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On the Dooley-Rice contraction of the principal series

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Abstract. In [A. H. Dooley and J. W. Rice: *On contractions of semisimple Lie groups*, Trans. Am. Math. Soc., **289** (1985), 185–202], Dooley and Rice introduced a contraction of the principal series representations of a non-compact semi-simple Lie group to the unitary irreducible representations of its Cartan motion group. We study here this contraction by using non-compact realizations of these representations.

Keywords: Contraction of Lie groups; contraction of representations; semi-simple Lie group; semi-direct product; Cartan motion group; unitary representation; principal series; coadjoint orbit.

MSC 2010 classification: primary 22E46, secondary 22E45, 22E70, 22E15

Introduction

Since the pioneering paper of Inönü and Wigner [19], the contractions of Lie group representations have been studied by many authors, see in particular [27], [24], [14], [11] and [12].

In [16], Dooley and Rice introduced an important contraction of the principal series of a non-compact semi-simple Lie group to the unitary irreducible representations of its Cartan motion group which recovered many known examples and also illustrated the Mackey Analogy between semi-simple Lie groups and semi-direct products [23].

In [15], Dooley suggested interpreting contractions of representations in the setting of the Kirillov-Kostant method of orbits [20], [22] and, in [13], Cotton and Dooley showed how to recover contraction results by using the Weyl correspondence. In this spirit, we established in [7] a contraction of the discrete series of a non-compact semi-simple Lie group to the unitary irreducible representations of a Heisenberg group (see also [26], [3], [6] and [4]) and, in [10], we study the Dooley-Rice contraction of the principal series at the infinitesimal level. We also refer the reader to [17] for recent developpements on contractions of representations.

Let G be a non-compact semi-simple Lie group with finite center and let K be a maximal compact subgroup of G. The corresponding Cartan motion group G_0 is the semi-direct product $V \rtimes K$, where V is the orthogonal complement of the Lie algebra of K in the Lie algebra of G with respect to the Killing form. In [16], the Dooley-Rice contraction of the principal series was established by using the compact picture for the principal series representations [21], p. 169. Here, we present the analogous contraction results in the non-compact picture.

This note is organized as follows. In Section 1 and Section 2, we introduce the non-compact realizations of the representations of the principal series of G and of the generic unitary irreducible representations of G_0, following [9] and [10]. In Section 3, we study the Dooley-Rice contraction of the principal series in the non-compact picture. Our main result is then Proposition 4 which is analogous to Theorem 1 in [16]. Finally, in Section 4, we establish similar results for the corresponding derived representations.

1 Principal series representations

In this section, we introduce the non-compact realization of a principal series representation. We follow the exposition of [9] and [10] which is mainly based on [21], Chapter 7, [30], Chapter 8 and [18], Chapter VI. We use standard notation.

Let G be a connected non-compact semi-simple real Lie group with finite center. Let $\mathfrak{g}$ be the Lie algebra of G. We can identify G-equivariantly $\mathfrak{g}$ to its dual space $\mathfrak{g}^*$ by using the Killing form β of $\mathfrak{g}$ defined by $\beta(X,Y) = \mathrm{Tr}(\mathrm{ad}\, X\, \mathrm{ad}\, Y)$ for X and Y in $\mathfrak{g}$. Let θ be a Cartan involution of $\mathfrak{g}$ and let $\mathfrak{g} = \mathfrak{k} \oplus V$ be the corresponding Cartan decomposition of $\mathfrak{g}$. Let K be the connected compact (maximal) subgroup of G with Lie algebra $\mathfrak{k}$. Let $\mathfrak{a}$ be a maximal abelian subalgebra of V and let M be the centralizer of $\mathfrak{a}$ in K. Let $\mathfrak{m}$ denote the Lie algebra of M. We can decompose $\mathfrak{g}$ under the adjoint action of $\mathfrak{a}$:

$$\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{m} \oplus \sum_{\lambda \in \Delta} \mathfrak{g}_\lambda$$

where Δ is the set of restricted roots. We fix a Weyl chamber in $\mathfrak{a}$ and we denote by Δ^+ the corresponding set of positive roots. We set $\mathfrak{n} = \sum_{\lambda\in\Delta^+} \mathfrak{g}_\lambda$ and $\bar{\mathfrak{n}} = \sum_{\lambda\in\Delta^+} \mathfrak{g}_{-\lambda}$. Then $\bar{\mathfrak{n}} = \theta(\mathfrak{n})$. Let A, N and $\bar{N}$ denote the analytic subgroups of G with algebras $\mathfrak{a}$, $\mathfrak{n}$ and $\bar{\mathfrak{n}}$. We fix a regular element ξ_1 in $\mathfrak{a}$, that is, $\lambda(\xi_1) \neq 0$ for each $\lambda \in \Delta$ and an element ξ_2 in $\mathfrak{m}$. Let $\xi_0 = \xi_1 + \xi_2$. Denote by $O(\xi_0)$ the orbit of ξ_0 in $\mathfrak{g}^* \simeq \mathfrak{g}$ under the (co)adjoint action of G and by $o(\xi_2)$ the orbit of ξ_2 in $\mathfrak{m}$ under the adjoint action of M.

