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Classification of book spreads in PG(5, 2)

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Abstract. We classify all line spreads $\mathcal{S}_{21}$ in PG(5, 2) of a special kind, namely those which are *book spreads*. We show that up to isomorphism there are precisely nine different kinds of book spreads and describe the automorphism groups which stabilize them. Most of the main results are obtained in two independent ways, namely theoretically and by computer.

Keywords: line spread, PG(5, 2), combinatorial design

MSC 2010 classification: 51E20, 05B25, 05B05

1 Introduction

A line spread in the projective space $\mathrm{PG}(5, q)$ consists of a set of $q^4 + q^2 + 1$ lines which partition the points of the space. The task of classifying all line spreads in $\mathrm{PG}(5, q)$ is an extremely formidable one, and certainly requires computer help even for low values of q. Line spreads in PG(5, 2) were considered in [13], where, with computer assistance, 131044 inequivalent spreads were found. Most of these spreads have very little symmetry and presumably their properties do not warrant further consideration. Indeed, see [13, Table I], as many as 128474 different kinds of line spreads in PG(5, 2) have trivial automorphism group!

In this paper we classify all line spreads $\mathcal{S}_{21}$ in PG(5, 2) of a special kind, namely those which are *book spreads*. We claim that book spreads are some of the most interesting kinds of spreads. For since their lines partition not only the whole projective space, but also subspaces covering it, they have rich automor-

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phism groups and are therefore a suitable type of spreads to study for higher parameters, for which computer classification is not possible.

In $\mathrm{PG}(5,2)$ there are no other spreads which partition at least five 3-dimensional subspaces, book spreads account for 9 of the 26 orbits of line spreads having an automorphism group of order greater than 35, and for 8 of the 16 orbits of line spreads having an automorphism group of order greater than 71. Moreover, the structure of book spreads makes them interesting for various constructions based on spreads. In particular, in [13] spreads in $\mathrm{PG}(5,2)$ were used to obtain affine 2-(64,16,5) designs by Rahilly's construction [15]. The aim was to find 2-(64,16,5) designs with the smallest possible 2-rank and thus look for new counter examples to Hamada's conjecture [6] (see also [7], [14]). Only two minimal rank designs were found [13, Table III] and they were both constructed from book spreads.

We expect that the $\mathrm{PG}(5,2)$ results in the present paper will be of use in some future work where we intend to investigate certain kinds of book spreads in $\mathrm{PG}(7,2)$, see Remark 1(i) below.

Spreads in projective spaces have been widely studied in the last several decades and very many constructions of spreads have been found [12]. Classification results are known for spreads in $\mathrm{PG}(3,q)$ with certain automorphisms [9], [10], [11], for maximal partial spreads in $\mathrm{PG}(3,2)$ [17], $\mathrm{PG}(3,3)$[17], $\mathrm{PG}(3,4)$ [17], [18], and $\mathrm{PG}(4,2)$ [5], for spreads in $\mathrm{PG}(5,2)$ [13], and for maximal partial spreads of size 45 in $\mathrm{PG}(3,7)$ [2].

We identify the nonzero elements of the GF(2)-vector space V_{n+1} with the points of the associated projective space $\mathbb{P}V_{n+1} = \mathrm{PG}(n,2)$ and hence the group $\mathrm{GL}(V_{n+1})$ with the collineation group $\mathrm{PGL}(n+1,2)$ of $\mathrm{PG}(n,2)$. We use $\langle u, v, \ldots\rangle$ for the flat (projective subspace) generated by projective points $u, v, \ldots$. Also, for any geometric structure $\mathcal{D}$ in $\mathrm{PG}(n,2)$, we denote by $\mathcal{G}(\mathcal{D})$ the subgroup of $\mathrm{PGL}(n+1,2)$ which stabilizes $\mathcal{D}$.

1.1 Books, quatrain books and book spreads in $\mathrm{PG}(5,2)$

Let μ be a line of $\mathrm{PG}(5,2)$ and let $\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ be a set of five solids (3-flats) σ_i in $\mathrm{PG}(5,2)$ such that each 3-flat σ_i contains μ and such that each of the 60 points in the complement $\mu^c := \mathrm{PG}(5,2)\setminus\mu$ of μ lies in one (and only one) of the five solids of $\mathcal{B}$. We will call $\mathcal{B}$ a *book of solids with spine* μ, and we refer to the elements $\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5$ of $\mathcal{B}$ as the *pages* of the book $\mathcal{B}$.

A *quatrain* $Q^{(i)}$ of σ_i is a partial spread of four lines such that $\Sigma^{(i)} := \{\mu\} \cup Q^{(i)}$ is a spread of σ_i. Suppose that in each of the five pages of the book $\mathcal{B}$ we 'write', i.e. choose, a quatrain $Q^{(i)}$. Thus equipped, we will refer to $\mathcal{B}$ as a *quatrain book*, or a *Qbook*, and denote it by ${}^5\mathcal{B}$. Each quatrain book ${}^5\mathcal{B}$ thus determines a line spread $\mathcal{S}_{21} = \mathcal{S}_{21}({}^5\mathcal{B})$ for PG(5, 2), whose elements are the lines of the five quatrains together with the spine of $\mathcal{B}$:

$$\mathcal{S}_{21} = \{\mu\} \cup Q^{(1)} \cup Q^{(2)} \cup Q^{(3)} \cup Q^{(4)} \cup Q^{(5)}. \tag{1.1}$$

Line spreads for PG(5, 2) of this kind will be referred to as *book spreads.*

In PG(5, 2) there are 651 choices for the spine μ of a book $\mathcal{B}$, and there are 56 choices for the set $\mathcal{B}$ of five solids σ_i through μ. Consequently, in PG(5, 2) there exist $651 \times 56 = 36,456 = 2^3.3.7^2.31$ books. Now the spine μ belongs to eight distinct spreads in each σ_i, so there are eight choices of quatrain for each of the five pages of the 36,456 books. Hence in PG(5, 2) there exist $36,456 \times 8^5 = 1,194,590,208 = 2^{18}.3.7^2.31$ quatrain books.

Remark 1. In the present paper we confine our attention solely to book spreads in PG(5, 2). It is clear that it is considerably more difficult to classify book spreads in PG(5, q), $q > 2$.

Concerning higher dimensional generalizations, these could also involve plane spreads. Thus in PG(8, 2) one could consider plane spreads contained in books consisting of 9 pages of 5-flats sharing a common plane as spine. In each of these nine PG(5, 2) pages we need to choose a spread of nine planes (one plane being the spine), and in PG(5, 2) *there exists only one kind of spread of nine planes*: see [16, Theorem 4.1].

1.2 Useful background material

1.2.1 Desarguesian spreads in *(i)* PG(3, 2) and in *(ii)* PG(5, 2)

The subgroup of GL(4, 2) preserving a Desarguesian spread in PG(3, 2) is clearly $\Gamma\mathrm{L}(2, 4)$. The center $\mathcal{Z}$ of GL(2, 4) preserves each component of the spread and together with the zero map is a field, the kernel of the spread, which is isomorphic to GF(4) as the spread is Desarguesian. The group GL(2, 4) preserves exactly one spread of PG(3, 2) inducing Alt(5) on it.

