \partial^n : \mathcal{A}^n \to \mathcal{A}^n \otimes_{\mathcal{A}} \Omega^1 \equiv \Omega^1(\mathcal{A}^n) \cong (\Omega^1)^n \equiv \Omega^n,$$

such that

$$(1.10)\qquad \partial^n := \underbrace{\partial \oplus \ldots \oplus \partial}_{n-\text{ times}}, \qquad n \geq 1.$$

One easily proves that (1.10) *yields an* $\mathcal{A}$*-connection of* (the $\mathcal{A}$-module) $\mathcal{A}^n$. As a matter of fact, (1.10) is a special case of an analogous formula entailing the *induced A-conncction* fora finite Whitney sum of A-modules each endowed with an A-connection (cf. [31]: chapt. VII; section 3]).

On the other hand, one defines the following *morphism of sheaves of (abelian)groups*

$$(1.11)\qquad \tilde{\partial} : \mathcal{A}^{\bullet} \to \Omega^1$$

such that

$$(1.12)\qquad \tilde{\partial}(\alpha) := \alpha^{-1} \cdot \partial(\alpha),$$

for any (local) section $\alpha \in \mathcal{A}^{\bullet}(U) \cong (\mathcal{A}(U))^{\bullet}$, U open in X; here $\mathcal{A}^{\bullet}$ stands for the sheaf *of units of* $\mathcal{A}$, defined by the (complctc) presheaf of (abelian) groups on X

$$(1.13)\qquad U \mapsto (\mathcal{A}(U))^{\bullet}, \quad U \text{ open in } X,$$

the target of (1.13) being the (abelian) group of units of the $\mathbb{C}$-algebra $\mathcal{A}(U)$ (according to our hypothesis for $\mathcal{A}$, see (1.2), the latter algebra is commutative and unital). Motivated by the classical situation, we call (1.11) the *logarithmic derivation* of $\mathcal{A}$ associated with ∂. Now, justifying our claim for $\tilde{\partial}$ one proves that

$$\tilde{\partial}(s \cdot t) = \tilde{\partial}(s) + \tilde{\partial}(t), \tag{1.14}$$

for any s, t in $\mathcal{A}^{\bullet}(U)$ and U open in X.

In this respect, we still note for later use that one can extend the previous situation by taking the *matrix* algebra *sheaf* $M_n(\mathcal{A})$, $n \geq 1$, generated by the (complete) presheaf

$$U \mapsto M_n(\mathcal{A}(U)), \quad U \text{ open in } X, \tag{1.15}$$

as well as the $\mathcal{A}$-module (in fact, $M_n(\mathcal{A})$*-module*, see also (1.18) below)

$$M_n(\Omega^1) := M_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1 \cong (\Omega^1)^{n^2}, \qquad n \geq 1. \tag{1.16}$$

So one can extend the operators ∂ and $\tilde{\partial}$ (we retain, however, the same notation), according to the relations

$$\partial(\boldsymbol{\alpha}) := \left(\partial\left(\alpha_{ij}\right)\right), \tag{1.17}$$

for any $\boldsymbol{\alpha} \equiv (\alpha_{ij}) \in M_n(\mathcal{A})(U) = M_n(\mathcal{A}(U))$, U open in X, and

$$\tilde{\partial}(\boldsymbol{\alpha}) := \boldsymbol{\alpha}^{-1} \cdot \partial(\boldsymbol{\alpha}), \tag{1.18}$$

for any $\boldsymbol{\alpha} \in \mathcal{GL}(n, \mathcal{A})(U) = GL(n, \mathcal{A}(U))$; here $\mathcal{GL}(n, \mathcal{A})$ denotes the sheaf of units of $M_n(\mathcal{A})$, defined by the complete presheaf

$$U \mapsto GL(n, \mathcal{A}(U)) = M_n(\mathcal{A}(U))^{\bullet}, \quad U \text{ open in } X \tag{1.19}$$

So one has (see also (1.15))

$$M_n(\mathcal{A})^{\bullet} = \mathcal{GL}(n, \mathcal{A}), \qquad n \geq 1, \tag{1.20}$$

called the *general linear group* sheaf of $\mathcal{A}$ of order n (for convenience, we have not stuck here with the bold type notation for sections of matrix sheaves, the distinction being clear, otherwise, from the context).

Before we proceed further, we comment briefly on the following example relating the preceding with the standard differential geometric context.

Example 1.1. Considera (real finite dimensional) C^∞*-manifold* X and let

$$\mathcal{A} \equiv C_X^\infty \tag{1.21}$$

be the *sheaf of germs of* $\mathbb{C}$*-valued* C^∞*-functions* on X. We denote by

$$\mathcal{T}_{\mathbb{C}}(X) := \mathcal{T}(X) \otimes_{\mathbb{R}} \mathbb{C} \tag{1.22}$$

the *complexified tangent bundle* of X and let Ω_X^1 be the *sheaf of germs of* $C^\infty - 1$*-forms* on X (*dual* of the sheafof germs of sections of the $(C^\infty -)$ $\mathbb{C}$-vector bundle on X (1.22)). Thus, setting

$$\mathcal{E} \equiv (\Omega_X^1)^*, \tag{1.23}$$

namely, by considering now the *sheaf of germs of* $(C^\infty -)$ *vector fields* on X, one gets that:

(1.24) a C_X^∞ – *connection*, in the sense of Definition 1.1, is a standard *(linear)* C^∞ – *connection on* X.

See, for instance, [33, p. 2891. Of course, one can consider, more generaily, any (finite dimensionai $C^\infty -$) $\mathbb{C}$*-vector bundle* E on X and the corresponding *sheaves of germs*, as above, *of* IE*-valued* C^∞*-sections*, so that one then gets the classical notion of a *(linear)* C^∞*-connection of* IE (ibid.). Now, it is a standard result that *any such vector bundle on* X, when the latter space is *paracompact, admits a* (linear $C^\infty -$) *connection* (cf., for instance, [46, p. 76, Proposition 1.111). As we note below (see Theorem 2.1) this is due to our last hypothesis for X along with a subtle cohomologicai property of C_X^∞ (the latter is thus a *fine sheaf* on X, hence *acyclic*; in turn, *this is* still *the case for any* C_X^∞*-module* on X, as for example, for Ω_X^1 above. See e.g. [5, p. 49, Theorem 9.8 and p. 50, Theorem 9.121).

On the other hand, this is not, for instance, always the case for *holomorphic vector bundles* on a *complex* (analytic) *manifold,* according to a standard result of M.F. Atiyah [1]: as we shall see, this is due again to the non-vanishing, in generai, of a certain cohomology class, the so-cailed *Atiyah (obstruction) class* (of the bundle under consideration; see e.g. [20, **p.** 119] or yet (2.7) below). However, we **do** have *holomorphic connections* for any (holomorphic) vector bundle on a *Stein manifold* X; in this case the above class is zero as a result of *Cartan's Theorem B* in conjunction with the *coherence of* Ω_X^1, the *sheaf of germs of holomorphic*

1-*forms* on X (for the terminology applied cf., for example, [18, p. 230, Theorem B and p. 274] or yet [10, p. 67, Corollary]). **So** we do *not always have $\mathcal{A}$-connections* in case of *a complex* (analytic) *manifold*, with $\boldsymbol{A} \equiv \mathcal{O}_X$ the sheaf *of germs* of *holomorphic functions* on X.

Now, as already said, our treatment is quite sheaf-theoretic in character so that in place of vector bundles we consider (in effect, equivalently, in that case, see e.g. [26, p. 406, Theorem 1.11]) the corresponding sheaves of sections. This point of view seems to be also in agreement with recent trends in the domain of applications of differential geometry (fiber bundle theory) in theoretical physics (elementary particle physics, gauge theories); thus cf., for example, [32, p. 38] and/or [3].

So as follows from the previous Example 1.1, not every vector bundle, in general admits an $\mathcal{A}$-connection for any $\boldsymbol{A}$ whatsoever. On the other hand, motivated by the important particular case of a ($\mathbb{C}-$) vector bundle, we further adopt throughout the sequel the following terminology:

Thus, given the $\mathbb{C}$-algebraized space $(X,\mathcal{A})$, as above (cf. (1.1)), a locally free $\mathcal{A}$-module of finite rank over X is called a ***vector*** sheaf on X. In particular, by a *line* sheaf on X we mean a locally free A-module of rank one (this terminology was inspired, in effect, by a similar one applied by S. Lang, see [22, p. 1]).

According to the preceding, we thus conclude that: not *every* vector sheaf on a topological space X, as above (cf. (1.1), (1.2)), *admits* an *$\mathcal{A}$-connection,* for any $\boldsymbol{A}$ in general (so see the next section 2).

Now, in view of the aforementioned applications, we call a pair

$$(\mathcal{L}, D) \tag{1.25}$$

consisting of a line sheaf $\mathcal{L}$ on X and an A-connection D of $\mathcal{L}$ (see Definition 1.1), a *Maxwell field* on X. **As** a fundamental example of **(1.25)** one can consider, of course, the *electromagnetic field* of a *(free)* photon; in this respect, see also the final section 9 in the sequel. On the other hand, a pair

$$(\mathcal{E}, D) \tag{1.26}$$

with $\mathcal{E}$ a vector sheaf on X and D an A-connection of E is called, in general, a *Yang-Mills field* on X (in this concern, apart from section 9 below, see also, for instance, [32, p. 72] or yet [6, p. 454]).

We briefly discuss in the next section conditions guaranteeing the existence of $\mathcal{A}$-connections that will be of use in the sequel. On the other hand, a detailed account hereby is given in [31, chapt. VII].

2. EXISTENCE OF A-CONNECTIONS

We explain below, in condensed form, the way one is led to the definition of the Atiyah class, as well as to the notion of a *Levi-Civita connection* for a vector sheaf on X. We are going to apply all this constantly in the ensuing discussion.

Thus, suppose we are given a *vector sheaf* $\mathcal{E}$ on X, say, of *rank* n; moreover, let $\mathcal{U} = (U_\alpha)_{\alpha\in I}$ be an *open covering* of X such that

$$\mathcal{E}_{|U_\alpha} = \mathcal{A}^n_{|U_\alpha}, \quad \alpha \in I, \tag{2.1}$$

within an *isomorphism* ofthe $\mathcal{A}_{|U_\alpha}$*-modules* concerned. We call such an opcn set $U_\alpha \subseteq X$ a *local gauge* of $\mathcal{E}$, whereas $\mathcal{U}$ is then called a *local frame* of $\mathcal{E}$. Now, it is easy to see that:

(2.2) *the set of all local frames of a given vector sheaf $\mathcal{E}$ on X is a cofinal subset in the set of all proper open coverings of X* (the lattcr set being directed under refincment).

(In this respect, see also [14, p. 171]).