Consider a unitary irreducible representation σ of M on a complex (finite-dimensional) vector space E. Henceforth we assume that σ is associated with the orbit $o(\xi_2)$ in the following sense, see [31], Section 4. For a maximal torus T of M with Lie algebra $\mathfrak{t}$, $i\beta(\xi_2, \cdot) \in i\mathfrak{t}^*$ is a highest weight for σ.

Now we consider the unitarily induced representation

$$\hat{\pi} = \mathrm{Ind}_{\mathrm{MAN}}^{\mathrm{G}} \left(\sigma \otimes \exp(\mathrm{i}\nu) \otimes 1_{\mathrm{N}}\right)$$

where $\nu = \beta(\xi_1, \cdot) \in \mathfrak{a}^*$. The representation $\hat{\pi}$ lies in the unitary principal series of G and is usually realized on the space $L^2(\bar{N}, E)$ which is the Hilbert space completion of the space $C_0^\infty(\bar{N}, E)$ of compactly supported smooth functions $\phi : \bar{N} \to E$ with respect to the norm defined by

$$\|\phi\|^2 = \int_{\bar{N}} \langle \phi\,(y), \phi(y) \rangle_E \, dy$$

where dy is the Haar measure on $\bar{N}$ normalized as follows. Let $(E_i)_{1 \le i \le n}$ be an orthonormal basis for $\bar{\mathfrak{n}}$ with respect to the scalar product defined by $(\,Y, Z\,) := -\beta(Y, \theta(Z))$. Denote by $(Y_1, Y_2, \ldots, Y_n)$ the coordinates of $Y \in \bar{\mathfrak{n}}$ in this basis and let $dY = dY_1 dY_2 \ldots dY_n$ be the Euclidean measure on $\bar{\mathfrak{n}}$. The exponential map exp is a diffeomorphism from $\bar{\mathfrak{n}}$ onto $\bar{N}$ and we set $dy = \log^*(dY)$ where $\log = \exp^{-1}$.

Note that $\hat{\pi}$ is associated with $O(\xi_0)$ by the method of orbits, see [2] and [5].

Recall that $\bar{N}MAN$ is a open subset of G whose complement has Haar measure zero. We denote by $g = \bar{n}(g)m(g)a(g)n(g)$ the decomposition of $g \in \bar{N}MAN$. For $g \in G$ the action of the operator $\hat{\pi}(g)$ is given by

$$\big(\hat{\pi}(g)\phi\big)(y) = e^{-(\rho + i\nu)\log a(g^{-1}y)} \sigma\big(m(g^{-1}y)\big)^{-1} \phi\big(\bar{n}(g^{-1}y)\big) \tag{1.1}$$

where $\rho(H) := \frac{1}{2}\operatorname{Tr}_{\bar{\mathfrak{n}}}(\operatorname{ad} H) = \frac{1}{2}\sum_{\lambda \in \Delta^+} \lambda$.

Recall that we have the Iwasawa decomposition $G = KAN$. We denote by $g = \tilde{k}(g)\tilde{a}(g)\tilde{n}(g)$ the decomposition of $g \in G$.

In order to simplify the study of the contraction, we slightly modify the preceding realization of $\hat{\pi}$ as follows. Let I be the unitary isomorphism of $L^2(\bar{N}, E)$ defined by

$$(I\phi)(y) = e^{-i\nu(\log \tilde{a}(y))}\phi(y).$$

Then we introduce the realization π of $\hat{\pi}$ defined by $\pi(g) := I^{-1}\hat{\pi}(g)I$ for each $g \in G$. We immediately obtain

$$\big(\pi(g)\phi\big)(y) = e^{i\nu(\log \tilde{a}(y) - \log \tilde{a}(\bar{n}(g^{-1}y))} e^{-(\rho + i\nu)\log a(g^{-1}y)} \sigma\big(m(g^{-1}y)\big)^{-1} \phi\big(\bar{n}(g^{-1}y)\big). \tag{1.2}$$

For $g \in G$ and $y \in \bar{N}$, we have

$$\begin{aligned} g^{-1}y &= \bar{n}(g^{-1}y)m(g^{-1}y)a(g^{-1}y)n(g^{-1}y) \\ &= \tilde{k}(\bar{n}(g^{-1}y))\tilde{a}(\bar{n}(g^{-1}y))\tilde{n}(\bar{n}(g^{-1}y))m(g^{-1}y)a(g^{-1}y)n(g^{-1}y). \end{aligned}$$

Then we get

$$\tilde{a}(g^{-1}y) = \tilde{a}(\bar{n}(g^{-1}y))a(g^{-1}y).$$

Hence we obtain

$$\begin{aligned} \big(\pi(g)\phi\big)(y) = e^{i\nu(\log \tilde{a}(y) - \log \tilde{a}(g^{-1}y))} e^{-\rho(\log a(g^{-1}y))} \sigma\big(m(g^{-1}y)\big)^{-1} \\ \phi\big(\bar{n}(g^{-1}y)\big). \end{aligned} \tag{1.3}$$