Similarly, the subgroup of GL(6, 2) preserving a Desarguesian spread in PG(5, 2) is $\Gamma\mathrm{L}(3, 4)$. Also, the center of GL(3, 4) together with the zero map is the kernel of the spread and is isomorphic to GF(4). All these subgroups are conjugate; this can be derived, for example, from [12, Theorem I.3]. Therefore, the number of distinct Desarguesian spreads of PG(5, 2) is $[\mathrm{GL}(6, 2) : \Gamma\mathrm{L}(3, 4)] = 2^8.7.31$.

Remark 2. Through a given point m of PG(2, 4) there pass five lines, each line containing four further points. We may consider the point m as the spine μ of a book $\mathcal{B}$ whose five PG(3, 2) pages arise from the five lines on m in PG(2, 4). Moreover, the four points other than m on a such a line equip each PG(3, 2) page with a quatrain, and so we have a quatrain book ${}^5\mathcal{B}$. Consequently, a

Desarguesian spread in $\mathrm{PG}(5,2)$ is an example of a book spread. Observe that such a book spread $\mathcal{S}_{21}$ has a highly unusual feature, namely that it may be viewed as a quatrain book ${}^5\mathcal{B}$ in 21 different ways, since any of its lines may serve as the spine. Consequently, the order of the stabilizer $\mathcal{G}({}^5\mathcal{B})$ of such a quatrain book is

$$|\mathcal{G}({}^5\mathcal{B})| = |\mathcal{G}(\mathcal{S}_{21})|/21 = 2^7.3^3.5 = 17,280. \tag{1.2}$$

Of course these Desarguesian spreads account for just one of the 131044 different kinds of $\mathrm{PG}(5,2)$ line spreads in the classification [13]. Incidentally, the order 362880 of their stabilizer group dwarfs the size, namely 5760 (see [13, Table I]) of the second largest stabilizer group.

1.2.2 Our standard spread Σ in $\mathrm{PG}(3,2)$

Choose a Singer cycle $A \in \mathrm{GL}(4,2)$, see for example [4, Table 3]. Then A has order 15 and, without loss of generality, we may suppose that A satisfies $A^4 = A + I$. We label the points of $\mathrm{PG}(3,2)$ so that A induces the permutation given in (1.3) on these points. We define the elements $B := A^6$ and $W := A^{10}$, of orders 5 and 3, respectively. Then $A = BW = WB$ and B and W induce the permutations given in (1.3) on the points of $\mathrm{PG}(3,2)$.

$$\begin{array}{lcl} A & : & (a_1, b_2, c_3, a_4, b_5, c_1, a_2, b_3, c_4, a_5, b_1, c_2, a_3, b_4, c_5), \\ B & : & (a_1, a_2, a_3, a_4, a_5)(b_1, b_2, b_3, b_4, b_5)(c_1, c_2, c_3, c_4, c_5), \\ W & : & (a_1, b_1, c_1)(a_2, b_2, c_2)(a_3, b_3, c_3)(a_4, b_4, c_4)(a_5, b_5, c_5). \end{array} \tag{1.3}$$

Moreover, the $\langle W\rangle$-orbits are lines of $\mathrm{PG}(3,2)$ which are permuted transitively by $\langle B\rangle$. For $i \in \{1,\ldots,5\}$, we define $\kappa_i = \{a_i, b_i, c_i\}$, and refer to

$$\Sigma := \{\kappa_1, \ldots, \kappa_5\}, \qquad \text{where } \kappa_i = \{a_i, b_i, c_i\}, \tag{1.4}$$

as the *standard spread* for $\mathrm{PG}(3,2)$. Since $\langle W\rangle$ is the unique subgroup of $\mathrm{GL}(4,2)$ which fixes each of the lines κ_i of Σ, we will refer to it as the *distinguished group* of Σ.

As $\langle B\rangle$ has no fixed points it permutes the 35 lines of $\mathrm{PG}(3,2)$ in 7 orbits, each of size 5. One of these orbits is the standard spread Σ, and the other six are given by the following six kinds of linear relations:

$$\begin{array}{lll} a_i + a_{i+1} = b_{i+3}; & b_i + b_{i+1} = c_{i+3}; & c_i + c_{i+1} = a_{i+3}; \\ a_i + c_{i+1} = a_{i+2}; & b_i + a_{i+1} = b_{i+2}; & c_i + b_{i+1} = c_{i+2}. \end{array} \tag{1.5}$$

In (1.5) and in many situations throughout the paper the indices i are to be read mod 5.

We now identify some particular elements of $\mathcal{G}(\Sigma)$ which we will use later. Since Sym(5) is isomorphic to SL(2, 4) extended by the field automorphism, it may be embedded in GL(4, 2). We describe one such embedding by $\pi \mapsto N_\pi$ for $\pi \in \mathrm{Sym}(5)$ and give explicit descriptions of the action N_π on σ when π is a transposition. In this description, we will see that π also corresponds to the action N_π induces on Σ.
Let $N_{(i,i+1)}$ be the mapping which interchanges a_i with c_{i+1} and b_i with b_{i+1}. Then $N_{(i,i+1)}$ induces the permutation

$$N_{(i,i+1)} : (a_i, c_{i+1})(b_i, b_{i+1})(c_i, a_{i+1})(b_{i+2}, c_{i+2})(a_{i+3}, b_{i+3})(b_{i+4}, c_{i+4}) \tag{1.6}$$

on points and the permutation $(\kappa_i, \kappa_{i+1})(\kappa_{i+2})(\kappa_{i+3})(\kappa_{i+4})$ on Σ.
Let $N_{(i,i+2)}$ be the mapping which interchanges a_i with a_{i+2} and a_{i+1} with b_{i+1}. Then $N_{(i,i+2)}$ induces the permutation

$$N_{(i,i+2)} : (a_i, a_{i+2})(b_i, c_{i+2})(c_i, b_{i+2})(a_{i+1}, b_{i+1})(a_{i+3}, c_{i+3})(a_{i+4}, c_{i+4}) \tag{1.7}$$

on points and the transposition (κ_i, κ_{i+2}) on Σ.

Defining $\mathcal{G}_1(\Sigma) := \langle N_{(ij)}, 1 \leq i < j \leq 5\rangle$, we see that $\mathcal{G}_1(\Sigma) \cong \mathrm{Sym}(5)$ acting naturally on Σ.