Accordingly, by applying (1.9), onc gets the following commutative diagram (*«Levi-Civita diagram»*)

$$\begin{array}{ccc} \mathcal{E}_{|U_\alpha} & \overset{D_\alpha}{\dashrightarrow} & \mathcal{E}\otimes_{\mathcal{A}}\Omega^1_{|U_\alpha} \\ \eta_\alpha \downarrow & & \downarrow \eta_\alpha^{-1}\otimes 1_{\Omega^1} \\ \mathcal{A}^n_{|U_\alpha} & \xrightarrow{\partial^n_\alpha \equiv \partial^n_{|U_\alpha}} & (\Omega^1)^n_{|U_\alpha} \cong \mathcal{A}^n \otimes_{\mathcal{A}} \Omega^1_{|U_\alpha} \end{array} \tag{2.3}$$

in the sense that one sets

$$D_\alpha := \left(\eta_\alpha^{-1} \otimes 1_{\Omega^1}\right) \circ \partial^n_\alpha \circ \eta_\alpha, \quad \alpha \in I \tag{2.4}$$

So, as in the classical case, we do have here too that: *the local trivialization of a given vector sheaf yields* (*always*, **due** to our assumption for ∂, cf. (1.3), **(1.4)**) *a (local)* $\mathcal{A}_{|U_\alpha}$*-connection* (alternatively, *«local A-connections always exist»*). Thus, the existence of a (global) A-connections of $\mathcal{E}$ depends on the following 1*-cocycle (A-connection difference)*

$$\delta\left((D_\alpha)\right) = \left(D_\beta - D_\alpha\right) \in Z^1\left(\mathcal{U}, \Omega^1(\mathcal{E}nd\,\mathcal{E})\right); \tag{2.5}$$

we call (2.5) the *Levi-Civita 1-cocycle of* E which is thus associated with any given local frame $\mathcal{U}$ of $\mathcal{E}$: indeed, setting $U_{\alpha\beta} \equiv U_\alpha \cap U_\beta (\neq \emptyset)$, with α, β in I, one obtains

$$(2.6)\quad \begin{aligned} D_\beta - D_\alpha &\in \mathcal{H}om_{\mathcal{A}}(\mathcal{E}, \mathcal{E} \otimes_{\mathcal{A}} \Omega^1)\left(U_{\alpha\beta}\right) = (\mathcal{H}om_{\mathcal{A}}(\mathcal{E},\mathcal{E}) \otimes_{\mathcal{A}} \Omega^1)\left(U_{\alpha\beta}\right) \equiv \\ &\equiv \Omega^1(\mathcal{E}nd\,\mathcal{E})\left(U_{\alpha\beta}\right) = (\mathcal{H}om_{\mathcal{A}}(\mathcal{A}^n, \mathcal{A}^n) \otimes_{\mathcal{A}} \Omega^1)\left(U_{\alpha\beta}\right) = \\ &= (M_n(\mathcal{A}) \otimes_{\mathcal{A}} \Omega^1)\left(U_{\alpha\beta}\right) = M_n(\Omega^1)(U_{\alpha\beta}) = M_n\left(\Omega^1\left(U_{\alpha\beta}\right)\right). \end{aligned}$$

Thus, the $\mathcal{A}$-module $M_n(\Omega^1)$ (cf. (1.16)) is in our case the *sheaf of A-connections coefficients,* while by virtue of (2.6) the corresponding *Atiyah class of* E is given by

$$(2.7)\qquad D(\mathcal{E}) := \left[\left(D_\beta - D_\alpha\right)\right] \in H^1(X, M_n(\Omega^1)).$$

We employ at this place *sheaf cohomology* which, however, at the final stage (Weil's theorem, see section 7 below) can be taken as *Čech cohomology,* since at that point we assume our space X to be *paracompact.*

Now, the fact that mostly concerns us here is the following result which, nevertheless for brevity's sake, we state without proof. For a full account of it cf. instead [31, Lemma 7.1, Theorem 9.1 and Theorem 10.21. So one gets the next.

Theorem 2.1. *Let* $(\mathcal{A}, \partial, \Omega^1)$ *be a given differential triad on a topological space* X *with* Ω^1 *being, in particular, a vector sheaf on* X. *Moreover, let* $\mathcal{E}$ *be a vector sheaf on* X. *Then,* E *admits an A-connection if, and only if, the corresponding Atiyah class of* E (cf. (2.7) above) *vanishes.* ∎

However, we comment a bit more on the previous theorem by pointing out certain particular items of its proof that will be also of use in the sequel: thus, in establishing Theorem 2.1, one also employs two further *equivalent versions of the notion of an A-cunnection;* namely, one such equivalent interpretation of an $\mathcal{A}$-connection is to consider it (viz. its existence) as (equivaient to) a *splitting of the short exact* $\mathcal{A}$*-sequence* (i.e., *exact sequence of* $\mathcal{A}$*-modules*)

$$(2.8)\qquad 0 \longrightarrow \mathcal{E}nd\ \mathcal{E} \longrightarrow \mathcal{S} \longrightarrow (\Omega^1)^* \longrightarrow 0$$

Here we set $(\Omega^1)^* := \mathcal{H}om_{\mathcal{A}}(\Omega^1, \mathcal{A})$ for the *dual* $\mathcal{A}$-module of Ω^1 (another *vector sheaf* on X, in view of our hypothesis for Ω^1; cf. Theorem 2.1). So one gets the relation

$$(2.9)\qquad Ext^1_{\mathcal{A}}\left((\Omega^1))^*,\ End\ E\right) = H^1(X, M_n(\Omega^1))$$

(within an *isomorphism of* (abelian) *groups*; cf. also, for instance, [16, p. 352, Example 4] or yet [31]). The last formula relates $\mathcal{A}$-*extensions* of $(\mathrm{R}^1)^*$ by *End* E, as in (2.8), with (2.7), the $\mathcal{A}$-module $M_n(\Omega^1)$ being also the *structure* sheaf of any $\mathcal{A}$-*extension* of the *form* (2.8).

In particular, by restricting ourselves to *paracompact* spaces one can apply *«liftable (Čech) cohomology»* (cf. section 4 below); thus, by virtue of (2.9), we relate the Atiyah class of a given vector sheaf $\mathcal{E}$ on X, as above, with a coordinate 1-*cocycle* of E, say,

$$(2.10) \qquad \left(g_{\alpha\beta}\right) \in Z^1(\mathcal{U}, \mathcal{GL}(n, \mathcal{A}))$$

(see also (1.20)), as well as with *the characteristic class* of *(2.8)*, denoted by $\delta(1)$; so one gets

$$(2.11) \qquad \delta(1) = \left[\left(D_\beta - D_\alpha\right)\right] = \left[\left(\tilde{\partial}\left(g_{\alpha\beta}\right)\right)\right] \in H^1(X, M_n(\Omega^1))$$

(Details are given in [31, chapt. VII, Theorem 9.21).

On the other hand, one can still consider an $\mathcal{A}$-*connection* of $\mathcal{E}$ as an *equivalent notion* to that of a *splitting* of the short exact A-*sequence*

$$(2.12) \qquad 0 \longrightarrow \Omega^1(\mathcal{E}) \longrightarrow \mathcal{J}^1(\mathcal{E}) \xrightarrow{\pi} \mathcal{E} \longrightarrow 0.$$

Here we denote by

$$(2.13) \qquad \mathcal{J}^1_{\mathcal{A}}(\mathcal{E}) \equiv \mathcal{J}^1(\mathcal{E}) := \mathcal{E} \oplus \Omega^1(\mathcal{E})$$

the $\mathcal{A}$-*module* (in fact, *vector sheaf*) on X of the corresponding *jet-line sheaf* (of yet *jet* sheaf *of order*, or *1 st jet sheaf*) of E, whose $\mathcal{A}$-module structure is given by the relation (warning!)

$$(2.14) \qquad \alpha \cdot (s \oplus t) := \alpha s \oplus (\alpha t + s \otimes \partial\alpha),$$

for any $\alpha \in \mathrm{A}(U)$, $s \in E(U)$ and $t \in \Omega^1(\mathcal{E})(U)$. Thus, one proves the *equivalence of* the $\mathcal{A}$-*extensions* (2.8) and *(2.12)*, since one gets (see also (2.9))

$$(2.15) \qquad Ext^1_{\mathcal{A}}\left(\mathcal{E}, \Omega^1(\mathcal{E})\right) = H^1(X, M_n(\Omega^1)) = Ext^1_{\mathcal{A}}\left((\Omega^1)^*, \mathcal{E}nd\ E\right).$$

So one concludes that:

(2.16) *any splitting of either one of (2.8) or (2.12) is equivalenf with the existence of a Levi-Civita* $\mathcal{A}$-*connection of* E, *hence with the vanishing of the Atiyah class of* & *in* (2.7), *as well*.

(in this respect, see [31, chapt. VII, Theorem 10.11). As a matter of fact, one realizes that any *splitting of (2.12)*, say

(2.17) $$D' : \mathcal{E} \longrightarrow \mathcal{J}^1(\mathcal{E}),$$

(viz. an $\mathcal{A}$-*morphism*, as above, such that $\pi \circ D' = 1_{\mathcal{E}}$, with π denoting the (canonical) $\mathcal{A}$-morphism, projection of $\mathcal{J}^1(\mathcal{E})$ onto $\mathcal{E}$, cf. (2.12) and (2.14)) *is of the form*

(2.18) $$D' = 1_{\mathcal{E}} \oplus D \equiv (1_{\mathcal{E}}, D),$$

for some uniquely defined $\mathcal{A}$*-connection* D *of* $\mathcal{E}$. (Indeed, *(2.18) is* a *characterization of being* D' *a splitting of (2.12)*; ibid. Proposition 10.1). As a result (see also ibid. Theorem 10.2), one finally concludes, within the framework of the previous Theorem 2.1, that:

(2.19) *a given vector sheaf* $\mathcal{E}$ *on* X *admits an* $\mathcal{A}$*-connection if, and only if, it also admits a Levi-Civita* $\mathcal{A}$*-connection; hence* (cf. (2.16) above), *if, and only if, the Atiyah class of* $\mathcal{E}$ *vanishes.*

The preceding constitute, in fact, the highlights of the proof of Theorem 2.1, that will be of help below. On the other hand, concerning the classical counterpart of the above, consult, for instance, [1], [6, p. 438], [32, p. 36 f and p. 38, Proposition 10], [17, pp. 338, 340].

Now, another item within the previous context, which will be of use in the sequel, is the *local form of* an $\mathcal{A}$-connection: thus, by analogy with the classical case of Differential Geometry, given a *vector sheaf* $\mathcal{E}$ on X and a *local gauge* $U \subseteq X$ (cf. (2.1)), one concludes that:

(2.20) *any given* $\mathcal{A}$*-connection* D *of* $\mathcal{E}$ *is locally* (viz. its restriction to an open set $U \subseteq X$, as above) *uniquely determined by a matrix (of «1-forms» on* U*)*

$$\omega \equiv (\omega_{ij}) \in M_n(\Omega^1(U)) = M_n(\Omega^1)(U)$$

i.e., by a *(local) section of the sheaf of* $\mathcal{A}$*-connection coefficients.*

Therefore, one can consider (the flat $\mathcal{A}$-connection) ∂ in (1.3), in effect the $\mathcal{A}$-connection (1.9), as the *«origin»*, in order to identify further the *affine* space *of* $\mathcal{A}$*-connections* of a given vector sheaf $\mathcal{E}$ on X (in case, of course, the latter space is non-trivial), *modeled* on (the $\mathcal{A}(X)$-module) $\Omega^1(\mathcal{E}nd\ \mathcal{E})(X)$, hence *locally* on $M_n(\Omega^1)(U)$ (see also (2.6)). Thus, by considering a *local gauge* U *of* $\mathcal{E}$, as above, and an $\mathcal{A}$*-connection* D of $\mathcal{E}$, one obtains (as a *local form of* D) the relation

(2.21) $$D = \partial + \omega,$$

for some ω, given by (2.20). In this respect, see also [31, chapt. VII; section 4 and Lemma 11.11.