Now, we compute the derived representation $d\pi$. We introduce some additional notation. If H is a Lie group and X is an element of the Lie algebra of H then we denote by X^+ the right-invariant vector field generated by X, that is, $X^+(h) = \frac{d}{dt}(\exp(tX))h|_{t=0}$ for $h \in H$. We denote by $p_{\mathfrak{a}}$, $p_{\mathfrak{m}}$ and $p_{\bar{\mathfrak{n}}}$ the projection operators of $\mathfrak{g}$ on $\mathfrak{a}$, $\mathfrak{m}$ and $\bar{\mathfrak{n}}$ associated with the decomposition $\mathfrak{g} = \bar{\mathfrak{n}} \oplus \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. Moreover, we also denote by $\tilde{p}_{\mathfrak{a}}$ the projection operator of $\mathfrak{g}$ on $\mathfrak{a}$ associated with the decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$. By differentiating the multiplication map $\bar{N} \times M \times A \times N \to \bar{N}MAN$, we easily get the following lemma, see [5].

Lemma 1. *1) For each $X \in \mathfrak{g}$ and each $y \in \bar{N}$, we have*

$$\begin{aligned} &\frac{d}{dt}\, a(\exp(tX)y)|_{t=0} = p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})X) \\ &\frac{d}{dt}\, m(\exp(tX)y)|_{t=0} = p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})X) \\ &\frac{d}{dt}\, \bar{n}(\exp(tX)y)|_{t=0} = \big(\mathrm{Ad}(y)\, p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\, X)\big)^+ (y). \end{aligned}$$

2) For each $X \in \mathfrak{g}$ and each $g \in G$, we have

$$\frac{d}{dt}\, \tilde{a}(\exp(tX)g)|_{t=0} = \Big(\tilde{p}_{\mathfrak{a}}\big(\mathrm{Ad}(\tilde{k}(g)^{-1})X\big)\Big)^+ (\tilde{a}(g)).$$

From this lemma, we easily obtain the following proposition.

Proposition 1. *For $X \in \mathfrak{g}$, $\phi \in C^\infty(\bar{N}, E)$ and $y \in \bar{N}$, we have*

$$\begin{aligned}(d\pi(X)\phi)(y) &= i\nu\left(\tilde{p}_{\mathfrak{a}}\big(\mathrm{Ad}(\tilde{k}(y)^{-1})X\big)\right)\phi(y)\\ &+ \rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})X)\right)\phi(y) + d\sigma\left(p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})X)\right)\phi(y)\\ &- d\phi(y)\left(\mathrm{Ad}(y)\,p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,X)\right)^{+}(y).\end{aligned}$$

2 Generic representations of the Cartan motion group

In this section we review some results from [9] and [10]. Recall that we have the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus V$ where V is the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to the Killing form β. We denote by $p^c_{\mathfrak{k}}$ and p^c_V the projections of $\mathfrak{g}$ on $\mathfrak{k}$ and V associated with the Cartan decomposition.

We can form the semi-direct product $G_0 := V \rtimes K$. The multiplication of G_0 is given by

$$(v,k).(v',k') = (v + \mathrm{Ad}(k)v', kk')$$

for v, v' in V and k, k' in K. The Lie algebra $\mathfrak{g}_0$ of G_0 is the space $V \times \mathfrak{k}$ endowed with the Lie bracket

$$[(w,U),(w',U')]_0 = ([U,w'] - [U',w], [U,U'])$$

for w, w' in V and U, U' in $\mathfrak{k}$.

Recall that β is positive definite on V and negative definite on $\mathfrak{k}$ [18], p. 184. Then, by using β, we can identify V^* to V and $\mathfrak{k}^*$ to $\mathfrak{k}$, hence $\mathfrak{g}_0^* \simeq V^* \times \mathfrak{k}^*$ to $V \times \mathfrak{k}$. Under this identification, the coadjoint action of G_0 on $\mathfrak{g}_0^* \simeq V \times \mathfrak{k}$ is then given by

$$(v,k)\cdot(w,U) = (\mathrm{Ad}(k)w, \mathrm{Ad}(k)U + [v, \mathrm{Ad}(k)w])$$

for v, w in V, k in K and U in $\mathfrak{k}$, see [25].

The coadjoint orbits of the semi-direct product of a Lie group by a vector space were described by Rawnsley in [25]. For each $(w,U) \in \mathfrak{g}_0^* \simeq \mathfrak{g}_0$, we denote by $O(w,U)$ the orbit of (w,U) under the coadjoint action of G_0. In [9], we proved the following lemma.

Lemma 2. *1) Let $\mathcal{O}$ be a coadjoint orbit for the coadjoint action of G_0 on $\mathfrak{g}_0^* \simeq \mathfrak{g}_0$. Then there exists an element of $\mathcal{O}$ of the form (ξ_1, U) with $\xi_1 \in \mathfrak{a}$. Moreover, if ξ_1 is regular then there exists $\xi_2 \in \mathfrak{m}$ such that $(\xi_1,\xi_2) \in \mathcal{O}$.*

2) Let ξ_1 be a regular element of $\mathfrak{a}$. Then M is the stabilizer of ξ_1 in K.