2 The standard book $\mathcal{B}$ and its quatrains

Since all books are GL(6, 2)-equivalent, we will use a particular decomposition $V_6 = V_4 \oplus V_2$ and a book $\mathcal{B} = \{\sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5\}$ in $\mathrm{PG}(5, 2) = \mathbb{P}V_6$ with spine $\mu = \mathbb{P}V_2 = \{u, v, w\}$, $u + v + w = 0$ for the rest of the paper. We also choose a solid $\sigma = \mathbb{P}V_4$, skew to μ, and label its elements so that the spread $\Sigma := \{\kappa_1, \ldots, \kappa_5\}$ induced by $\mathcal{B}$, setting $\kappa_i = \sigma \cap \sigma_i$ for $i = 1, \ldots, 5$, is consistent with the labelling in Section 1.2.2. Since each line in PG(3, 2) belongs to eight distinct spreads, there is a choice of eight possible quatrains $Q_1^{(i)}, Q_2^{(i)}, \ldots, Q_8^{(i)}$ for each page σ_i of our book $\mathcal{B}$. In displaying these quatrains it helps to adopt abbreviations of the kind

$$\begin{pmatrix} u & w & v \\ 0 & u & u \\ w & 0 & w \\ v & v & 0 \end{pmatrix}^{(i)} := \begin{pmatrix} a_i{+}u & b_i{+}w & c_i{+}v \\ a_i & b_i{+}u & c_i{+}u \\ a_i{+}w & b_i & c_i{+}w \\ a_i{+}v & b_i{+}v & c_i \end{pmatrix}. \tag{2.1}$$

For the page $\sigma_i = \langle a_i, b_i, u, v\rangle$, four of the quatrains are $Q_1^{(i)}, Q_2^{(i)}, Q_3^{(i)}, Q_4^{(i)}$, as given by the respective four arrays

$$\begin{pmatrix} 0 & 0 & 0 \\ u & v & w \\ v & w & u \\ w & u & v \end{pmatrix}^{(i)}, \begin{pmatrix} u & w & v \\ 0 & u & u \\ w & 0 & w \\ v & v & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} w & v & u \\ 0 & w & w \\ v & 0 & v \\ u & u & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} v & u & w \\ 0 & v & v \\ u & 0 & u \\ w & w & 0 \end{pmatrix}^{(i)}, \tag{2.2}$$

the four lines of a quatrain being given by adding (a_i, b_i, c_i) to each row of the array. The remaining four quatrains $Q_5^{(i)}, Q_6^{(i)}, Q_7^{(i)}, Q_8^{(i)}$ are then those given by the respective four arrays

$$\begin{pmatrix} 0 & 0 & 0 \\ u & w & v \\ w & v & u \\ v & u & w \end{pmatrix}^{(i)}, \begin{pmatrix} u & v & w \\ 0 & u & u \\ v & 0 & v \\ w & w & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} v & w & u \\ 0 & v & v \\ w & 0 & w \\ u & u & 0 \end{pmatrix}^{(i)}, \begin{pmatrix} w & u & v \\ 0 & w & w \\ u & 0 & u \\ v & v & 0 \end{pmatrix}^{(i)}. \tag{2.3}$$

For $1 \le i \le 5$, we define $\mathcal{Q}_+^{(i)} := \{Q_r^{(i)}\}_{r\in\{1,2,3,4\}}$, $\mathcal{Q}_-^{(i)} := \{Q_r^{(i)}\}_{r\in\{5,6,7,8\}}$, and $\mathcal{Q}^{(i)} := \mathcal{Q}_+^{(i)} \cup \mathcal{Q}_-^{(i)}$. We also define $\mathcal{Q}_+ := \bigcup_{1\le i\le 5} \mathcal{Q}_+^{(i)}$, $\mathcal{Q}_- := \bigcup_{1\le i\le 5} \mathcal{Q}_-^{(i)}$, and $\mathcal{Q} := \mathcal{Q}_+ \cup \mathcal{Q}_-$.

3 Aspects of the groups $\mathcal{G}(\mathcal{B})$, $\mathcal{G}_0(\mathcal{B})$ and $\mathcal{G}_0({}^2\mathcal{B})$

Since $\mathrm{GL}(6,2)$ is transitive on lines of $\mathrm{PG}(5,2)$, any Qbook is $\mathrm{GL}(6,2)$-equivalent to a Qbook in $\mathcal{B}$. Moreover, any two Qbooks in $\mathcal{B}$ which are $\mathrm{GL}(6,2)$-equivalent are $\mathcal{G}(\mathcal{B})$-equivalent, since an element mapping one to the other fixes the spine. Consequently, the $\mathrm{GL}(6,2)$-orbits of Qbooks are the $\mathrm{GL}(6,2)$-orbits of the representatives of the $\mathcal{G}(\mathcal{B})$-orbits of Qbooks.

3.1 The groups $\mathcal{G}(\mathcal{B})$, $\mathcal{G}_0(\mathcal{B})$

A general element $A \in \mathcal{G}(\mathcal{B})$ has the block form

$$A = \begin{pmatrix} A_4 & 0 \\ X & A_2 \end{pmatrix}, \text{ where } A_4 \in \mathcal{G}(\Sigma), A_2 \in \mathrm{GL}(V_2), X \in M_{2,4}(GF(2)). \tag{3.1}$$

Consequently,

$$\mathcal{G}(\mathcal{B}) \cong Z_2^8 : (\Gamma\,\mathrm{L}(2,4) \times \mathrm{GL}(2,2)), \text{ and } |\mathcal{G}(\mathcal{B})| = 2^{12}.3^3.5 = 552,960. \tag{3.2}$$

It follows that the number of books in $\mathrm{PG}(5,2)$ is $|\,\mathrm{GL}(6,2)|/|\mathcal{G}(\mathcal{B})| = 36,456$.

Let B_4 and W_4 denote the elements of $\mathcal{G}(\Sigma)$ relative to V_4 denoted by B and W in (1.3). In particular, $\langle W_4 \rangle$ is the distinguished subgroup of the spread Σ. Also we denote by W_2 the element of order 3 in $\mathrm{GL}(V_2)$ inducing the permutation (uvw) on the spine μ.

Define three elements W, W^* and B of $\mathrm{GL}(V_6)$, where $V_6 = V_4 \oplus V_2$, by

$$W = W_4 \oplus W_2, \quad W^* = W_4 \oplus (W_2)^{-1}, \quad B = B_4 \oplus I_2. \tag{3.3}$$

For each $\pi \in \mathrm{Sym}(5)$, we define an element $\bar{N}_\pi \in \mathrm{GL}(6,2)$ which fixes σ and μ, acting on σ as N_π and on μ as $v \leftrightarrow w$ if π is odd and trivially if π is even. We also define an involution $J_{(vw)}$ of $\mathrm{GL}(V_6)$ which fixes σ pointwise and fixes μ, interchanging v and w.

Let $\mathcal{G}_0(\mathcal{B})$ denote the subgroup of $\mathcal{G}(\mathcal{B})$ of those elements which fix each page of the book $\mathcal{B}$. Then $A \in \mathcal{G}_0$, with A as in (3.1), if, and only if, A_4 is in the distinguished subgroup of Σ; that is, $A_4 \in \langle W_4 \rangle$. Consequently,

$$\mathcal{G}_0(\mathcal{B}) \cong Z_2^8 : (Z_3 \times GL(2,2)), \text{ and } |\mathcal{G}_0(\mathcal{B})| = 2^9.3^2 = 4608. \tag{3.4}$$

Let $\mathcal{G}_{256} \cong \mathbb{Z}_2^8$ be the normal subgroup of $\mathcal{G}_0(\mathcal{B})$ consisting of those elements of the form (3.1) with $A_4 = I_4$ and $A_2 = I_2$.