3. CURVATURE

Another concept which will be of a particular concem to us in the sequel, is that of the curvature of an A-connection. In this regard, we note that, by contrast with what happened before, conceming the existence of A-connections, one can always define the curvature of a given A-connection (even of an A-module, $\mathcal{E}$, in general), once we have an appropnately enriched framework than that afforded by **(1.2)** (see **(3.7)** below). **So** we presently explain this, in brief, below contributing thus to the comprehensiveness of the later text (for more details we still refer instead to **[31**, chap. **VIII])**.

So assume that we have a *differential triad* (A, ∂, Ω^1), as above, and let

$$\Omega^2 := \Omega^1 \wedge \Omega^1 \equiv \bigwedge^2 \Omega^1 \tag{3.1}$$

be the *$\mathcal{A}$-module* on ***X***, *2 nd exterior power* of Ω^1 (thus, this now explains our index «1» on the corresponding A-module in **(1.2))**.

Furthermore, assume that we are given a *$\mathbb{C}$-linear morphism*

$$d^1 \equiv d : \Omega^1 \longrightarrow \Omega^2 \tag{3.2}$$

which first satisfies the following condition

$$d(\alpha \cdot s) = \alpha \cdot d s - s \wedge \partial\alpha \tag{3.3}$$

for any (local) sections $\alpha \in A(U)$ and $s \in \Omega^1(V)$ (V stands here, as always, for an *opcn sct* in X). Second, we still assume that the given operator d^1 $(\equiv d)$ obeys the relation

$$d \circ \partial = 0, \tag{3.4}$$

viz., equivalently,

$$\operatorname{im} \partial \subseteq \ker d. \tag{3.5}$$

(Indeed, the stronger condition of equality in **(3.5)** will be adopted later on; cf., for instance, *(5.5)* below). Now, **as** a result of **(3.4)** and **(3.3)** (see also (1.14)), we note for later use that

$$d \circ \widetilde{\partial} = 0 \tag{3.6}$$

Hereafter, we assume, of course, that:

(3.6') *the map $d^1 \equiv d$ in (3.2) **is** meant along with the properries (3.3) and (3.4),* unless otherwise specified; cf. for example *(5.5)* in the sequel.

The previous map **(3.2)** is our *1st exterior derivative operator*. **Yet,** the following finite sequence (cf. also **(3.6)**)

$$(3.7) \qquad (\mathcal{A}, \partial, \Omega^1, d, \Omega^2)$$

is (deliberately) called a *curvature datum* on X; the terminology applied hereby is justified by the fact that the preceding framework will provide us with the notion of the curvature of a given $\mathcal{A}$-connection, as we shall presently see.

Thus, given an $\mathcal{A}$-*module* $\mathcal{E}$ on X endowed with an $\mathcal{A}$-*connection* D (cf. Definition **1.1**), consider the following map:

$$(3.8) \qquad D^1 : \Omega^1(\mathcal{E}) \longrightarrow \Omega^2(\mathcal{E}) \equiv \mathcal{E} \otimes_{\mathcal{A}} \Omega^2$$

given by the relation

$$(3.9) \qquad D^1(s \otimes t) := s \otimes dt + Ds \wedge t$$

for any $s \in \mathcal{E}(U)$ and $t \in \Omega^1(U)$.

Of course, one can extend by $\mathbb{C}$-linearity the previous relation (3.9) to $\mathcal{E} \otimes_{\mathbb{C}} \Omega^1$. But, the second member of the same relation defines, in fact, a *balanced map,* say, ρ; that is, one has

$$(3.10) \qquad \rho(\alpha s, t) = \rho(s, \alpha t),$$

for any $\alpha \in \mathcal{A}(U)$ and s, t as above. Therefore, one can actually extend **(3.9)** to the whole of $\Omega^1(\mathcal{E})$, by «$\mathcal{A}$-*linearity*», so that one concludes that

$$(3.11) \qquad D^1 \in Hom_{\mathbb{C}}(\Omega^1(\mathcal{E}), \Omega^2(\mathcal{E}));$$

hence, D^1 *is a* $\mathbb{C}$*-linear morphism* for the underlying structures of sheaves of $\mathbb{C}$-vector spases of the $\mathcal{A}$-modules concerned).

Now, by extending the classical terminology, we call D^1 the *1st covariant exterior derivative operator* (with respect to **(3.7)**) or yet the *1st prolongation of the* given $\mathcal{A}$-*connection* D.

On the other hand, the following property of the operator D^1 is needed right below; so one has

$$(3.12) \qquad D^1(\alpha \cdot Ds) = \alpha \cdot D^1(Ds) - Ds \wedge \partial\alpha$$

for any $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$.

We come now to define, within the preceding framework of **(3.7)**. the following important operator which is associated with any given dconnection D; **it** concerns, at first view **a** $\mathbb{C}$*-lincar morphism*, in fact, something much more is actuaily true, as we shall presently see. Thus, one has the following diagram

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{D} & \Omega^1(\mathcal{E}) \\ & {\scriptstyle D^1 \circ D} \searrow & \downarrow {\scriptstyle D^1} \\ & & \Omega^2(\mathcal{E}) \end{array} \tag{3.13}$$

We call the operator

$$R(D) \equiv R := D^1 \circ D, \tag{3.14}$$

as defined above, the *curvature* of the d-connection D.

In this regard, a given dconnection D is said to be *flat*, if one has

$$R(D) \equiv R = 0. \tag{3.15}$$

Thus, one can prove, for instance, that

$$R(\partial) = 0, \tag{3.16}$$

concerning the d-connection $\partial : \mathcal{A} \longrightarrow \Omega^1$ of $\mathcal{A}$ considered by **(1.13)** (in this respect, see also the comments after Definition 1.1 above; so this now justifies the terminology for ∂, adopted at the beginning).

Now, on the strength of **(3.12)**. one further proves that

$$R(\alpha \cdot s) = \alpha \cdot R(s), \tag{3.17}$$

for any $\alpha \in \mathcal{A}(U)$ and $s \in \mathcal{E}(U)$. **So** R *is*, in effect, *an $\mathcal{A}$-morphism* of the d-modules concemed (cf. **(3.13))**; that **is,** one obtains

$$R \in Hom_{\mathcal{A}}\left(\mathcal{E}, \Omega^2(\mathcal{E})\right) = \mathcal{H}om_{\mathcal{A}}\left(\mathcal{E}, \Omega^2(\mathcal{E})\right)(X). \tag{3.18}$$

In particular, *if $\mathcal{E}$ is* **a** *vector sheaf on* X, then *(3.18) yields* (see also, for example, (2.6))

$$R \in \Omega^2(\mathcal{E}nd\ \mathcal{E})(X) \cong Z^0\left(\mathcal{U}, \Omega^2(\mathcal{E}nd\ \mathcal{E})\right); \tag{3.19}$$

that is, one concludes that:

(3.20) *the curvature R of a given A-connection D of a vector sheaf $\mathcal{E}$ on X defines a global section (0-cocycle) of the vector sheaf $\Omega^2(End\ E)$.*

The above provides now the familiar form one has from the classical theory for the curvature of a connection, in case one considers a *local gauge* $U \subseteq X$ of E: namely, one then obtain (see also (2.6))

$$R|_U \in \Omega^2(\mathcal{E}nd\ \mathcal{E})(U) = \Omega^2(U) \otimes_{\mathcal{A}(U)} (\mathcal{E}nd\ \mathcal{E})(U) = = \Omega^2(U) \otimes_{\mathcal{A}(U)} M_n(\mathcal{A})(U) = M_n(\Omega^2)(U) = M_n(\Omega^2(U)) \tag{3.21}$$

That is, *one has*

$$R|_U \equiv (R_{ij}) \in M_n(\Omega^2(U)) = M_n(\Omega^2)(U), \tag{3.22}$$

for any U, as above. So by rephrazing the classical fact, one infers that:

(3.23) *over a local gauge of a vector* **sheaf** *E, the curvature of a given A-connection of E is expressed* **as** *a matrix of «2-forms»* (or yet as **a** *matrix-valued «2-forms»)*

On the other hand, if $w \equiv (\omega_{ij}) \in M_n(\Omega^1)(U)$ is the analogous matrix of «1-forms» defining the given A-connection D of E, locally on U (cf. (2.20), one obtains the following relation: *Cartan's structural equation*

$$R = d\omega + w \wedge \omega, \tag{3.24}$$

valid (locally) on any $U \subseteq X$, as above (in this respect, cf. also [31, chapt. VIII; (2.8)]. The particular form of (3.24) in case of *line sheaves* will be considered in the sequel **(see** (6.15)).

4. LIFTABLE (SHEAF) COHOMOLOGY

Before we proceed further to be engaged in our main concern which is, of course, the theorem in the title of this paper, we comment below, for clarity's sake, **a** bit more on certain sheaf cohomology-theoretic concepts that we apply several times in the sequel. Indeed, we use the fact that:

(4.1) *in a paracompact (Hausdorff) space X sheaf cohomology can be defined by means of the so-called «liftable cochains»*

Thus, consider the following *short exact* $\mathcal{A}$*-sequence* on X

$$(4.2)\qquad 0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0\,;$$

then, for any *integer* $p \geq 0$, the corresponding sequence of p-cochains is, in generai, only left exact. That is, one gets the *exact sequence*

$$(4.3)\qquad 0 \longrightarrow C^p(\mathcal{U},\mathcal{F}) \xrightarrow{\phi} C^p(\mathcal{U},\mathcal{S}) \xrightarrow{\psi} C^p(\mathcal{U},\mathcal{E}),$$

where $\mathcal{U}$ stands for any (proper) open covering of X. So by taking the *image of* ψ *in (4.3), say,*

$$(4.4)\qquad \operatorname{im}\psi \equiv C^p_\alpha(\mathcal{U},\mathcal{E}) \subseteq C^p(\mathcal{U},\mathcal{E})$$

($\mathbb{C}$-vector space of *liftable* p*-cochains*), one obtains the following *short exact sequence* (of $\mathbb{C}$-vector spaces)

$$(4.5)\qquad 0 \longrightarrow C^p(\mathcal{U},\mathcal{F}) \xrightarrow{\phi} C^p(\mathcal{U},\mathcal{S}) \xrightarrow{\psi} C^p_\alpha(\mathcal{U},\mathcal{E}) \longrightarrow 0.$$

(By an obvious abuse of notation, we use for simplicity the same symbol for the maps ϕ, ψ in all the above three last relations). Thus, by setting

$$(4.6)\qquad H^p_\alpha(\mathcal{U},\mathcal{E}) := Z^p_\alpha(\mathcal{U},\mathcal{E})/\delta C^{p-1}_\alpha(\mathcal{U},\mathcal{E}),$$

one obtains

$$(4.7)\qquad H^p(X,\mathcal{E}) = \varinjlim_{U} H^p_\alpha(\mathcal{U},\mathcal{E}),$$

which explains (4.1) (in this regard, see also, for instance, [15, p. 180, Proposition 7.3.5]). Here we consider *Čech cohomology* (aithough *unadorned*) to which, of course, any other (sheaf) cohomology theory can (isomorphically) be reduced in case of paracompact spaces (see e.g. [44, p. 184, Corollary] or yet [18, p. 215, Theorem 50.21).

Now, in view of (2.2), we further note that:

(4.8) *in case* $\mathcal{E}$ *is a vector sheaf on* X*, one can consider in (4.7) the open covering U as ranging over the (proper) local frames of* &.

The previous fact has been applied, for instance, in (2.7) as a result of the calculations in (2.6).