In the rest of this note, we consider the orbit $O(\xi_1,\xi_2)$ of $(\xi_1,\xi_2)\in\mathfrak{a}\times\mathfrak{m}\subset\mathfrak{g}_0^*\simeq\mathfrak{g}_0$ under the coadjoint action of G_0. As in Section 1, we assume that ξ_1 is a regular element of $\mathfrak{a}$ and that the adjoint orbit $o(\xi_2)$ of ξ_2 in $\mathfrak{m}$ is associated with a unitary irreducible representation σ of M which is realized on a (finite-dimensional) Hilbert space E. Then $O(\xi_1,\xi_2)$ is associated with the unitarily induced representation

$$\hat{\pi}_0=\mathrm{Ind}_{\mathrm{V}\times\mathrm{M}}^{\mathrm{G}_0}\left(\mathrm{e}^{\mathrm{i}\nu}\otimes\sigma\right)$$

where $\nu=\beta(\xi_1,\cdot)\in\mathfrak{a}^*$ (see [22] and [25]). By a result of Mackey, $\hat{\pi}_0$ is irreducible since σ is irreducible [29]. We say that the orbit $O(\xi_1,\xi_2)$ is *generic* and that the associated representation $\hat{\pi}_0$ is *generic.*

Let $O_V(\xi_1)$ be the orbit of ξ_1 in V under the action of K. We denote by μ the K-invariant measure on $O_V(\xi_1)\simeq K/M$. We denote by $\tilde{\pi}_0$ the usual realization of $\hat{\pi}_0$ on the space of square-integrable sections of a Hermitian vector bundle over $O_V(\xi_1)$ [22], [28], [25]. Let us briefly describe the construction of $\tilde{\pi}_0$. We introduce the Hilbert G_0-bundle $L:=G_0\times_{e^{i\nu}\otimes\sigma}E$ over $O_V(\xi_1)\simeq K/M$. Recall that an element of L is an equivalence class

$$[g,u]=\{(\,g.(v,m),e^{-i\nu(v)}\sigma(m)^{-1}u)\mid v\in V,\,m\in M\,\}$$

where $g\in G_0$, $u\in E$ and that G_0 acts on L by left translations: $g\,[g',u]:=[gg',u]$. The action of G_0 on $O_V(\xi_1)\simeq K/M$ being given by $(v,k).\xi=\mathrm{Ad}(k)\xi$, the projection map $[(v,k),u]\to\mathrm{Ad}(k)\xi_1$ is G_0-equivariant. The G_0-invariant Hermitian structure on L is given by

$$\langle[g,u],[g,u']\rangle=\langle u,u'\rangle_E$$

where $g\in G_0$ and u, $u'\in E$. Let $\mathcal{H}_0$ be the space of sections s of L which are square-integrable with respect to the measure μ, that is,

$$\|s\|^2_{\mathcal{H}_0}=\int_{O_V(\xi_1)}\langle s(\xi)\,,\,s(\xi)\rangle\,d\mu(\xi)<+\infty.$$

Then $\tilde{\pi}_0$ is the action of G_0 on $\mathcal{H}_0$ defined by

$$(\tilde{\pi}_0(g)\,s)(\xi)=g\,s(g^{-1}.\xi).$$

Now we introduce a non-compact realization of $\hat{\pi}_0$. We consider the map $\tau:y\to\mathrm{Ad}(\tilde{k}(y))\xi_1$ which is a diffeomorphism from $\bar{N}$ onto a dense open subset of $O_V(\xi_1)$ [30], Lemma 7.6.8. We denote by $k\cdot y$ the action of $k\in K$ on $y\in\bar{N}$ defined by $\tau(k\cdot y)=\mathrm{Ad}(k)\tau(y)$ or, equivalently, by $k\cdot y=\bar{n}(ky)$. Then the K-invariant measure on $\bar{N}$ is given by $(\tau^{-1})^*(\mu)=e^{-2\rho(\log\tilde{a}(y))}dy$ [30], Lemma 7.6.8. We associate with each $s\in\mathcal{H}_0$ the function $\phi_s:\bar{N}\to E$ defined by

$$s(\tau(y))=[(0,\tilde{k}(y))\,,\,e^{\rho(\log\tilde{a}(y))}\phi_s(y)].$$

We can easily verify that $J : s \to \phi_s$ is a unitary operator from $\mathcal{H}_0$ to $L^2(\bar{N}, E)$ and we set $\pi_0(v,k) := J\tilde{\pi}_0(v,k)J^{-1}$ for $(v,k) \in G_0$. Then we obtain

$$(\pi_0(v,k)\phi)(y) = e^{-\rho(\log a(k^{-1}y))+i\beta(\mathrm{Ad}(\tilde{k}(y))\xi_1,v)}\,\sigma\left(m(k^{-1}y)\right)^{-1}\phi\left(\bar{n}(k^{-1}y)\right), \tag{2.1}$$

see [10].