Lemma 1. *(i) For $1 \leq i \leq 5$, the elements W, W^* and $J_{(vw)}$ of $\mathcal{G}_0(\mathcal{B})$ act on $\mathcal{Q}^{(i)}$ inducing respectively the permutations*

$$(Q_2^{(i)} Q_3^{(i)} Q_4^{(i)}), (Q_6^{(i)} Q_7^{(i)} Q_8^{(i)}) \text{ and } (Q_1^{(i)} Q_5^{(i)})(Q_2^{(i)} Q_6^{(i)})(Q_3^{(i)} Q_7^{(i)})(Q_4^{(i)} Q_8^{(i)}). \tag{3.5}$$

(ii) B is an element of $\mathcal{G}(\mathcal{B})$ which permutes the pages of $\mathcal{B}$ transitively and induces the permutation $\prod_{1 \leq r \leq 8}(Q_r^{(1)} Q_r^{(2)} Q_r^{(3)} Q_r^{(4)} Q_r^{(5)})$ on the 40 quatrains of $\mathcal{B}$.

Proof. Immediate, from (2.2), (2.3). $\boxed{QED}$

3.2 Some subgroups of $\mathcal{G}_0(\mathcal{B})$

3.2.1 Some elementary Abelian subgroups

Since $\{a_i, b_i, a_{i+1}, b_{i+1}, u, v\}$ is, for each $i \in \{1,2,3,4,5\} \pmod 5$, a basis for V_6, we may consider involutions $J_i, J_i', J_i'' \in \mathrm{GL}(6,2)$ such that each of them fixes the page $\sigma_i = \langle a_i, b_i, u, v \rangle$ pointwise and satisfy

$$\begin{aligned} J_i &: a_{i+1} \mapsto a_{i+1} + u, \quad b_{i+1} \mapsto b_{i+1} + v, \quad c_{i+1} \mapsto c_{i+1} + w, \\ J_i' &: a_{i+1} \mapsto a_{i+1} + v, \quad b_{i+1} \mapsto b_{i+1} + w, \quad c_{i+1} \mapsto c_{i+1} + u, \\ J_i'' &: a_{i+1} \mapsto a_{i+1} + w, \quad b_{i+1} \mapsto b_{i+1} + u, \quad c_{i+1} \mapsto c_{i+1} + v. \end{aligned} \tag{3.6}$$

The involutions $J_i, J'_i, J''_i \in \mathcal{G}_{256}$ also preserve the other pages $\sigma_{i-1}, \sigma_{i-2}$, and σ_{i+2}, since they act trivially on V_6/V_2. In particular, J_i induces the following maps on these pages:

$$\begin{aligned} &a_{i-1} \mapsto a_{i-1} + u, \quad b_{i-1} \mapsto b_{i-1} + v, \quad c_{i-1} \mapsto c_{i-1} + w, \\ &a_{i-2} \mapsto a_{i-2} + w, \quad b_{i-2} \mapsto b_{i-2} + u, \quad c_{i-2} \mapsto c_{i-2} + v, \\ &a_{i+2} \mapsto a_{i+2} + w, \quad b_{i+2} \mapsto b_{i+2} + u, \quad c_{i+2} \mapsto c_{i+2} + v. \end{aligned} \tag{3.7}$$

As $J''_i = J_i J'_i$, we have

$$\mathcal{A}_4^{(i)} := \langle J_i, J'_i \rangle = \{I, J_i, J'_i, J''_i\} \cong (Z_2)^2, \quad i = 1,2,3,4,5. \tag{3.8}$$

Define

$$\begin{aligned} &K_i := J_{(vw)} J_i J_{(vw)}, \ K'_i := J_{(vw)} J'_i J_{(vw)}, \ K''_i := J_{(vw)} J''_i J_{(vw)}, \\ &\mathcal{C}_4^{(i)} := J_{(vw)} \mathcal{A}_4^{(i)} J_{(vw)} = \{I, K_i, K'_i, K''_i\} \cong (Z_2)^2, \quad i = 1,2,3,4,5. \end{aligned} \tag{3.9}$$

So K_i, K'_i, K''_i keep pointwise fixed the page σ_i and their action upon the other pages is given by interchanging v and w in (3.6), (3.7). It is easy to see that

$$\begin{aligned} W^* J_i (W^*)^{-1} = J'_i, \qquad W^* J'_i (W^*)^{-1} = J''_i, \\ W K_i W^{-1} = K'_i, \qquad W K'_i W^{-1} = K''_i. \end{aligned} \tag{3.10}$$

Define $\mathcal{A}_{16} := \langle J_i, J'_i, J''_i \colon 1 \le i \le 5 \rangle$ and $\mathcal{C}_{16} := J_{(uv)} \mathcal{A}_{16} J_{(uv)}$. Then

Lemma 2. *(i)* $\mathcal{A}_{16} = \{I\} \cup \{J_i, J'_i, J''_i \colon 1 \le i \le 5\} \cong (Z_2)^4$, $\mathcal{C}_{16} = \{I\} \cup \{K_i, K'_i, K''_i \colon 1 \le i \le 5\} \cong (Z_2)^4$, *and if* $i \ne j$ *then*

$$\mathcal{A}_{16} = \mathcal{A}_4^{(i)} \times \mathcal{A}_4^{(j)}, \qquad \mathcal{C}_{16} = \mathcal{C}_4^{(i)} \times \mathcal{C}_4^{(j)}. \tag{3.11}$$

Moreover, $\mathcal{G}_{256} = \mathcal{A}_{16} \times \mathcal{C}_{16}$.

(ii) $\mathcal{A}_{16}$ *fixes each quatrain in* $\mathcal{Q}_+$ *and has the orbits* $\mathcal{Q}_-^{(i)}$, $1 \le i \le 5$, *in* $\mathcal{Q}_-$. $\mathcal{C}_{16}$ *fixes each quatrain in* $\mathcal{Q}_-$ *and has the orbits* $\mathcal{Q}_+^{(i)}$, $1 \le i \le 5$, *in* $\mathcal{Q}_+$.

(iii) For $1 \le i \le 5$, $\mathcal{A}_4^{(i)}$ *and* $\mathcal{C}_4^{(i)}$ *fix* σ_i *pointwise. For* $1 \le i, j \le 5$ *and* $j \ne i$, $\mathcal{Q}_-^{(j)}$ *is a* $\mathcal{A}_4^{(i)}$*-orbit in* $\mathcal{Q}_-$ *and* $\mathcal{Q}_+^{(j)}$ *is a* $\mathcal{C}_4^{(i)}$*-orbit in* $\mathcal{Q}_+$.