In this concern, we still note that a crucial point in obtaining (4.7) is the argument expressed by the following lemma. We use it too systematically below. So we have:

Lemma 4.1. *Let X be a paracompact Hausdorff space, and*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\phi} \mathcal{S} \xrightarrow{\psi} \mathcal{E} \longrightarrow 0 \tag{4.9}$$

a short exact $\mathcal{A}$-sequence. Moreover, let $\mathcal{U} = (U_\alpha)_{\alpha \in I}$ be an open covering of X. Then, for every p-cocycle

$$g \in Z^p(\mathcal{U}, \mathcal{E}), \qquad p \geq 0, \tag{4.10}$$

there exists a refinement $\mathcal{V} = (\mathcal{V}_\beta)_{\beta \in J}$ of $\mathcal{U}$ and a p-cochain

$$h \in C^p(\mathcal{V}, \mathcal{S}) \tag{4.11}$$

such that

$$\rho g = \psi h \tag{4.12}$$

(here $\rho : J \to I$ denotes the *refinement map*), *and*

$$\delta h = \phi f \tag{4.13}$$

where

$$f \in Z^{p+1}(\mathcal{V}, \mathcal{F}) \tag{4.14}$$

(The map δ in (4.13) stands, of course, for the corresponding *Bockstein operator*).

Proof. Cf. [15, p. 180, Lemma 7.3.6] or yet [12, p. 33 f, proof of Theorem 1]. ∎

Looking now at the corresponding cohomology classes, *one obtains*

$$\delta^*([g]) = [\delta(h)] = \left[\phi^{-1}(\delta(h))\right] = [f] \in H^{p+1}(X, \mathcal{F}), \tag{4.15}$$

for any $[g] \in H^p(X, \mathcal{E})$, *for some* $[f] \in H^{p+1}(X, \mathcal{F})$. The crucial fact at this point is that, by virtue of (4.12), one actually has that $[g] \in H^p_\alpha(X, E)$, the latter space being, in effect, $H^p(X, E)$ as follows from (4.7). Yet, δ^* in (4.15) denotes the corresponding Bockstein operator for the long exact cohomology sequence, associated with (4.9).

Finally, we still recall for latter use the following *cohomological classification of vector sheaves* on X; that is, one has

$$\Phi^n_{\mathcal{A}}(X) = H^1(X, \mathcal{GL}(n, \mathcal{A})), \tag{4.16}$$

within a *bijection.* Here the first member of the last relation stands for the *set of isomorphism classes of vector sheaves of rank n on X*, whereas the second one denotes the usual *1st cohomology set of X* with coefficients in the sheaf of (non-abelian, in general) groups $\mathcal{GL}(n, \mathcal{A})$ (see also (1.20)). In this concern, cf., for instance [13, p. 11, Theorem 1] or yet [31, chapt. V].

5. PRE-WEIL SPACES

In the preceding, for completeness' sake, we exhibited the necessary material which constitutes the appropriate context for the rest of our discussion. Now, the spaces in the title of this section ***are,*** as we shall presently see, the suitable framework in order to deai with a substantial ingredient of our final target (Weil's theorem. Therefore, the employed terminology; but see also the next section).

Thus, suppose we are given a *curvature datum* on a topological space X (cf. (3.7), and also (3.6')), and let

$$\Omega^3 := \wedge^3 \Omega^1 \tag{5.1}$$

be the $\mathcal{A}$*-module* on X, *3 rd exterior power of* Ω^1. Now, we further define a $\mathbb{C}$*-linear morphism*

$$d^2 \equiv d : \Omega^2 \longrightarrow \Omega^3 \tag{5.2}$$

by the relation

$$d^2(s \wedge t) := d^1 s \wedge t - s \wedge d^1 t \equiv ds \wedge t - s \wedge dt, \tag{5.3}$$

for any $s, t \in \Omega^1(U)$: namely, in a similar manner as for the map **(3.8)** (see also (3.9)), one proves that (5.3) *defines a bafanced map* (see e.g. (3.10)); hence. one can then extend it, **by** $\mathcal{A}$*-linearity*, to the whole of Ω^2 (by definition, 2 nd exterior power of the given $\mathcal{A}$-module Ω^1).

Thus, we are now in the position to set the following

Definition 5.1. *By a* pre-Weil space *we mean a paracompact Hausdorff space* X *endowed with a curvature datum (3.7), where* Ω^1 *is a vector sheaf on* X. *Moreover, we assume that the previous data yield the following exact sequence (of sheaves of* $\mathbb{C}$*-vector spaces)*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{\partial} \Omega^1 \xrightarrow{d^1 \equiv d} \Omega^2 \xrightarrow{d^2 \equiv d} d\Omega^2 \longrightarrow 0. \tag{5.4}$$

In this regard, we finally suppose that all the previous operators appeared in **(5.4)** commute with the corresponding Bockstein (coboundary) operator, whenever one considers the Čech cohomology on X.

Accordingly, *in Case of a pre-Weil space* X, we **first** adopt (by definition) that the preceding relation (3.5) *holds*, in essence, *as an equality*; that is, *one has* (by hypothesis) *the relation*

$$\ker d^1 = \partial(\mathcal{A}), \tag{5.5}$$

as well as the relation

$$\ker d^2 = d^1(\Omega^1) \equiv d\Omega^1. \tag{5.6}$$

We come now to our first basic observation, concerning the main objective of this paper. That is, one gets the following result (but see also Scholium 5.1 in the sequel).

Proposition **5.1.** *Let X be a pre-Weil space and w a closed 2-form on X; viz. we assume that*

$$\omega \in \Omega^2(X) \quad \textit{such that} \quad dw = 0. \tag{5.7}$$

Then, one can associate with w a 2-dimensional complex cohomology class of X, say.

$$c(\omega) \in \check{H}^2(X, \mathbb{C}) \equiv H^2(X, \mathbb{C}). \tag{5.8}$$

Proof. As a first extract from (5.4), one obtains the following *short exact sequence* (of sheaves of $\mathbb{C}$-vector spaces; cf. (3.2))

$$0 \longrightarrow d\Omega^1 \longrightarrow \Omega^2 \longrightarrow d\Omega^2 \longrightarrow 0. \tag{5.9}$$

Therefore, by considering the corresponding *long exact cohomology sequence*, associated with (5.9) (cf., for instance, [44, p. 177, (5.18); **(a)**, (c)]), one obtains

$$\begin{aligned} 0 &\longrightarrow \Gamma(X, d\Omega^1) \longrightarrow \Gamma(X, \Omega^2) \longrightarrow \Gamma(X, d\Omega^2) \longrightarrow \\ &\longrightarrow H^1(X, d\Omega^1) \longrightarrow \dots \end{aligned} \tag{5.10}$$

Accordingly, in view of our hypothesis for w (see (5.7)),

$$w \in \ker\left(\Gamma(X, \Omega^2) \longrightarrow \Gamma(X, d\Omega^2)\right) \cong \Gamma(X, d\Omega^1) \equiv (d\Omega^1)(X); \tag{5.11}$$

that is, *the hypothesis that w is closed entails, in fact, that*

$$w \in (d\Omega^1)(X) \overset{\subseteq}{\rightarrow} \Omega^2(X). \tag{5.12}$$

(Indeed, *under the assumption* ***of*** *(5.4),* the previous conditions (5.7) ***and*** *(5.12) are, in effect, equivalent;* cf. also the next Scholium 5.1).

Now, by virtue of (5.4), one further obtains the following *short exact sequence*

$$0 \longrightarrow \partial\mathcal{A} \longrightarrow \Omega^1 \longrightarrow d\Omega^1 \longrightarrow 0. \tag{5.13}$$

Thus, according to (5.12), one gets

$$\omega \in (d\Omega^1)(X) \equiv \Gamma(X, d\Omega^1) \cong Z^0(\mathcal{U}, d\Omega^1), \tag{5.14}$$

for any open covering $\mathcal{U}$ of X (see, for example, [12, p. 28, Lemma 4]). Therefore, on the basis of Lemma 4.1, we can find (modulo, eventually, a refinement $\mathcal{V}$ of $\mathcal{U}$) a 0*-cochain* (*of 1-forms*)

$$(\theta_\alpha) \in C^0(\mathcal{V}, \Omega^1) \tag{5.15}$$

such that one has

$$\omega = d((\theta_\alpha)) := (d(\theta_\alpha)) \equiv (d\theta_\alpha) \tag{5.16}$$

Moreover (cf. (4.13), (4.14)), one gets

$$\delta((\theta_\alpha)) \equiv \delta(\theta_\alpha) = (\theta_\beta - \theta_\alpha) \in Z^1(\mathcal{V}, \partial\mathcal{A}) \tag{5.17}$$

As a final extract from (5.4), one now gets the following *short exact sequence*

$$0 \longrightarrow \mathbb{C} \stackrel{\varepsilon}{\longrightarrow} \mathcal{A} \longrightarrow \partial\mathcal{A} \longrightarrow 0 \tag{5.18}$$

Thus, in view of (5.17) and Lemma 4.1, we conclude (modulo, in general, a further refinement, say $\mathcal{W}$, of $\mathcal{V}$) the existence of a 0*-cochain* of $\mathcal{A}$, say,

$$(f_{\alpha\beta}) \in C^1(\mathcal{W}, \mathcal{A}), \tag{5.19}$$

such that

$$\delta(\theta_\alpha) = (\theta_\beta - \theta_\alpha) = \partial(f_{\alpha\beta}) \equiv (\partial f_{\alpha\beta}) \in Z^1(\mathcal{W}, \partial\mathcal{A}) \tag{5.20}$$

(In this respect, cf. also (5.16) for the notation applied in the last term of (5.20). Furthermore, we should also point out at this piace the obvious abuse of notation employed above, concerning, namely, the *indices* of the open coverings of X considered). **Yet** (the same lemma) one obtains

$$\delta(f_{\alpha\beta}) \equiv (\lambda_{\alpha\beta\gamma}) \in Z^1(\mathcal{W}, \mathbb{C}); \tag{5.21}$$

therefore. one now sets

$$c(\omega) := \left[\left(\lambda_{\alpha\beta\gamma}\right)\right] \in H^2(X, \mathbb{C}), \tag{5.22}$$

which also finishes the proof. ∎

The previous result could also be obtained *for any closed p-form* on X, viz. for any element $\omega \in \Omega^p(X)$ $(:= (\bigwedge^p \Omega^1)(X))$ with $d\omega (\equiv d^p(\omega)) = 0$, undera further suitable extension of (5.4). However this, in conjunction with its application in defining *Chern classes* of vector sheaves on X, will be taken up elsewhere (see, for instance, [31]).

Now, a closed 2-forms ω (on a pre-Weil space X, cf. (5.7)) is said to be *integral*, in case one has (see also (5.22))

$$c(\omega) \in \operatorname{im} \left(H^2(X, \mathbb{Z}) \to H^2(X, \mathbb{C})\right). \tag{5.23}$$

(The last part of the previous relation is, of course, the one derived from the *canonical inclusion* of the corresponding *constant sheaves* $\mathbb{Z} \overset{\subseteq}{\to} \mathbb{C}$; cf. also, for instance, [4, p. 92, Corollaire 1] as well as (4.7) above).