The computation of $d\pi_0$ is similar to that of $d\pi$. By using Lemma 1, we obtain the following proposition.

Proposition 2. *For $(v,U) \in \mathfrak{g}_0$, $\phi \in C^\infty(\bar{N}, E)$ and $y \in \bar{N}$, we have*

$$\begin{aligned}(d\pi_0(v,U)\phi)(y) &= i\beta\left(\mathrm{Ad}(\tilde{k}(y))\xi_1, v\right)\phi(y)\\ &+ \rho\left(p_{\mathfrak{a}}(\mathrm{Ad}(y^{-1})U)\right)\phi(y) + d\sigma\left(p_{\mathfrak{m}}(\mathrm{Ad}(y^{-1})U)\right)\phi(y)\\ &- d\phi(y)\left(\mathrm{Ad}(y)\,p_{\bar{\mathfrak{n}}}(\mathrm{Ad}(y^{-1})\,U)\right)^+(y).\end{aligned}$$

3 Contraction of group representations

Let us consider the family of maps $c_r : G_0 \to G$ defined by

$$c_r(v,k) = \exp(rv)\,k$$

for $v \in V$, $k \in K$ and indexed by $r \in]0,1]$. One can easily show that

$$\lim_{r\to 0} c_r^{-1}(c_r(g)\,c_r(g')) = g\,g'$$

for each g, g' in G_0. Then the family (c_r) is a group contraction of G to G_0 in the sense of [24].

Let $(\xi_1, \xi_2) \in \mathfrak{g}_0$ as in Section 2. Recall that π_0 is a unitary irreducible representation of G_0 associated with (ξ_1, ξ_2). For each $r \in]0,1]$, we set $\xi_r := (1/r)\xi_1 + \xi_2$ and we denote by π_r the principal series representation of G corresponding to $O(\xi_r)$.

We have to take into account some technicalities due to the fact that the projection maps a, m and $\bar{n}$ are not defined on G but just on $\bar{N}MAN$. We begin by the following lemma.

Lemma 3. *For each $k \in K$, the set $U_k := \{\, y \in \bar{N} \mid k^{-1}y \in \bar{N}MAN \,\}$ is an open subset of $\bar{N}$ whose complement in $\bar{N}$ has measure zero.*

Proof. For each $k \in K$, we set $V_k := \bar{N}MAN \cap k\bar{N}MAN$. Note that $G \setminus V_k = (G \setminus \bar{N}MAN) \cup (G \setminus k\bar{N}MAN)$ has Haar measure zero. On the other hand, we have $V_k = U_k MAN$ hence $G \setminus V_k = (G \setminus \bar{N}MAN) \cup (\bar{N} \setminus U_k)MAN$. Thus we see that $(\bar{N} \setminus U_k)MAN$ also has Haar measure zero. Since the restriction of the Haar measure dg on G to $\bar{N}MAN$ is $e^{2\rho(\log a)}dy\,da\,dm\,dn$ where dy, da, dm and dn are Haar measures on $\bar{N}$, A, M and N [30], p. 179, we conclude that $\bar{N} \setminus U_k$ has measure zero. QED

We denote by $C_0(\bar{N}, E)$ the space of compactly supported continuous functions $\phi : \bar{N} \to E$ and by $C_0^\infty(\bar{N}, E)$ the space of compactly supported smooth functions $\phi : \bar{N} \to E$. We have the following proposition.

Proposition 3. *For each $(v, k) \in G_0$, $\phi \in C_0(\bar{N}, E)$ and $y \in U_k$, we have*

$$\lim_{r\to 0} \pi_r(c_r(v,k))\phi\,(y) = \pi_0(v,k)\,\phi(y).$$

Proof. By using the expressions for π_r and π_0 given in Section 1 and Section 2, we have just to verify that

$$\lim_{r\to 0} \frac{1}{r}\beta\Big(\xi_1\,,\,\log\tilde{a}(y) - \log\tilde{a}(k^{-1}\exp(-rv)y)\Big) = \beta\big(\mathrm{Ad}(\tilde{k}(y))\xi_1, v\big).$$

But we have

$$\begin{aligned}\tilde{a}(k^{-1}\exp(-rv)y) &= \tilde{a}(\exp(-rv)y) = \tilde{a}(\exp(-rv)\tilde{k}(y)\tilde{a}(y)\tilde{n}(y)) \\ &= \tilde{a}(\exp(-r\,\mathrm{Ad}(\tilde{k}(y)^{-1})v))\tilde{a}(y).\end{aligned}$$

Then we get

$$\tilde{a}(y)\tilde{a}(k^{-1}\exp(-rv)y)^{-1} = \tilde{a}(\exp(-r\,\mathrm{Ad}(\tilde{k}(y)^{-1})v))^{-1}.$$

Thus, by using Lemma 1, we have

$$\frac{d}{dr}\log\,\tilde{a}(y)\tilde{a}(k^{-1}\exp(-rv)y)^{-1}|_{r=0} = p_{\tilde{a}}(\mathrm{Ad}(\tilde{k}(y)^{-1})v).$$

Hence the result follows. QED

In order to establish the L^2-convergence, we need the following lemma.