(iv) For $1 \le i, j \le 5$ *and* $j \ne i$, $\mathcal{A}_{16}$ *acts transitively on the sixteen pairs of quatrains* $\{Q_r^{(i)}, Q_s^{(j)}\}_{5 \le r,s \le 8}$, *and* $\mathcal{C}_{16}$ *acts transitively on the sixteen pairs of quatrains* $\{Q_r^{(i)}, Q_s^{(j)}\}_{1 \le r,s \le 4}$.

Proof. A somewhat lengthy, but straightforward, verification. Concerning (3.11), we may verify relations such as $J_1 J_2 = J'_4$. Concerning (iv), this follows from (iii) on account of the direct product structures (3.11). $\boxed{QED}$

The partition $\mathcal{Q}_+ \cup \mathcal{Q}_-$ defines an equivalence relation on $\mathcal{Q}$. We say that two elements of $\mathcal{Q}$ are in *harmony* with one another if both are in $\mathcal{Q}_+$ or both are in $\mathcal{Q}_-$. That is, two quatrains of $\mathcal{B}$ are in harmony if they are both fixed by $\mathcal{A}_{16}$ or both fixed by $\mathcal{C}_{16}$.

Remark 3. Let $\mathcal{L}_{16}^{(i)}$ denote the set of sixteen lines of the page σ_i which are skew to the spine μ. In the 4+4 splitting $\mathcal{Q}^{(i)} = \mathcal{Q}_+^{(i)} \cup \mathcal{Q}_-^{(i)}$ of the eight quatrains for the page σ_i, see (2.2) and (2.3), each line $\lambda \in \mathcal{L}_{16}^{(i)}$ appears in precisely two quatrains, one a member of $\mathcal{Q}_+^{(i)}$ and one a member of $\mathcal{Q}_-^{(i)}$. This ties in with the fact that the skew pair of lines $\{\lambda, \mu\}$ lies in two spreads in PG(3, 2) whose distinguished subgroups act differently upon the two lines.

By using harmony considerations we may classify full Qbooks ${}^5\mathcal{B}$ into three broad harmony types:

$\mathcal{T}(5, 0)$: all five quatrains of ${}^5\mathcal{B}$ are in harmony;
$\mathcal{T}(4, 1)$: precisely four quatrains of ${}^5\mathcal{B}$ are in harmony;
$\mathcal{T}(3, 2)$: the quatrains of ${}^5\mathcal{B}$ split $(3, 2)$ or $(2, 3)$ between $\mathcal{Q}_+$ and $\mathcal{Q}_-$.

Furthermore, since the involution $J_{(vw)}$ interchanges the quatrain sets $\mathcal{Q}_+$ and $\mathcal{Q}_-$, we may restrict our attention to Qbooks ${}^5\mathcal{B}$ such that at least three of its quatrains belong to $\mathcal{Q}_+$, without loss of generality. Moreover, as the group induced on the set of pages of $\mathcal{B}$ is Sym(5), we may restrict attention to those Qbooks ${}^5\mathcal{B}$ for which *the quatrains in its first three pages* σ_1, σ_2 *and* σ_3 *all belong to* $\mathcal{Q}_+$.

3.2.2 The subgroups $\mathcal{A}_{12}^{(i)}$, $\mathcal{G}_{36}^{(i)}$, $\mathcal{G}_{48}$, $\mathcal{G}_{48}^*$ and $\mathcal{G}_{144}$ of $\mathcal{G}_0(\mathcal{B})$

Since $\mathcal{A}_{16}$ is centralized by W and normalized by W^*, we may define

$$\mathcal{G}_{48} := \langle \mathcal{A}_{16}, W \rangle, \quad \mathcal{G}_{48}^* := \langle \mathcal{A}_{16}, W^* \rangle, \quad \mathcal{G}_{144} := \langle \mathcal{G}_{48}, W^* \rangle. \tag{3.12}$$

The groups $\mathcal{A}_{16}, \mathcal{G}_{48}, \mathcal{G}_{48}^*$ and $\mathcal{G}_{144}$ are all subgroups of $\mathcal{G}_0(\mathcal{B})$ and are isomorphic to $(Z_2)^4$, $Z_2^4 \times Z_3$, $Z_2^4 : Z_3$ and $(Z_2^4 : Z_3) \times Z_3$, respectively. From the definition of $\mathcal{A}_{16}$, it is easy to see that $\mathcal{G}_{144} = \langle J_1, J_2, , W, W^* \rangle$. The five subgroups $\mathcal{A}_4^{(i)}$ of $\mathcal{A}_{16}$ give rise to ten subgroups of $\mathcal{G}_{144}$ defined by:

$$\mathcal{A}_{12}^{(i)} := \langle \mathcal{A}_4^{(i)}, W^* \rangle = \langle J_i, W^* \rangle, \quad \mathcal{G}_{36}^{(i)} := \langle \mathcal{A}_{12}^{(i)}, W \rangle \quad 1 \le i \le 5. \tag{3.13}$$

Structurally, $\mathcal{A}_{12}^{(i)} \cong \mathrm{Alt}(4)$ and $\mathcal{G}_{36}^{(i)} \cong \mathrm{Alt}(4) \times Z_3$ for $1 \le i \le 5$.

Lemma 3. *(i) For* $1 \le i \le 5$*, The action of* $\mathcal{G}_{144}$ *on the set of eight quatrains* $\{Q_r^{(i)}\}_{1 \le r \le 8}$ *has three orbits*

$$\Omega_A^{(i)} = \{Q_1^{(i)}\}, \Omega_B^{(i)} = \{Q_2^{(i)}, Q_3^{(i)}, Q_4^{(i)}\}, \Omega_C^{(i)} = \{Q_5^{(i)}, Q_6^{(i)}, Q_7^{(i)}, Q_8^{(i)}\}. \tag{3.14}$$

The stabilizers in $\mathcal{G}_{144}$ *of any* $Q_r^{(i)} \in \Omega_B^{(i)}$*, and* $Q_5^{(i)}$ *are* $\mathcal{G}_{48}^*$*, and* $\mathcal{G}_{36}^{(i)}$*, respectively.*

(ii) For $1 \le i \le 5$*,* $\mathcal{G}_{48}^*$ *acts transitively on* $\Omega_C^{(i)}$ *and fixes each of the quatrains* $\{Q_r^{(i)}\}_{1 \le r \le 4}$.

The stabilizer in $\mathcal{G}_{48}^*$ *of* $Q_5^{(i)}$ *is* $\langle \mathcal{A}_4^{(i)}, W^* \rangle$.

(iii) For $1 \le i, j \le 5$ *with* $j \ne i$*,* $\mathcal{G}_{36}^{(i)}$ *fixes* $Q_1^{(i)}$*,* $Q_5^{(i)}$ *and* $Q_5^{(j)}$*, and has the orbits* $\Omega_B^{(i)}$*,* $\Omega_B^{(j)}$*,* $\{Q_r^{(i)} : 6 \le r \le 8\}$*, and* $\Omega_C^{(j)}$.

The stabilizer in $\mathcal{G}_{36}^{(i)}$ *of any* $Q_r^{(i)} \in \Omega_B^{(i)}$ *and of any* $Q_r^{(j)} \in \Omega_B^{(j)}$ *is* $\mathcal{A}_{12}^{(i)}$*, and the stabilizer of* $Q_5^{(j)}$ *is* $\langle W \rangle \times \langle W^* \rangle$.