Scholium 5.1. As already remarked, *the exactness of (5.4) yields the relation*

$$(d\Omega^1)(X) \equiv (d^1 \Omega^1)(X) = \ker d^2 \subseteq \Omega^2(X). \tag{5.24}$$

(Otherwise, this is also a consequence of the very definition of the *exactness of (5.4) at* Ω^2; *see*, for example, [9, p. 114, § 1.6 and also p. 132, § 2.5]). On the other hand, by looking more closely at the proof of Proposition 5.1, we realize that (under (5.16))

(5.25)
one actually obtains a map
$c : (d\Omega^1)(X) \longrightarrow H^2(X, \mathbb{C}),$
whenever one has the exactness only of the two sequences
(5.13) and (5.18);

hence, not necessarily that of (5.9), as well. Now, this (slight) weakening of the conditions of Proposition 5.1 might be of a particular significance for the applications, if one seeks for *the (full) exactness of (5.4).* Furthermore, *it is the map (5.25) that essentially appears when one deals with line sheaves;* the latter will be our main concern in the next section (see e.g. (6.20), as well as the Appendix).

Now, our next objective is to characterize the situation when, in the particular case considered, a given *«2-form»* ω on X as in (5.7) *satisfies (5.23)* (of course, this is actually the content of Weil's theorem, as well). But to this end we still need some more terminology, which we are going thus to establish in the following section.

6. SEMI-WEIL SPACES

We start with exhibiting the necessary complementary material to that aiready provided by the previous Definition **5.1,** in **order** to be able to get the desired form of Weil's theorem. Thus, we first have the following.

Definition 6.1. *Let $(\mathcal{A}, \partial, \Omega^1)$ be a given differential triad on a topological space X, $\mathcal{A}^{\bullet}$ the sheaf of units of $\mathcal{A}$ and $\tilde{\partial}$ the corresponding logarithmic derivation of $\mathcal{A}$, associared with ∂ (cf. section 1). Now, by an exponential sheaf* **diagram** *on X, we mean the following short exact sequence of sheaves (of abelian groups on X) along with the associated iriangle (of sheaf morphisms) which we also assume to be commutative*

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{A} \xrightarrow{e} \mathcal{A}^{\bullet} \longrightarrow 1, \qquad \partial : \mathcal{A} \to \Omega^1, \quad \frac{1}{2\pi i}\tilde{\partial} : \mathcal{A}^{\bullet} \to \Omega^1 \tag{6.1}$$

So conceming the previous definition, we accept, in particular, that

$$\tilde{\partial} \circ e = 2\pi i \cdot \partial, \tag{6.2}$$

where $i \equiv (-1)^{\frac{1}{2}}$. Indeed, we need below the preceding concept combined with that in **(3.7).** Thus, we still set the next.

Definition 6.2. *By a* semi-Weil space *we mean a paracompact Hausdorff space endowed with a curvature datum (cf. (3.7)) and an exponential sheaf diagram, as above. Here we still assume that the operator (exponential sheaf morphism) e commutes with the Bockstein operator δ (so this operator too is cohomologically acceptable).*

We discuss in the sequel (see thus section 8) a concrete particular case where a situation like that described by the above definition can be occurred (for more relevant details we refer, however, to [31, chapt. 6]). On the other hand, as a first consequence of the preceding terminology, we can give now the next fundamental result. So we have.

Lemma 6.1. *Let $(\mathcal{L}, D)$ be a Maxwell field* (see (1.25)) *on a semi-Weil space X. Then, modulo an eventual translation of D by an element of $\Omega^1(\mathcal{E}nd\ \mathcal{L})(X) \cong \Omega^1(X)$, the corresponding curvature form, say R, of the $\mathcal{A}$-connection of $\mathcal{L}$ under consideration, yields an integral 2-form on X; viz. one gets*

$$c(R) \in H^2(X, \mathbb{Z}). \tag{6.3}$$

Proof. By hypothesis $\mathcal{L}$ (a *line sheaf* on X) admits an $\mathcal{A}$-*connection* D, so that (cf. Theorem 2.1) the corresponding Atiyah class of $\mathcal{L}$ vanishes; i.e., we have

$$D(\mathcal{L}) := \left[\left(D_\beta - D_\alpha\right)\right] = \left[\tilde{\partial}\left(g_{\alpha\beta}\right)\right] = 0 \in H^1\left(X, \Omega^1\right) \tag{6.4}$$

(see also (2.7) for $n = 1$). In this regard, we recall (cf. section 2) that

$$\left(D_\alpha\right) \in Hom_{\mathbb{C}}\left(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{A}} \Omega^1\right) \tag{6.5}$$

is a 0-*cochain* derived from a Levi-Civita diagram for $\mathcal{L}$ (cf. (2.3)), which can be associated with any given *local frame*, say $\mathcal{U} = (U_\alpha)_{\alpha \in I}$, of $\mathcal{L}$. Furthermore, $\mathcal{L}$ being a line sheaf on X, one gets in particular that

$$\mathcal{H}om_{\mathcal{A}}\left(\mathcal{L}, \mathcal{L} \otimes_{\mathcal{A}} \Omega^1\right) = \mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \otimes_{\mathcal{A}} \Omega^1 = \Omega^1(\mathcal{E}nd\ \mathcal{L}) \equiv \mathcal{A} \otimes_{\mathcal{A}} \Omega^1 = \Omega^1, \tag{6.6}$$

since one has

$$\mathcal{H}om_{\mathcal{A}}(\mathcal{L}, \mathcal{L}) \equiv \mathcal{E}nd\ \mathcal{L} \cong \mathcal{A}, \tag{6.7}$$

within an *isomorphism of* $\mathcal{A}$-*modules;* cf., for instance, [11, p. 116; (5.4.3.17)] or yet [31, chapt. 4]. Thus, by considering the Levi-Civita 1-cocycle of $\mathcal{L}$, which corresponds to (6.5), one obtains

$$\delta\left(D_\alpha\right) = \left(D_\beta - D_\alpha\right) \in Z^1\left(\mathcal{U}, \Omega^1\right) \tag{6.8}$$

(cf. also (2.5) or yet (2.6) for $n = 1$). Yet by referring to (6.4)

$$\left(g_{\alpha\beta}\right) \in Z^1\left(\mathcal{U}, \mathcal{A}^\bullet\right) \tag{6.9}$$

stands there for a coordinate *1-cocycle* of $\mathcal{L}$, associated with the given local frame $\mathcal{U}$ of $\mathcal{L}$, as above.

On the other hand, by virtue of (2.19) and (6.4), one infers that $\mathcal{L}$ *admits a* Levi-Civita $\mathcal{A}$-*connection, as well, say* $\tilde{D}$, *whose 1-cocycle* ($\mathcal{A}$-connection difference) *is* given *by (6.8).* Now, on the strength again of (6.4), one gets

$$\left(D_\beta - D_\alpha\right) = \delta\left(\theta_\alpha\right) \in Z^1\left(\mathcal{U}, \Omega^1\right), \tag{6.10}$$

for some 0-*cochain* (of 1-forms)

$$(6.11) \qquad (\theta_\alpha) \in C^0(\mathcal{U}, \Omega^1).$$

At this point we also remark (for latter use, as well) that one may still assume, in addition to (6.10), the following relation

$$(6.12) \qquad \tilde{\partial}\left(g_{\alpha\beta}\right) = \delta(\theta_\alpha) \in Z^1(\mathcal{U}, \Omega^1),$$

with (θ_α) given by (6.11) (in this respect, one actually proves that the two cocycles $(\tilde{\partial}(g_{\alpha\beta}))$ and $\delta(D_\alpha)$ are *«gauge equivalent»*, via the (local) *«coordinatization»* of $\mathcal{C}$ (cf. (2.3)), hence the relation (6.12), in view of (6.4) and (6.10). For details see, however, [31, chapt. VII, section 8]).

Now, the *local form of* $\tilde{D}$, with respect to $\mathcal{U}$, is given by the relation

$$(6.13) \qquad \tilde{D}|_{U_\alpha} \equiv \tilde{D}_\alpha = D_\alpha - \theta_\alpha, \quad \alpha \in I,$$

so that *the 0-cochain* (θ_α) *may be considered as the* corresponding *local $\mathcal{A}$-connection form representing* $\tilde{D}$ (cf. also (2.20), (2.21)). Moreover, *the two A-connections* D *and* $\tilde{D}$ *of* $\mathcal{C}$, considered hitherto, *differ by an element of* Ω^1 *(and* $\mathcal{C}$*)*(X) = (cf. (6.6)) $\Omega^1(X)$, *viz. one has*

$$(6.14) \qquad D - \tilde{D} \in \Omega^1(X) \cong Z^0(\mathcal{U}, \Omega^1)$$

(cf. the comments before (2.21) or yet [31, chapt. VII, Proposition 4.11). This also explains our claim in the statement of the lemma, by further considering the curvature form of $\tilde{D}$; thus, we have

$$(6.15) \qquad R(\tilde{D}) \in R = (d\theta_\alpha).$$

(The last relation is a consequence of (3.24) and our previous conclusion for the 0-cochain (θ_α)).

At this point we still remark that *one can also directly conclude* (apart from (3.19)) *that* R *defines a global 2-form on* X : namely, one gets in view of (6.12) and (3.6) (see also (3.6')) ;

$$(6.16) \qquad \delta(d\theta_\alpha) = d(\delta(\theta_\alpha)) = d\left(\tilde{\partial}\left(g_{\alpha\beta}\right)\right) = 0,$$

so that one obtains

$$R = (d\,\theta_\alpha) \in Z^0(\mathcal{U}, d\,\Omega^1) \cong (d\,\Omega^1)(X) \hookrightarrow \Omega^2(X), \tag{6.17}$$

as asserted.

Yet, we shall prove that R *is integral*, in the sense that it provides an integral (2-dimensional) cohomology class of X (cf. (6.21) below; in this respect, see also the next Scholium 6.1): indeed, based now on (6.9), the short exact (exponential sheaf) sequence in (6.1) and on Lemma 4.1, in conjunction with our hypothesis for X, we conclude the existence of a *1-cochain*

$$(f_{\alpha\beta}) \in C^1(\mathcal{V}, \mathcal{A}) \tag{6.18}$$

such that

$$e((f_{\alpha\beta})) := (e(f_{\alpha\beta})) = (g_{\alpha\beta}) \tag{6.19}$$

(In (6.18) $\mathcal{V}$ stands for an eventual refinement of $\mathcal{U}$, while we still abuse the notation for the indices). Furthermore (again Lemma 4.1), one gets

$$\delta(f_{\alpha\beta}) \equiv (\lambda_{\alpha\beta\gamma}) \in Z^2(\mathcal{V}, \ker e) \cong Z^2(\mathcal{V}, \mathbb{Z}) \tag{6.20}$$

(since, in view of (6.1), one has $\ker e \cong \mathbb{Z}$). The previous 2-cocycle determines now the desired ***2-dimensional integral cohomology class of*** X, which is thus provided by $R = (d\,\theta_\alpha)$ (cf. also e.g. (5.25)); i.e., we set

$$c(R) := [\delta(f_{\alpha\beta})] \in H^2(X, \mathbb{Z}), \tag{6.21}$$

and this also terminates the proof of the lemma. ∎

Scholium 6.1. Suppose we are given a *topological* space X endowed *with* a curvature *datum* (cf. (3.7)) and let $(\mathcal{L}, D)$ be a *Maxwell field* on X. Now, if $(\theta_\alpha) \in C^0(\mathcal{U}, \Omega^1)$ is a *0-cochain of 1-forms of* X *which locally represents* D (see (2.20) for $n = 1$ or yet (2.21)), then the curvature *of* D is given by a similar relation to (6.17) (yet, within the present more general context); viz. one has

$$R(D) \equiv R = (d\,\theta_\alpha) \in (d\,\Omega^1)(X) \hookrightarrow \Omega^2(X) \tag{6.22}$$

(cf. also (3.24)). On the other hand, the (global) 2-form provided by R, as before, can be considered as a *closed form on X* , as follows: namely, by definition, this means that

(6.23) $$dR = 0,$$

where $d \equiv d^2 : \Omega^2 \rightarrow \Omega^3$ is the $\mathbb{C}$*-linear morphism* defined by (5.3). So if we further assume that

(6.24) $$d^2 \circ d' = 0,$$

then, as immediately follows from (6.22), *R is a closed 2-form on X* .