Lemma 4. *Let U be an open subset of $\bar{N}$ such that $\bar{N} \setminus U$ has measure zero. Then, for each $\phi \in C_0(\bar{N}, E)$ and each $\varepsilon > 0$, there exists $\psi \in C_0(\bar{N}, E)$ such that* $\operatorname{supp}\psi \subset U$ *and* $\|\psi - \phi\| \leq \varepsilon$.

Proof. Let $|\cdot|$ denote the Euclidean norm on $\bar{\mathfrak{n}}$ (see Section 1). We endow $\bar{N}$ with the distance d defined by $d(y,y') = |\log y - \log y'|$. Let $\phi \in C_0(\bar{N}, E)$ and $\varepsilon > 0$. Let C be a compact subset of $\bar{N}$ such that $C \subset U \cap \operatorname{supp}\phi$ and

$$\int_{(U\cap \operatorname{supp}\phi)\setminus C} dy \leq (1 + 4\sup_{y\in\bar{N}} \|\phi(y)\|_E^2)^{-1}\,\varepsilon.$$

In particular, we have $\delta := d(C, \bar{N}\setminus U) > 0$. Let $V := \{y \in \bar{N} \,:\, d(y,C) < \delta/2\}$. Then V is an open set such that $C \subset V \subset \bar{V} \subset U$. Consider now the function $\psi : \bar{N} \to E$ defined by

$$\psi(y) = \frac{d(y, \bar{N}\setminus V)}{d(y,C) + d(y,\bar{N}\setminus V)}\,\phi(y).$$

Note that ψ is well-defined since the intersection of C with the adherence of $\bar{N}\setminus V$ in $\bar{N}$ is empty. Moreover, we have the following properties

(1) $\operatorname{supp}\psi \subset \bar{V} \subset U$ and $\operatorname{supp}\psi \subset \operatorname{supp}\phi$;

(2) $\sup_{y\in\bar{N}} \|\psi(y)\|_E \leq \sup_{y\in\bar{N}} \|\phi(y)\|_E$;

(3) $\psi(y) = \phi(y)$ for each $y \in C$.

This implies that

$$\begin{aligned}\int_{\bar{N}} \|\psi(y) - \phi(y)\|_E^2\,dy &= \int_{U\cap\operatorname{supp}\phi} \|\psi(y) - \phi(y)\|_E^2\,dy \\ &= \int_{(U\cap\operatorname{supp}\phi)\setminus C} \|\psi(y) - \phi(y)\|_E^2\,dy \\ &\leq 4\sup_{y\in\bar{N}} \|\phi(y)\|_E^2 \int_{(U\cap\operatorname{supp}\phi)\setminus C} dy \\ &\leq \varepsilon.\end{aligned}$$

QED

Proposition 4. *1) Let ϕ, ψ in $L^2(\bar{N}, E)$ and $(v,k) \in G_0$. Then we have*

$$\lim_{r\to 0} \langle \pi_r(c_r(v,k))\phi, \psi\rangle = \langle \pi_0(v,k)\phi, \psi\rangle.$$

2) Let $\phi \in L^2(\bar{N}, E)$ and $(v,k) \in G_0$. Then we have

$$\lim_{r\to 0} \|\pi_r(c_r(v,k))\phi - \pi_0(v,k)\phi\| = 0.$$

Proof. 1) We can assume without loss of generality that $\phi,\,\psi \in C_0(\bar{N},E)$. Moreover, by Lemma 4, we can also assume that $\operatorname{supp}\phi \subset U_k$. We have

$$\langle \pi_r(c_r(v,k))\phi,\psi\rangle = \int_{\operatorname{supp}\psi} \langle \pi_r(c_r(v,k))\phi(y),\psi(y)\rangle_E\, dy.$$

By Proposition 3, for each $y \in \operatorname{supp}\psi$, the integrand

$$I_r(y) := \langle \pi_r(c_r(v,k))\phi(y),\psi(y)\rangle_E$$

converges to $\langle \pi_0(v,k)\phi(y),\psi(y)\rangle_E$ when $r \to 0$. In order to obtain the desired result, it suffices to verify that the dominated convergence theorem can be applied. This can be done as follows.

First we claim that there exists $r_0 > 0$ such that for each $r \in [0,r_0]$ and each $y \in \operatorname{supp}\psi$, we have $k^{-1}\exp(-rv)y \in \bar{N}MAN$. Indeed, if this is not the case, then there exists a sequence $r_n > 0$ converging to 0 and a sequence $y_n \in \operatorname{supp}\psi$ such that $k^{-1}\exp(-r_n v)y_n \in G \setminus \bar{N}MAN$ for each n. Since $\operatorname{supp}\psi$ is compact, we can also assume that y_n converges to an element $y \in \operatorname{supp}\psi$. Then we get $k^{-1}y \in G \setminus \bar{N}MAN$. This a contradiction.