Proof. This is a straightforward exercise using Lemmas 1 and 2. $\boxed{QED}$

3.3 Harmonious quatrains and normal pentads

If a quatrain book ${}^5\mathcal{B}$ has the quatrain $Q_{r_i}^{(i)}$, $r_i \in \{1, 2, \ldots, 8\}$, in its ith page, and so determines the book spread $\mathcal{S}_{21}({}^5\mathcal{B}) = \{\mu\} \cup Q_{r_1}^{(1)} \cup Q_{r_2}^{(2)} \cup Q_{r_3}^{(3)} \cup Q_{r_4}^{(4)} \cup Q_{r_5}^{(5)}$ of $\mathrm{PG}(5, 2)$, then we refer to the ordered pentad of quatrains

$$Q_{r_1 r_2 r_3 r_4 r_5} := (Q_{r_1}^{(1)}, Q_{r_2}^{(2)}, Q_{r_3}^{(3)}, Q_{r_4}^{(4)}, Q_{r_5}^{(5)}) \tag{3.15}$$

as the *content* of the Qbook ${}^5\mathcal{B}$, and we use ${}^5\mathcal{B}_{r_1 r_2 r_3 r_4 r_5}$ to denote this Qbook. If we write quatrains $Q_{r_1}^{(1)}$ and $Q_{r_2}^{(2)}$ in the first two pages of our book $\mathcal{B}$, and leave the other pages blank, then we will say that we have a *2-quatrain book,* or *2Qbook,* ${}^2\mathcal{B}$ whose content is $Q_{r_1 r_2}$, and we use the notation ${}^2\mathcal{B}_{r_1 r_2}$ for this 2Qbook. We define in an analogous fashion a *3-quatrain book,* or *3Qbook,* ${}^3\mathcal{B}_{r_1 r_2 r_3}$ whose content is $Q_{r_1 r_2 r_3}$ and a *4Qbook* ${}^4\mathcal{B}_{r_1 r_2 r_3 r_4}$ whose content is $Q_{r_1 r_2 r_3 r_4}$.

Lemma 4. *The line spreads of the Qbooks* ${}^5\mathcal{B}_{11111}$ *and* ${}^5\mathcal{B}_{55555}$ *are both Desarguesian.*

Proof. From (2.2) we see that the element W cycles through the points of each line of each of the quatrains $Q_1^{(i)}$, $1 \le i \le 5$, and so $\langle W \rangle$ serves as the required distinguished subgroup. From (2.3) the element W^* serves similarly for the quatrains $Q_5^{(i)}$, $1 \le i \le 5$. $\boxed{QED}$

A pentad $Q_{r_1 r_2 r_3 r_4 r_5}$ will be termed a *normal pentad* of quatrains if it is the content of a Qbook ${}^5\mathcal{B}$ whose book spread $\mathcal{S}_{21}({}^5\mathcal{B})$ is a Desarguesian spread. Subsets of a normal pentad $Q_{r_1 r_2 r_3 r_4 r_5}$ of sizes 2, 3 and 4 will be termed *normal duads, triads and tetrads.*

Lemma 5. *If the quatrains* $Q_r^{(i)}, Q_s^{(j)}$*,* $i \ne j$*, are in harmony then the pair* $\{Q_r^{(i)}, Q_s^{(j)}\}$ *is a normal duad.*

Proof. This follows immediately from Lemmas 2(iv) and 4. $\boxed{QED}$

Lemma 6. *A book $\mathcal{B}$ supports precisely 16 normal pentads with quatrains in $\mathcal{Q}_+$ and 16 normal pentads with quatrains in $\mathcal{Q}_-$.*

Proof. Within $\mathcal{Q}_+$ there are $\binom{5}{2} \times 4 \times 4 = 160$ duads. But each normal pentad contains $\binom{5}{2}$ duads. $\boxed{QED}$

3.4 The standard 2Qbook $^2\mathcal{B}$ and the group $\mathcal{G}_0(^2\mathcal{B})$

As explained in Section 3.2.1, in classifying the book spreads in PG(5, 2) we may restrict attention to those Qbooks $^5\mathcal{B}$, based on our standard book $\mathcal{B}$, for which the quatrains in its first three pages all belong to $\mathcal{Q}_+$. Since $\mathcal{C}_{16}$ is transitive on the sixteen pairs of quatrains $\{Q_r^{(1)}, Q_s^{(2)}\}_{1\le r,s\le 4}$, by Lemma 2(iv), we may suppose that the quatrains on the first two pages are $Q_1^{(1)}$ and $Q_1^{(2)}$, respectively. Thus equipped, the book $\mathcal{B}$ becomes our *standard 2Qbook* $^2\mathcal{B}_{11}$, whose content is Q_{11}.

Lemma 7. *The subgroup of $\mathcal{G}_0(\mathcal{B})$ which stabilizes our standard 2Qbook $^2\mathcal{B}_{11}$ is $\mathcal{G}_{144}$.*

Proof. Let $\mathcal{G}_0(^2\mathcal{B}_{11})$ be the stabilizer of the standard 2Qbook $^2\mathcal{B}_{11}$ in $\mathcal{G}_0(\mathcal{B})$. By Lemma 1(i) and Lemma 2(ii), $\mathcal{G}_{144} \le \mathcal{G}_0(^2\mathcal{B}_{11})$. None of the involutions in $\mathcal{C}_{16}$ fixes $Q_1^{(1)}$ or $Q_1^{(2)}$. Hence, $\mathcal{G}_0(^2\mathcal{B}_{11}) \cap (\mathcal{A}_{16} \times \mathcal{C}_{16}) = \mathcal{A}_{16}$. As $\mathcal{G}_0(^2\mathcal{B}_{11})/\mathcal{A}_{16} \le$ GL(2, 2) and $J_{(uv)} \notin \mathcal{G}_0(^2\mathcal{B}_{11})$, we have $\mathcal{G}_{144} = \mathcal{G}_0(^2\mathcal{B}_{11})$. $\boxed{QED}$

4 Extending the 2Qbook $^2\mathcal{B}$ to a full Qbook $^5\mathcal{B}$

Before extending our standard 2Qbook $^2\mathcal{B}_{11}$ with further pages, we study the subsets of normal pentads, whose quatrains are in $\mathcal{Q}_+$.