Now, by definition, a *Bianchi datum* on a topological space X, is a curvature datum (cf. (3.7)) which is further endowed with an operator $d^2 \equiv d$, as above, in such a way that (6.24) be satisfied (the termilology here is due, for instance, to the fact that within the previous framework one can obtain the corresponding to our case *Bianchi's identity*; but in this concem, we refer instead to [31, chapt. VIII, Theorem 3.11).

Thus, as a consequence of the preceding, one concludes that:

(6.25) *given a Bianchi datum on a topological space X and a Maxwell field* (C, D) *on X* (cf. (1.25)), *the curvature R of D yields a closed 2-form on X; viz. one has, in particular,* $R \in \Omega^2(X)$ *with* $dR = 0$.

More specifically, in case of a semi-Weil space X which is supplied, in particular, with a Bianchi datum, we call such a space a *semi-Weil-Bianchi space,* one concludes (by supplementing thus Lemma 6.1) that:

(6.26) *every Maxwell field* $(\mathcal{L}, D)$ *on a semi-Weil-Bianchi space X determines an integral closed 2-form* (cf. e.g. (5.23)); this is accomplished through the curvature form of an A-connection of C, an eventual suitable translate of the given A-connection D.

It is to be noted here that, as in the classical theory, the previous (integral) cohomology class corresponds to a suitably defined *Chern class* of $\mathcal{L}$, when in the particular case considered $\boldsymbol{A}$ fulfils the appropriate conditions [31].

Now, our final aim is to obtain a converse of the above Lemma 6.1 (in effect, of (6.26) within the appropriate framework); but to this end we shall need the whole machinery provided by all the previous notions, considered so far. So we discuss it in the next section.

7. WEIL SPACES

As already said we get in this section a converse of Lemma 6.1 (in fact, of (6.26)); thus, its combination with the latter result will supply us with the desired form of Weil's theorem. So, for convenience, we first set the following.

Definition 7.1. *A pre-Weil space X which is a semi-Weil space, as well, is called a* Weil space.

Thus, in a Weil space X being, by definition, a *paracompact Hausdorffspace*(see Definition 5.1), one is supplied with a curvature datum (ibid.) and, in fact, with a *Bianchi datum* (cf. Scholium 6.1), due to the *exact sequence (5.4)*. Moreover, X being a semi-Weil space one is also endowed with **an** *exponential sheafdiagram*,like **(6.1)**.

The preceding framework is presently applied in the next result. That is, we obtain the following.

Lemma 7.1. *Let X be a Weil space and w an integral closed 2-form on X; i.e., we assume that*

$$w \in \Omega^2(X) \quad \text{with} \quad dw = 0 \tag{7.1}$$

and such that (cf. (5.8) **and** (5.23))

$$c(\omega) \in H^2(X, \mathbb{Z}) \tag{7.2}$$

Then, there exists a Maxwell field $(\mathcal{L}, \widetilde{D})$ on X, having w as the curvature form of the A-connection $\widetilde{D}$.

Proof. Based on (7.1) **and** Proposition 5.1, we conclude that

$$w = (d\,\theta_\alpha) \in (d\,\Omega^1)(X) \hookrightarrow \Omega^2(X) \tag{7.3}$$

such that

$$(\theta_\alpha) \in C^0(\mathcal{U}, \Omega^1). \tag{7.4}$$

Moreover, one has (see (5.20))

$$\delta(\theta_\alpha) = (\partial f_{\alpha\beta}) \in Z^1(\mathcal{V}, \partial\mathcal{A}) \hookrightarrow Z^1(\mathcal{V}, \Omega^1), \tag{7.5}$$

for **some** I-cochain

$$(f_{\alpha\beta}) \in C^1(\mathcal{V}, \mathcal{A}) \tag{7.6}$$

such that, in view of (5.21) and (7.2), one obtains

$$c(\omega) := \left[\delta\left(f_{\alpha\beta}\right)\right] \in H^2(X,\mathbb{Z}) \hookrightarrow H^2(X,\mathbb{C}) \tag{7.7}$$

In particular, we thus assume in virtue of (7.2) that

$$\delta\left(f_{\alpha\beta}\right) \equiv \left(\lambda_{\alpha\beta\gamma}\right) \in Z^2(\mathcal{V},\mathbb{Z}). \tag{7.8}$$

(In the previous argument we considered an eventual refinement $\mathcal{V}$ of the open covering $\mathcal{U}$ appeared in (7.4); in this respect, see also the comments after the relation (5.20) in the preceding).

Now, by applying (6.1), we set

$$\left(g_{\alpha\beta}\right) := e\left(\left(f_{\alpha\beta}\right)\right) = \left(e\left(f_{\alpha\beta}\right)\right) \in C^1\left(\mathcal{V},\mathcal{A}^{\bullet}\right) \tag{7.9}$$

(see also (7.6)); but on the strength of (7.8) and since, in view of (6.1), $\mathbb{Z} \cong \ker e$, one obtains

$$\delta\left(g_{\alpha\beta}\right) = \delta\left(e\left(f_{\alpha\beta}\right)\right) = e\left(\delta\left(f_{\alpha\beta}\right)\right) = 1. \tag{7.10}$$

Therefore, *one* has *in particular* that

$$\left(g_{\alpha\beta}\right) \in Z^1\left(\mathcal{V},\mathcal{A}^{\bullet}\right) \tag{7.11}$$

So we can now define (see also (4.6) for $n = 1$)

$$\mathcal{L} := \left[\left(g_{\alpha\beta}\right)\right] \in H^1\left(X,\mathcal{A}^{\cdot}\right) \cong \Phi^1_{\mathcal{A}}(X), \tag{7.12}$$

which we contend thus to be the desired *line sheaf* on X:

Namely, we first prove that $\mathcal{L}$ *admits an* $\mathcal{A}$*-connection*. Indeed, one obtains

$$\delta\left(\theta_\alpha\right) = (\text{cf. } (7.5))\ \left(\partial f_{\alpha\beta}\right) = (\text{by } (6.1) \text{ and } (7.9))\ \frac{1}{2\pi i}\tilde{\partial}\left(g_{\alpha\beta}\right). \tag{7.13}$$

Accordingly, concerning the Atiyah class of $\mathcal{L}$ (see (2.7) and (2.11)), one gets

$$D(\mathcal{L}) = \left[\left(D_\beta - D_\alpha\right)\right] = \left[\tilde{\partial}\left(g_{\alpha\beta}\right)\right] = \left[2\pi i\delta\left(\theta_\alpha\right)\right] = 0 \in H^1\left(X,\Omega^1\right) \tag{7.14}$$

Hence (cf. Theorem 2.1), C *admits a Levi-Civita* $\mathbf{A}$*-connection*, say $\widetilde{D}$. Now, modulo a suitable change of the Levi-Civita 1-cocycle $(D_\beta - D_\alpha)$ (see (2.5)) and an eventual translation of the resulting $\mathcal{A}$-connection via an element of $\Omega^1(X)$ (cf. also the comments before (2.21)), one may assume that $\widetilde{D}$ is locally given by the relation

$$\widetilde{D}|_{U_\alpha} \equiv \widetilde{D}_\alpha = D_\alpha - \theta_\alpha, \quad \alpha \in I \tag{7.15}$$

(By an obvious abuse of notation, we retained above, for simplicity, the same symbol for $\widetilde{D}$). Thus, $(\theta_\alpha) \in C^0(\mathcal{U}, \Omega^1)$, as given by (7.4), may be considered as the *local* $\mathbf{A}$*-connection 1-form representing* $\widetilde{D}$ *(see* e.g. (2.20), for n = 1. Yet, here again a «perturbation» as before of the $\mathcal{A}$-connection involved might be in sight, concerning (2.21)). Therefore, the pair

$$(\mathcal{L}, \widetilde{D}) \tag{7.16}$$

is a *Maxwell field* on X such that (Cartan's structural equation, see (3.24)) one has

$$R(\widetilde{D}) \equiv R = (d\theta_\alpha) = \text{(by (7.3))}\ \omega, \tag{7.17}$$

and this terminates the proof. ■

As a consequence of the preceding two Lemmas 6.1 and 7.1 (cf. also (6.26)), we are now in the position to state the following theorem. On the other hand, concerning the standard form of this classical result, see [45, p. 90, Lemma 2] or yet [19, p. 133, Proposition 2.1.1]. ***So*** we have.

Theorem **7.1.** ***(A.** Weil). Let X be a Weil space and ω a closed 2-form on X. Then, the two following assertions are equivalent:*

1) ω is integral; viz. it determines a (2-dimensional) cohomology class of X, say, $c(\omega) \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \mathbb{C})$.

2) ω ***is*** *the curvature form of an* $\mathbf{A}$*-connection of a line sheaf on X.* ■

As a matter of fact, that which we actually proved above is the following more general statement (see also Definition 7.1):

(7.18) *Every Maxwell field $(\mathcal{L}, D)$ on a semi-Weil-Bianchi space X yields* (through the curvature of a possible suitable translate of D) *an integral closed 2-form. In particular, if X is a Weil space then, conversely, the above* ***is*** *the only way that such forms arise.*

8. EXAMPLES

As already said we consider below a certain particular instance, where the previous context can be applied. Indeed, the point of view exhibited by the ensuing discussion was also the initial motive to the abstract (axiomatic) framework, as this was presented in the preceding. Yet, analogous considerations of S.A. Selesnick in the context of commutative unital *Banach algebras* (see e.g. [36]) triggered off, in effect, our initial study referred to the material that will be discussed in the sequel.

So, as we shall presently see, *the spectrum of a certain appropriate class of topological algebras is a semi-Weil-Bianchi space.* Now, conditions under which this could also be made into a Weil space are yet unclear (cf., for instance, (8.11) below). On the other hand, concerning the generai terminology and results on *topological algebras* that we apply below we refer to [25].

Thus. suppose we have a *commutative Pták regular semi-simple locally m-convex* (topologicai) *$\mathbb{C}$-algebra* $\mathbf{A}$, having an *identity element* and an *equicontinuous spectrum*

$$(8.1) \qquad \mathcal{M}(\mathbf{A}) \equiv X$$

(ibid.); now, it has been proved in [27] (cf. also [23, p. 488, theorem 6.11) that $\mathbf{A}$ is a *geometric algebra,* in the sense that one has

$$(8.2) \qquad \mathbf{A} = \Gamma(X, \mathcal{A}) \equiv \mathcal{A}(X),$$

within an *isomorphism of $\mathbb{C}$-algebras.* Here $\mathcal{A}$ stands for a suitable *agebra sheaf* (the *Gel'fand sheaf* of $\mathbf{A}$) over X (indeed, $\mathcal{A}$ is a *sheaf of topological algebras* on X, viz. a sheaf generated by a *topological algebra presheaf* on X; cf., for instance, [27] or yet [31]).