Since the projection maps a and $\bar{n}$ are continuous on $\bar{N}MAN$, there exists $c > 0$ such that, for each $r < r_0$ and each $y \in \operatorname{supp}\psi$, we have

$$e^{-\rho(\log\, a(k^{-1}\exp(-rv)y))} \leq c.$$

Then, by taking into account the expression for $\pi_r(c_r(v,k))$, we get

$$|I_r(y)| \leq c.\sup_{z\in\bar{N}} \|\phi(z)\|_E.\|\psi(y)\|_E$$

for each $r < r_0$ and each $y \in \operatorname{supp}\psi$, hence the result.

2) Since π and π_0 are unitary, for each $\phi \in L^2(\bar{N},E)$ we have

$$\|\pi_r(c_r(v,k))\phi - \pi_0(v,k)\phi\|^2 = 2\|\phi\|^2 - 2\,\mathrm{Re}\,\langle \pi_r(c_r(v,k))\phi, \pi_0(v,k)\phi\rangle$$

which converges to $2\|\phi\|^2 - 2\,\mathrm{Re}\,\langle \pi_0(v,k)\phi, \pi_0(v,k)\phi\rangle$ when $r \to 0$ by 1). QED

Remarks (1) In fact, 2) of Proposition 4 asserts that π_0 is a contraction of (π_r) in the sense of [24] (see also [8]).

(2) By using the Bruhat decomposition $G = \bigcup_{w\in W} MANwMAN$ where W is the Weyl group, it is easy to see that $\bigcap_{k\in K} k\bar{N}MAN = \emptyset$ then the set of all elements $y \in \bar{N}$ such that $k^{-1}y \in \bar{N}MAN$ for each $k \in K$ is also empty. Hence, it seems to be difficult to get uniform convergence on the compact sets of G_0 in Proposition 4 as in Theorem 1 of [16].

4 Contraction of derived representations

In this section, we give similar contraction results for the derived representations.

For each $r \in]0,1]$, we denote by C_r the differential of c_r. Then the family (C_r) is a contraction of Lie algebras from $\mathfrak{g}$ onto $\mathfrak{g}_0$, that is,

$$\lim_{r\to 0} C_r^{-1}\big([C_r(X)\,,\,C_r(Y)]\big) = [X\,,\,Y]_0$$

for each X, $Y \in \mathfrak{g}_0$. We also denote by $C_r^* : \mathfrak{g}^* \simeq \mathfrak{g} \to \mathfrak{g}_0^* \simeq \mathfrak{g}_0$ the dual map of C_r. Then we note that $\lim_{r\to 0} C_r^*(\xi_r) = (\xi_1, \xi_2)$.

Proposition 5. *1) For each $(v,U) \in \mathfrak{g}_0$, $\phi \in C^\infty(\bar{N},E)$ and $y \in \bar{N}$, we have*

$$\lim_{r\to 0} d\pi_r(C_r(v,U))\phi\,(y) = d\pi_0(v,U)\,\phi(y).$$

2) For each $(v,U) \in \mathfrak{g}_0$ and ϕ, $\psi \in C_0^\infty(\bar{N},E)$, we have

$$\lim_{r\to 0} \langle d\pi_r(C_r(v,U))\phi, \psi\rangle = \langle d\pi_0(v,U)\phi, \psi\rangle.$$

3) For each $(v,U) \in \mathfrak{g}_0$ and $\phi \in C_0^\infty(\bar{N},E)$,we have

$$\lim_{r\to 0} \|d\pi_r(C_r(v,U))\phi - d\pi_0(v,U)\phi\| = 0.$$

Proof. We immediately deduce 1) from Proposition 1 and Proposition 2. Note that another proof of 1) by the Berezin-Weyl calculus can be found in [10]. Moreover, by using Proposition 1 and Proposition 2 again, we see that if $\phi \in C_0^\infty(\bar{N},E)$ then $d\pi_r(C_r(v,U))\phi$, $d\pi_0(v,U)\phi \in C_0^\infty(\bar{N},E) \subset L^2(\bar{N},E)$. Hence the expressions $\langle d\pi_r(C_r(v,U))\phi, \psi\rangle$ and $\langle d\pi_0(v,U)\phi, \psi\rangle$ make sense for ϕ, ψ in $C_0^\infty(\bar{N},E)$ and we easily obtain 2). Finally, to prove 3), we write

$$\begin{aligned}\|d\pi_r(C_r(v,U))\phi - d\pi_0(v,U)\phi\|^2 &= \|d\pi_r(C_r(v,U))\phi\|^2 + \|d\pi_0(v,U)\phi\|^2\\ &\quad - 2\,\mathrm{Re}\langle d\pi_r(C_r(v,U))\phi, d\pi_0(v,U)\rangle.\end{aligned}$$

By 2), we see that

$$\lim_{r\to 0} \langle d\pi_r(C_r(v,U))\phi, d\pi_0(v,U)\phi\rangle = \langle d\pi_0(v,U)\phi, d\pi_0(v,U)\phi\rangle.$$

By the same arguments, we verify that

$$\lim_{r\to 0} \|d\pi_r(C_r(v,U))\phi\|^2 = \|d\pi_0(v,U)\phi\|^2.$$

Then the result follows. QED

References

[1] F. A. Berezin: *Quantization*, Math. USSR Izv., **8**, n. 5, (1974), 1109–1165.