Lemma 8. *(i) For $r_i \in \{1, 2, 3, 4\}$ the ordered pentad*

$$Q_{r_1r_2r_3r_4r_5} := (Q_{r_1}^{(1)}, Q_{r_2}^{(2)}, Q_{r_3}^{(3)}, Q_{r_4}^{(4)}, Q_{r_5}^{(5)}) \tag{4.1}$$

is a normal pentad of quatrains if and only if either $r_i = 1$, $1 \le i \le 5$, or if $r_1r_2r_3r_4r_5$ takes one of the following fifteen values

$$\begin{aligned} &12442,\ 21244,\ 42124,\ 44212,\ 24421;\\ &13223,\ 31322,\ 23132,\ 22313,\ 32231;\\ &14334,\ 41433,\ 34143,\ 33414,\ 43341. \end{aligned} \tag{4.2}$$

(ii) Each of the 160 *pairs of quatrains* $\{Q_r^{(i)}, Q_s^{(j)}\}_{1\le r,s\le 4,\ 1\le i<j\le 5}$ *belongs to precisely one of these sixteen normal pentads.*

(iii) The sixteen normal pentads comprise a single $\mathcal{C}_{16}$*-orbit.*

Proof. (i) Since Q_{11111} is a normal pentad by Lemma 4 and the involution K_1 maps Q_{11111} to Q_{12442}, Q_{12442} is normal. From Lemma 1(ii) we see that B sends $Q_{r_1r_2r_3r_4r_5}$ to $Q_{r_2r_3r_4r_5r_1}$; so the three rows of (4.2) correspond to three B-orbits of pentads. But from Lemma 1(i) we see that W cyclically permutes the three rows (the fifteen pentads thus forming a single $\langle BW\rangle$-orbit). So it follows that all fifteen pentads given by (4.2) are normal. The sixteen normal pentads thus found exhaust the possibilities, since by Lemma 6 no further normal pentads exist with quatrains in $\mathcal{Q}_+$.

(ii) This is a simple verification; see proof of Lemma 6.

(iii) This follows from (ii) since, from Lemma 2(iv), $\mathcal{C}_{16}$ is transitive on the sixteen pairs $\{Q_r^{(i)}, Q_s^{(j)}\}_{1\le r,s\le 4}$ for any $i, j \in \{1,\dots,5\}$ with $i \ne j$. $\boxed{QED}$

Remark 4. From Lemma 3, the subgroup $\mathcal{G}_0({}^5\mathcal{B}_{11111})$ of $\mathcal{G}_0(\mathcal{B})$ which stabilizes the Qbook ${}^5\mathcal{B}_{11111}$ is $\mathcal{G}_{144}$.

We partition the set of pentads as follows. First we partition the set of pentads of harmony type $\mathcal{T}(5,0)$ into four classes:

$P \in \mathcal{P}_0$ if P is a normal pentad, e.g. Q_{11111};
$P \in \mathcal{P}_1$ if $P \notin \mathcal{P}_0$ but P contains a normal tetrad, e.g. Q_{11112};
$P \in \mathcal{P}_2$ if $P \notin \mathcal{P}_0 \cup \mathcal{P}_1$ but P contains a normal triad, e.g. Q_{11122};
$P \in \mathcal{P}_3$ if P contains no normal triads; e.g. Q_{11232}.

Lemma 9. *A harmonious pentad of type* $\mathcal{P}_2$ *contains precisely two normal triads.*

Proof. Without loss of generality we may assume that the pentad is Q_{111rs} for some $r, s \in \{2,3,4\}$. By Lemma 8(ii) the harmonious duad $\{Q_r^{(4)}, Q_s^{(5)}\}$ is a subset of precisely one of the normal pentads (4.2), and so is a subset of a unique normal triad $\{Q_1^{(i)}, Q_r^{(4)}, Q_s^{(5)}\}$ for some $i \in \{1,2,3\}$. Thus Q_{111rs} contains just the two normal triads $\{Q_1^{(1)}, Q_1^{(2)}, Q_1^{(3)}\}$ and $\{Q_1^{(i)}, Q_r^{(4)}, Q_s^{(5)}\}$. (For example, besides $\{Q_1^{(1)}, Q_1^{(2)}, Q_1^{(3)}\}$, the only other normal triad contained in Q_{11142} is $\{Q_1^{(1)}, Q_4^{(4)}, Q_2^{(5)}\}$, arising from the first entry in (4.2).) $\boxed{QED}$

Next we partition the set of pentads of harmony type $\mathcal{T}(4,1)$ into three classes:

$P \in \mathcal{P}'_1$ if P contains a normal tetrad, e.g. Q_{11115};
$P \in \mathcal{P}'_2$ if $P \notin \mathcal{P}_1$ but P contains a normal triad, e.g. Q_{11125};
$P \in \mathcal{P}'_3$ if P contains no normal triads, e.g. Q_{11235}.

Finally we partition the set of pentads of harmony type $\mathcal{T}(3,2)$ into two classes:
$P \in \mathcal{P}''_2$ if P contains a normal triad, e.g. Q_{11155};
$P \in \mathcal{P}''_3$ if P contains no normal triads, e.g. Q_{11255}.

Using Lemma 8(i), Lemma 9 and elementary calculations, we can determine the sizes of these nine classes of pentad. We list them in the following Lemma.

Lemma 10. *$|\mathcal{P}_0| = 32$, $|\mathcal{P}_1| = 480$, $|\mathcal{P}_2| = 1440$, $|\mathcal{P}_3| = 96$, $|\mathcal{P}'_0| = 640$, $|\mathcal{P}'_1| = 7680$, $|\mathcal{P}'_2| = 1920$, $|\mathcal{P}''_0| = 5120$, and $|\mathcal{P}''_1| = 15360$.*

In the remaining sections, we will show that there are precisely nine distinct GL(6, 2)-orbits of Qbooks and they are the nine classes listed above.

4.1 Extending the 2Qbook ${}^2\mathcal{B}$ to a 3Qbook ${}^3\mathcal{B}$

By making a choice of quatrain in $\mathcal{Q}_+$ for the page σ_3 of $\mathcal{B}$, we thereby extend our standard 2Qbook ${}^2\mathcal{B}_{11}$ to a 3Qbook ${}^3\mathcal{B}_{11r}$ with $1 \le r \le 4$. In fact, by Lemmas 7 and 3, we may suppose that $r \in \{1,2\}$ and we get

$$\mathcal{G}_0({}^3\mathcal{B}_{111}) = \mathcal{G}_{144}; \qquad \mathcal{G}_0({}^3\mathcal{B}_{112}) = \mathcal{G}^*_{48}. \tag{4.3}$$

4.2 Qbooks of type $\mathcal{T}(3,2)$

In the case of Qbooks of harmony type $\mathcal{T}(3,2)$, we may extend the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ to full Qbooks directly.