Now, following (within the above more generai framework) the reasoning of [36] (see also [35]), assume further that our topological algebra $\mathbf{A}$, as above, carries a *continuous involution* «*», with respect to which $\mathbf{A}$ is *self-adjoint*; by the latter term we mean that *the* corresponding *Gel'fand map of* $\mathbf{A}$ *is a* **-morphism* (see e.g. [25, p. 481 ff]).

Thus, by a *Selesnick algebra* we mean a commutative unital Pták regular semi-simple locally m-convex $\mathbb{C}$-algebra $\mathbf{A}$ which is also self-adjoint with respect to a continuous involution and has an equicontinuous spectrum $\mathcal{M}(\mathbf{A})$.

Of course, the hypothesis on a Selesnick algebra implies, among other things, that its spectrum $\mathcal{M}(\mathbf{A})$ is actually a *compact (Hausdorff) space* (cf. [25, p. 186, Corollary 1.51). So a Banach algebra of the previous type is, in particular, the *Fourier algebra* $A(G)$ *of a compact abelian group* G (cf. e.g. [35, p. 322, Theorem 3.31 and [37, p. 704]; see also, for instance, [34, p. 1041).

On the other hand, an important special instance of a Selesnick algebra (that *cannot* bc a normed one!), is the algebra

$$(8.3) \qquad \mathbf{A} = C^\infty(X),$$

of $\mathbb{C}$-valued C^∞-functions on a *compact* (Hausdorff) *2 nd countable smoolh* (viz. $C^\infty-$) *manifold* X, endowed with the respective C^∞-topology; thus, one gets $X = \mathcal{M}(\mathbf{A})$, within a *homeomorphism* (cf. [25, p. 227, Theorem 2.11. See also Example 1.1 in the foregoing).

Yet, another example of a Selesnic (*topological,* not necessarily normed) algebra from *function algebra theory,* is the algebra $\mathcal{O}(K)$ of $\mathbb{C}$-valued *«locally holomorphic»* functions on a *compacl Stein set* K of a *Stein manifold* X (in this regard, cf. [27]; see also [25, p. i34 ff]).

Thus, in connection with our considerations in the preceding, one can prove now the following.

(8.4) *the spectrum* $\mathcal{M}(\mathbf{A})$ *of any given Selesnick algebra* $\mathbf{A}$ *is a semi-Weil-Bianchi space*; here one employs the corresponding *Gel'-fand sheuf* $\mathcal{A}$ of $\mathbf{A}$ [27] and then the *de Rham – Kahler complex* (eventuaily not exact!) *of* $\mathcal{A}$ (see [28], [31]).

Namely, following [36] one proves the existence of an *exponential shcaf diagram* for $\mathcal{A}$, like (6.1) above. In this respect, one defines the *exponential sheafmorphism* $e : \mathcal{A} \to \mathcal{A}^\bullet$ by the relation

$$(8.5) \qquad e(s) := \exp 2\pi i s,$$

for any (local) section $s \in \mathcal{A}(U)$, U open in X; we note at this point that the previous relation makes sense, since *the exponential function «operates» on any complete* (in fact, *a-complete* is enough) *locally* m*-convex algebra* with *an identity element* (see e.g. [24, p. 492, (5.1)], as well as [28] or yet [31]).

Now, the corresponding «differential» framework is further established by employing a (sheaf-theoretic) *Kähler lheory of differentials for* $\mathcal{A}$. *So* the point of view developed in the preceding sections has here a special bearing; in this concern, cf. also, for instance, [28] and/or [31]. Thus, one gets in particular that:

(8.6) *for any Selesnick algebra* $\mathbf{A}$ *the conclusion of* (6.26) *holds good* (cf. also (7.18)), relative to its spectrum $\mathcal{M}(\mathbf{A})$.

In this regard, it is yet worth mentioning here that:

(8.7) *the Gel'fand sheuf* $\mathcal{A}$ *of any Pták-Šilov-Q* (for short, *PQŠ–*, so in particular of any *Selesnick algebra* $\mathbf{A}$ (see also [25]) *is always fine*

Cf. (271, [31] or yet [35, p. 319, Proposition 2.5, iii) for $p = 0$]. Therefore, given that *our base space* X, viz. the spectrum of **A** (cf. (8.1)), *is*, as already said above, *a compact (Hausdorff) space,* $\mathcal{A}$ *is acyclic* (cf. [5, p. 49, Theorem 9.81); hence, *the same holds true for every* $\mathcal{A}$*-module on* X (ibid., p. 50, Theorem 9.12). Thus, in view of Theorem 2.1, we further obtain that:

(8.8) *every vector sheaf on ihe spectrum* $\mathcal{M}(\mathbf{A})$ *of a* $PQ\check{S}$*-algebra* (hence, a fortiori of any Selesnick algebra) **A** *admiis an* $\mathcal{A}$*-connection, with* $\mathcal{A}$ *being the corresponding Gel'fand sheaf of* **A**.

Of course, in case of the algebra **A**, as in (8.3), the respective Gel'fand sheaf of **A** is the *structure sheaf* $\mathcal{C}_X^\infty$ of the manifold X under consideration (see also e.g. (1.21) above).

Scholium 8.1. In connection with the exact sequence in (5.4) one could consider, instead (in fact, more naturally!), the *cxaclness of thc following sequence (of shcaves of* $\mathbb{C}$*-vector spaces* on X)

$$(8.9) \qquad 0 \longrightarrow \ker\partial \longrightarrow \mathcal{A} \xrightarrow{\partial} \Omega^1 \xrightarrow{d} \Omega^2 \longrightarrow d\Omega^2 \longrightarrow 0$$

We recall that, in view of our hypothesis (cf. (1.8)), we always have $\mathbb{C} \xrightarrow{\subset} \ker\partial$. Thus, one gets an analogous result to Proposition 5.1 by an obvious rephrasing of (5.8); viz. one should then have

$$(8.10) \qquad c(\omega) \in H^2(X, \ker\partial),$$

for any closed 2-form on X. Now, the previous sequence *(8.9) provides,* of course, *exactness at* $\mathcal{A}$; however,

(8.11) *further exactness of (8.9), without, namely, postulating ii, might be in close conneciion with the very structure of a given topologicalalgebra* **A**, *whose Gel'fand sheaf is* $\mathcal{A}$ (take e.g. a Selesnick algebra, as above).

So in the important particular instance of the algebra (8.3), the exactness of the analogous sequence to (8.9) (and, in effect, to (5.4)) is, of course, a consequence of the *Poincaré Lemma* (see, for example, [9, p. 133, Example 2.5.1 and p. 181, Example 4.7.3] or yet [44, p. 190, (3) and p. 156, Corollary (a)], [43, p. 203]). Now, the latter depends on the topology of X, i.e., on that of the spectrum of (the topological algebra) **A**; hence, in turn (Šilov, Arens-Royden et al.), *on the structure of* **A**.

In this regard, an analogous study (however in case of discrete algebras) by S. Teleman (see e.g. [41], [42]), conceming an *abstract fonn of de Rham's theorem.* is here akin to the above problematic (speculation). Thus, the *locally acyclic algebras* of Teleman might give an example, when suitably topologized. In conclusion, we are thus in pursuit of:

(8.12) *topological algebras* **A** *satisfying* (8.2) (viz. *geometric* (topological) *algebras* [23, p.487; § 6], even *with acyclic Gel'fand sheaves)*, whose *spectra are Weil spaces in the modified sense of the sequence (8.9).*

So for an algebra **A** as above, by referring, in effect, to its Gel'fand sheaf $\mathcal{A}$, *the full fonn of Weil's theorem* (cf. Theorem 7.1), as modified now by the relation

$$c(\omega) \in H^2(X, \mathbb{Z}) \hookrightarrow H^2(X, \ker \partial), \tag{8.13}$$

would be in force.

Finally, within the same vein of thoughts, we further note that by considcring thc *polynomial algebra* **A** $\equiv \mathbb{C}[t_1, \ldots, t_n]$ in n variables, one gcts an example of **a** (discrete) $\mathbb{C}$-algebra for which the corresponding *de Rham-Kähler complex is exact.* Cf. [21, p. 612, Theorem **12.31.**

9. ELEMENTARY PARTICEES. PREQUANTIZATION

As a further outcome of the preceding, we give below a breezy discussion on certain thoughts to which we are led, in connection with a standard *application of Weil's integrality theorem to quantum mechanics* [19], [48] and an interpreiation of *elementary particles* in terms of *sections of vector sheaves.* Thus, our exposition here is aiming only at an indication of the interrelation of the above two facts, while we refer instead to [30] for further details.

So, according to an interpretation due to S.A. Selesnick (see e.g. [38]), *free elementary particles obeying Bose-Einstein statistics* (or yet *integral-spin* particles) may correspond to *mnk one projective* **A***-modules,* where **A** is given by **(8.3)** for a suitable (compact) C^∞-manifold (representing an empty finite universe). Furthermore, similar *modules of rank greater than one* may represent *free particles obeying Fermi-Dirak statistics.*

On the other hand, the C^∞-analogue of the classical *Serre-Swan theorem* (see, for instance, [23, p. 481, Theorem **4.21** for an ampie generalization of this result) identifies the previous modules with C^∞-(complex) vector bundles on X, having the respective dimension (rank). However, (C^∞–) *vector bundles on X* may, in tum, be iàentified with the corresponding *locally free sheaves* (of their C^∞-sections) , these being too *of finite rank,* that is with *vector sheaves on X* (here the *«coefficient sheaf»* $\mathcal{A}$ is given by (1.21); in this respect, see also e.g. [26, p. 406, Theorem 1.1]).

Now. it is also standard to consider (cf., for instance [38] or yet [7, pp. 364, 3751) the *field strength* of a given elementary particle as represented by the *curvature form* of an a p propriately defined connection form (*«gauge potential»*; see also e.g. (2.3). We recall that connections here always exist, yet globally, $\mathcal{A} \equiv C_X^\infty$ being a *fine sheaf* on the compact X; cf. also, for instance, Theorem 2.1 in the preceding). Accordingly one comes to the conclusion that:

(9.1) *every free boson (field) corresponds to a Maxwell field* $(\mathcal{L}, D)$ *on* X *which thus yields,* through its field strength (cf. (6.26) or yet (7.18)), *an integral cohomology class*. Therefore (by further employing a standard prequantization argument [48] one obtains that) *every such elementary particle is pre-quantizable.*

Furthermore, by viewing (free) *fermions* as corresponding to (sections of) Ω^1 (see [38], as well as (1.23) above), one may further consider a Yang-Mills field (Ω^1, D) (cf. also (1.26)) as representing the free fermion (field) involved. In this regard, the connection D can be derived by the effect on Ω^1 of a free boson, say $(\mathcal{L}, \widetilde{D})$ (*«mediating forces by the exchange of bosons»*); that is, mathematically speaking, we let the boson in question «act» on Ω^1 by tensoring, so that one has

$$\Omega^1 \otimes_{\mathcal{A}} \mathcal{L} \cong \Omega^1, \quad \textit{locally } (!). \tag{9.2}$$

Thus, the presence here of the (auxiliary) free boson $\mathcal{L}$ is *locally undiscernible*, its contribution being that it provides us, via its force (curvature of $\widetilde{D}$), with a (pre-)quantizing line sheaf for Ω^1. As a matter of fact, D yiels an integrai cohomology class, as well, by *pulling-back* that of $\widetilde{D}$ (see e.g. [9, p. 200], and also (7.18) above). Therefore, one again obtains that:

(9.3) *every free fermion is pre-quantizable.* Hence, by virtue of (9.1), we conclude that *the same holds good for every free elementary particle.*

So according to the preceding one gets already from the outset a *«pre-quantizing fine bundle* (or rather a *line sheaf*)*»* without thus to have first to look for the appropriate Hamiltonian framework. This is, of course, in agreement with the point of view of geometric quantization theory (see, for instance, [40]); yet, it might also be viewed as a further justification of the claim that *«quantization is provided by the physical law itself»* (cf. [47, p. 3231). Furthermore, one realizes that *sheaf cohomology* lies at the basis of the mathematical structure involved in field quantities, which are used to describe a *«geometrized field theory»*. In this respect, see also [32] and/or [2], [39].