[2] B. Cahen: *Deformation Program for Principal Series Representations*, Lett. Math. Phys., **36** (1996), 65–75.

[3] B. Cahen: *Quantification d'orbites coadjointes et théorie des contractions*, J. Lie Theory, **11** (2001), 257–272.

[4] B. Cahen: *Contractions of $SU(1,n)$ and $SU(n+1)$ via Berezin quantization*, J. Anal. Math., **97** (2005), 83–102.

[5] B. Cahen: *Weyl quantization for principal series*, Beiträge Algebra Geom., **48** (2007), 175–190.

[6] B. Cahen: *Contraction of compact semisimple Lie groups via Berezin quantization*, Illinois J. Math., **53**, n. 1, (2009), 265–288.

[7] B. Cahen: *Contraction of Discrete Series via Berezin Quantization*, J. Lie Theory, **19** (2009), 291–310.

[8] B. Cahen: *Some Remarks on the Notion of Contraction of Lie Group Representations*, Math. Moravica, **14** (2010), 35–46.

[9] B. Cahen: *Berezin-Weyl quantization for the Cartan motion groups*, Comment. Math. Univ. Carolin., **52**, n. 1, (2011), 127–137.

[10] B. Cahen: *Contraction of the principal series by Berezin-Weyl quantization*, Rocky Mountain J. Math., **43**, n. 2, (2013), 417–441.

[11] U. Cattaneo and W. F. Wreszinski: *Trotter limits of Lie algebra representations and coherent states*, Helv. Phys. Acta, **52** (1979), 313–329.

[12] U. Cattaneo and W. F. Wreszinski: *Contractions of Lie algebra representations*, Rev. Math. Phys., **11** (1999), 1179–1207.

[13] P. Cotton and A. H. Dooley: *Contraction of an Adapted Functional Calculus*, J. Lie Theory, **7**, (1997), 147–164.

[14] E. Celeghini and M. Tarlini: *Contractions of Group Representations-I*, Nuovo Cimento B, **31** (1981), 265–277.

[15] A. H. Dooley: *Contractions of Lie groups and applications to analysis*, In Topics in Modern Harmonic Analysis, Proc. Semin., Torino and Milano 1982, Vol. I , Ist. di Alta Mat, Rome, 1983, pp. 483–515.

[16] A. H. Dooley and J. W. Rice: *On contractions of semisimple Lie groups*, Trans. Am. Math. Soc., **289** (1985), 185–202.

[17] A. H. Dooley: *Intertwining operators, The Cayley transform and the contraction of K to NM*, Contemp. Math., **544**, (2011), 101–108.

[18] S. Helgason: Differential Geometry, Lie Groups and Symmetric Spaces, Graduate Studies in Mathematics, Vol. 34, American Mathematical Society, Providence, Rhode Island 2001.

[19] E. Inönü , E. P. Wigner: *On the contraction of groups and their representations*, Proc. Nat. Acad. Sci. USA, **39** (1953), 510–524.

[20] A. A. Kirillov: Lectures on the Orbit Method, Graduate Studies in Mathematics Vol. 64, American Mathematical Society, Providence, Rhode Island, 2004.

[21] A. W. Knapp: Representation theory of semisimple groups. An overview based on examples, Princeton Math. Series t. 36, 1986.

[22] B. Kostant: *Quantization and unitary representations*, in: Modern Analysis and Applications, Lecture Notes in Mathematics 170, Springer-Verlag, Berlin, Heidelberg, New-York, 1970, pp. 87–207.

[23] G. Mackey: *On the analogy between semisimple Lie groups and certain related semi-direct product groups*, in Lie groups and their representations, I. M. Gelfand Ed., Hilger, London, 1975.

[24] J. Mickelsson and J. Niederle: *Contractions of Representations of de Sitter Groups*, Commun. Math. Phys., **27** (1972), 167–180.

[25] J. H. Rawnsley: *Representations of a semi direct product by quantization*, Math. Proc. Camb. Phil. Soc., **78** (1975), 345–350.

[26] F. Ricci: *A Contraction of $SU(2)$ to the Heisenberg Group*, Monatsh. Math., **101**, (1986), 211–225.

[27] E. J. Saletan: *Contraction of Lie groups*, J. Math. Phys., **2** (1961), 1–21.

[28] D. J. Simms: Lie Groups and Quantum Mechanics, Lecture Notes in Mathematics 52, Springer-Verlag, Berlin, Heidelberg, New-York, 1968.

[29] M. E. Taylor: Noncommutative Harmonis Analysis, Mathematical Surveys and Monographs, Vol. 22, American Mathematical Society, Providence, Rhode Island 1986.

[30] N. R. Wallach: Harmonic Analysis on Homogeneous Spaces, Pure and Applied Mathematics, Vol. 19, Marcel Dekker, New-York 1973.

[31] N. J. Wildberger: *On the Fourier transform of a compact semisimple Lie group*, J. Austral. Math. Soc. A, **56** (1994), 64–116.