APPENDIX

Based on Scholium 5.1 and on the proofs of Lemmas 6.1 and 7.1, we can get the following variant of Weil's Theorem 7.1; conceming the terminology applied below see also the previous section **9.** Thus, we have the next.

Theorem. *Let* X *be a paracompact Hausdorff space enabwed wiih a curvature (Bianchi) datum* (cf. (6.24))

$$(\mathcal{A}, \partial, \Omega^1, d, \Omega^2),$$

where Ω^1 *is a vector sheaf on* X *, such that thefollowing sequence is exact*

$$0 \longrightarrow \mathbb{C} \xrightarrow{\varepsilon} \mathcal{A} \xrightarrow{\partial} \Omega^1 \xrightarrow{d} d\,\Omega^1 \longrightarrow 0 .$$

Moreover, suppose that we are given the next commutative exponential sheaf diagram;

$$\begin{array}{ccccccccc} 0 \longrightarrow \mathbb{Z} \longrightarrow & \mathcal{A} & \xrightarrow{e} & \mathcal{A}^{\bullet} & \longrightarrow 1 \\ & \partial \searrow & & \swarrow \frac{1}{2\pi i}\tilde{\partial} & \\ & & \Omega^1 & & \end{array}$$

Then, the only integral (closed 2-)forms on X *of the type* w E $Z^0(\mathcal{U}, d\,\Omega^1)$ *are field strengths of Maxwell fields on* X . ∎

The previous result reinforces Theorem 7.1 regarding the exact sequence (5.4), however, this at the cost of the type of the (ciosed) 2-forms on X considered; on the other hand, the particular type of these forms is aiways occurred in case of *line sheaves,* as a result of *Cartan's structural equation (see* (3.24)).

Remarks (added in proof). (i) By suitably modifying the proof **of** Theorem 2.1, we can dispense with the hypothesis that Ω^1 is a vector sheaf on X ; namely, *it suffices to be* just *an* $\mathcal{A}$*-module on* X, asin $(\mathcal{A}, \partial, \Omega^1)$.
(ii) Conceming the Theorem in the Appendix, one can finally state the following (more amply realizable, see below) version of *Weil's Theorem:* namely, *for any semi-Weil Bianchi space* X, *for which* $H^1(X, \mathcal{A}) = H^2(X, \mathcal{A}) = H^1(X, \Omega^1) = 0$, *any* $z \in H^2(X, \mathbb{Z})$ *is of the form* $z = z(w)$, *where* w *is the curvature form of an* $\mathcal{A}$*-connection* **of** *a line sheaf on* X *; i.e.,* w **E** $\Omega^2(X)$, *suchthatw = (do,), with* (θ_α) E $C^0(\mathcal{U}, \Omega^1)$. *And conversely,* the curvature **form** of **an** $\mathcal{A}$-connection of a line sheaf on X is of the above type. **As** an application, one can consider (in a canonical way) as X *the spectrum* **of** *any Selesnick algebra or* yet *that one of any self-adjoint Fréchet-Šilov algebra* (non-compact spectrum; *see* also e.g. [28]).

REFERENCES

[1] M.F. ATIYAH, *Complex analytic connections in fiber bundles*. Trans. Amer. Math. Soc., **85** (1957). pp. 181-207.

[2] S.J. AVIS, C.J. ISHAM, *Quantum field iheory and fiber bundles in general space-time*, in: «Recent Developments in Gravitation», Cargèse 1978, 347-401; M. Lévy - S. Deser (eds.), Plenum Press, New York, 1979.

[3] D. BLEECKER, *Gauge iheory and variaiional principles*. Addison-Wesley. Reading , Mass.. 1981.

[4] N. BOURBAKI, *Théorie des ensembles*, chap. 3, Hermann, Paris. 1967.

[5] G.L. BRENDON, *Sheafiheory*. McGraw-Hill, New York. 1967.

[6] N.P. BUCHDAHL, *Analysis on analytic spaces and non-self-dual Yang-Mills fields*, Trans. Amer. Math. Soc., ***288*** (1985), pp. 431-469.

[7] Y. COQUET-BRUHAT, C. DE WITT-MORETTE, M. DILLARD-BLEICK, *Analysis, manifolds and physics*, (revised ed.), North-Holland, Amsterdam, 1982.

[8] J.L. DUPONT, *Curvature and characteristic classes*, Lecture Notes in Mathemaiics. **640**, Springer-Verlag. Berlin. 1978.

[9] R. GODEMENT, *Topologie algébrique et théorie des faisceaux*, (3 ème éd.), Hermann, Paris, 1973.

[10] H. GRAUERT, R. REMMERT, *Theory of Siein spaces*. Springer-Verlag, Berlin, 1979.

[11] A. GROTHENDIECK, J.A. DIEUDONNÉ, *Éléments de géométrie algébrique, I*, Springer-Verlag, Berlin, 1971.

[12] R.C. GUNNING, *Leciures on Riemann surfaces*, Princeion Univ. Press. Princeion, NJ, 1966.

[13] R.C. GUNNING, *Lectures on vector bundles over Riemann surfaces*, Princeton Univ. Press. Princeton. NJ, 1%7.

[14] F. HIRZEBRUCH, *Topological methods in algebraic geomeiry*. (3rd ed.), Springer-Verlag. Berlin. 1978.

[15] L. HÖRMANDER, *An iniroduciion io complex analysis in several variables*, (2nd. ed.). North-Holland, Amsterdam. 1973.

[16] N. JACOBSON, *Basic algebra, II*, W.H. Freeman, San Francisco, Ca., 1980.

[17] M. KAROUBI, C. LERUSTE, *Algebraic topology via differential geomeiry*, Cambridge Univ. Press. Cambridge, 1987.

[18] L. KAUP, B. KAUP, *Holomorphic functions of several variables*, W. de Gruyter, Berlin, 1983.

[19] B. KOSTANT, *Quantization and unitary representations*, in: «Lectures in Modern Analysis and Applications, II»; Lecture Notes in Mathematics, **170** (1970). pp. 87-208, Springer-Verlag, Berlin.

[20] J.L. KOSZUL, *Leciures onfiber bundles and differential geomeiry*, Tata Inst. Fund. Res., Bombay. 1960.

[21] S. LANG, *Algebra*, (2nd ed.), Addison-Wesley, Rcading, Mass.. 1984.

[22] S. LANG, *Introduction ioArakelov Theory*. Springer-Verlag. New York, 1988.

[23] A. MALLIOS, *Vector bundles and K-iheory over iopological algebras*. J. Math. Anal. **Appl.. 92** (1983). pp. 452-506.

[24] A. MALLIOS, *Hermitian K-iheory over iopological *-algebras*, J. Math. Anal. Appl.. **106** (1985). pp. 454-539.

[25] A. MALLIOS, *Topological algebras: selected topics*, North-Holland, Amsterdam. 1986.

[26] A. MALLIOS, *Continuous vector bundles over iopological algebras, II*, J. Math. Anal. **Appl.. 132** (1988). pp. 401-423.

[27] A. MALLIOS, *Topological algebras: localizations and extensions*, (in preparaiion).

[28] A. MALLIOS, *On ihe de Rhan-Kahler complex of ihe Gel'fand sheaf of a topological algebra*. J. Math. Anal. Appl. 175, 1993. 143-168.

[29] A. MALLIOS, (i) *On ihe existence of A-conneciions*. abstracts Amer. Math. Soc., **9** (6) (1988). p. 509. (ii) ***A*** *- conneciioni as splitting extensions*, ibid., **10** (2) (1989). p. 186. (iii) *On an abstract form of Weil's integrality iheorem*. ibid.. **10** (4) (1989), p. 312. (iv) *On ihe curvature of an A-connection*, ibid., 10 (5) (1989), p. 410. (v) *Local form of A-connections. Structural equations*, ibid.. **11** (4) (1990). p. 354.

[30] A. MALLIOS, *Vector sheaves and second quantization*, (to appear).

[31] A. MALLIOS, *Geometry of vector sheaves*, (in preparation).

[32] YU.I. MA", *Gauge field theory and complex geometry*, Springer-Verlag. Berlin, 1988.

[33] J.W. MILNOR, J.D STASHEFF, *Characteristic classes*, Princeton Univ. Press, Princeton, NJ, 1974.

[34] H. REITER, *Classical harmonic analysis and locally compacf groups.* Oxford Univ. Press. Oxford. 1968.

[35] S.A. SELESNICK, *Watts cohomology for a class of Banach algebra and the duality of compacf abelian groups*, Math. Z., 130 (1973), pp. 313-323.

[36] S.A. SELESNICK, *Line bundles and harmonic analysis on compacf groups*, Math. Z., **146** (1976). pp. 53-67.

[37] S.A. SELESNICK, Rank *one projecfive modules over certain Fourier algebra*, in «Applications of Sheaves», Proc. Res. Symp., Darham (1977), pp. 702-713; Lecture Notes in Mathematics, 753 (1971), Springer-Verlag, Berlin.

[38] S.A. SELESNICK, *Second quantization, projecfive modules, and local gauge invariance*, Intem. *J. Theory* Phys.. **22** (1983), pp. 29-53.

[39] S.A. SELESNICK, *Geometrical symmetry breaking*, Nuovo Cimento. **83A** (1984), pp. 175-190.

[40] D.J. SIMMS, N.M.J. WOODHOUSE, *Lectures on geometric quantization*, Lecture Notes in Physics. 53 (1976), Springer-Verlag. Berlin.

[41] S. TELEMAN, *Theory of harmonic algebras with applications to von Neumann algebras and cohomology of locally compacf spaces (de Rham's fheorem)*, in «Lectures on the Applications of Sheaves to Ring Theory», pp. 99-315; Lecture Notes in Mathematichs. **248** (1971), Springer-Verlag, Berlin.

[42] S. TELEMAN, *The fheorem of de Rham for harmonic algebras*, J. Algebra, 23 (1972), pp. 271-290.

[43] I. VAISMAN, *Cohomology and differenfinl form*, Marcell Dekkem New York, 1973.

[44] EW. WARNER, *Foundations of differentiable manifolds and lie groups*, Springer-Verlag. New York, 1983.

[45] A. WEIL, *Introduction à l'étude des variétés kählériennes*, (nouv. éd.), Hermann, Paris, 1971.

[46] R.O. WELLS, *Differential analysis on complex manifolds*, Springer-Verlag, New York, 1980.

[47] C. VON WESTENHOLZ, *Differential form in mathematical physics*, (revised ed.), North-Holland, Amsterdam. 1981.

[48] N. WOODHOUSE, *Geometric quantization*, Oxford Univ. Press, Oxford. 1980.

Received September 12,1990

A. Mallios
Mathematical Institute
University of Athens
Panepistimiopolis
Athens 15784
Greece