Theorem 1. *There are exactly two* GL(6, 2)*-orbits of full Qbooks of type $\mathcal{T}(3,2)$ in* PG(5, 2)*. They are represented by ${}^5\mathcal{B}_{11155}$ and ${}^5\mathcal{B}_{11255}$, whose pentads belong to $\mathcal{P}''_2$ and $\mathcal{P}''_3$, respectively.*

Proof. $\mathcal{A}_{16}$ fixes the quatrains $\{Q^{(i)}_r\}_{1\le r\le 4}$ in each page σ_i and acts transitively on the sixteen pairs of quatrains $\{Q^{(4)}_r, Q^{(5)}_s\}_{5\le r,s\le 8}$ by Lemma 2. So, up to isomorphism, the extensions ${}^5\mathcal{B}_{111rs}$ and ${}^5\mathcal{B}_{112rs}$ of type $\mathcal{T}(3,2)$ to the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ are represented by those with $r = s = 5$. $\boxed{QED}$

From Lemma 3, we have

$$\mathcal{G}_0({}^5\mathcal{B}_{11155}) = \langle W, W^*\rangle; \qquad \mathcal{G}_0({}^5\mathcal{B}_{11255}) = \langle W^*\rangle. \tag{4.4}$$

4.3 Qbooks of type $\mathcal{T}(5,0)$

We now determine up to isomorphism those extensions of the 3Qbooks ${}^3\mathcal{B}_{111}$ and ${}^3\mathcal{B}_{112}$ to full Qbooks ${}^5\mathcal{B}_{111rs}$ and ${}^5\mathcal{B}_{112rs}$ of type $\mathcal{T}(5,0)$.

By (4.3), $\mathcal{G}_0({}^3\mathcal{B}_{111}) = \mathcal{G}_{144}$ and, by (3.14), the $\mathcal{G}_0({}^3\mathcal{B}_{111})$-orbits of quatrains for the page σ_4 are $\Omega^{(4)}_A, \Omega^{(4)}_B$ and $\Omega^{(4)}_C$. Hence, up to isomorphism, there are thus just two kinds of 4Qbook which extend ${}^3\mathcal{B}_{111}$ such that the four quatrains are in harmony. They are represented by ${}^4\mathcal{B}_{1111}$ and ${}^4\mathcal{B}_{1112}$. From Lemma 3, it follows that the two subgroups $\mathcal{G}_0({}^4\mathcal{B})$ of $\mathcal{G}_0(\mathcal{B})$ which stabilize these two 4Qbooks are

$$\mathcal{G}_0({}^4\mathcal{B}_{1111}) = \mathcal{G}_{144}; \qquad \mathcal{G}_0({}^4\mathcal{B}_{1112}) = \mathcal{G}^*_{48}. \tag{4.5}$$

Since $\mathcal{G}_0({}^3\mathcal{B}_{111}) = \mathcal{G}_{144}$, the $\mathcal{G}_{144}$-orbits involving quatrains for the pages σ_j, $j \in \{4,5\}$, in harmony with those of ${}^3\mathcal{B}_{111}$ lie in $\Omega_A^{(j)}$ and $\Omega_B^{(j)}$. Furthermore, $\mathcal{G}^*_{48}$ fixes each quatrain in $\Omega_B^{(j)}$ as $\mathcal{G}^*_{48} \lhd \mathcal{G}_{144}$. Thus, up to isomorphism, there are at most four extensions of ${}^3\mathcal{B}_{111}$ to a full Qbook of type $\mathcal{T}(5,0)$, and these are ${}^3\mathcal{B}_{11111}$, ${}^3\mathcal{B}_{11112}$, ${}^3\mathcal{B}_{11121}$ and ${}^3\mathcal{B}_{11122}$. Since B maps ${}^3\mathcal{B}_{11121}$ to ${}^3\mathcal{B}_{11112}$, there are at most three extensions. However, the pentads of the three Qbooks ${}^5\mathcal{B}_{11111}$, ${}^5\mathcal{B}_{11112}$, and ${}^5\mathcal{B}_{11122}$ are of different types; indeed,

$$Q_{11111} \in \mathcal{P}_0, \quad Q_{11112} \in \mathcal{P}_1, \quad Q_{11122} \in \mathcal{P}_2. \tag{4.6}$$

Consequently, there are exactly three GL$(6,2)$-orbits of extension of the 3Qbook ${}^3\mathcal{B}_{111}$ to a full Qbook of type $\mathcal{T}(5,0)$, and they are represented by ${}^5\mathcal{B}_{11111}$, ${}^5\mathcal{B}_{11112}$, and ${}^5\mathcal{B}_{11122}$.

Next we look at extensions of the 3Qbook ${}^3\mathcal{B}_{112}$ to a full Qbook ${}^5\mathcal{B}_{112rs}$ with $r,s \in \{1,2,3,4\}$. By Lemma 1(ii), B^2 maps ${}^5\mathcal{B}_{11211}$ to ${}^5\mathcal{B}_{11112}$ and B maps ${}^5\mathcal{B}_{112r1}$ to ${}^5\mathcal{B}_{1112r}$, $r \in \{2,3,4\}$, which as extensions of ${}^3\mathcal{B}_{111}$ have already been considered. Then, $\bar{N}_{(45)}$, defined in Section 3.1 and (1.6), maps ${}^5\mathcal{B}_{1121s}$ for $s \in \{2,3,4\}$ to ${}^5\mathcal{B}_{112s'1}$ for some $s' \in \{2,3,4\}$, and this has already been considered. Of the remaining nine extensions ${}^5\mathcal{B}_{112rs}$, $r,s \in \{2,3,4\}$, of ${}^3\mathcal{B}_{112}$, we see from (4.2) that eight have pentads of types $\mathcal{P}_1$ and $\mathcal{P}_2$, and the ninth, ${}^5\mathcal{B}_{11232}$, has a harmonious pentad of type $\mathcal{P}_3$.

Lemma 11. *For $k = 0,1,2,3$ all pentads in $\mathcal{P}_k$ are isomorphic.*

Proof. If $P_1 \in \mathcal{P}_1$ then let $P_0 \in \mathcal{P}_0$ be the normal pentad which shares four of its quatrains with P_1. By Lemma 8(iii) there exists $K \in \mathcal{C}_{16}$ such that $K(P_0) = Q_{11111}$. Hence $K(P_1)$ agrees with Q_{11111} in four of its places, and so for some power B^h of B, see Lemma 1(ii), we have $B^h K(P_1) = Q_{1111r}$ for some $r \in \{2,3,4\}$. Hence each $P_1 \in \mathcal{P}_1$ is isomorphic to Q_{11112}. Similar considerations, using appropriate elements of $\mathcal{G}(\mathcal{B})$, allow us to prove that all pentads in $\mathcal{P}_2$ are isomorphic. Finally recall, see the preamble to the Lemma, that the 3Qbook ${}^3\mathcal{B}_{112}$ has a *unique* extension to a Qbook having a harmonious pentad of kind $\mathcal{P}_3$. Consequently, all pentads in $\mathcal{P}_3$ are isomorphic. $\boxed{QED}$

Our results for Qbooks of type $\mathcal{T}(5,0)$ are thus as in the next theorem.

Theorem 2. *There exist in* PG(5, 2) *just four* GL(6, 2)*-orbits of full Qbooks of type* $\mathcal{T}(5,0)$*, with representatives:*

$$^5\mathcal{B}_{11111}, \quad ^5\mathcal{B}_{11112}, \quad ^5\mathcal{B}_{11122} \text{ and } \quad ^5\mathcal{B}_{11232}. \tag{4.7}$$

The pentads corresponding to the Qbooks in (4.7) belong to $\mathcal{P}_0$*,* $\mathcal{P}_1$*,* $\mathcal{P}_2$*, and* $\mathcal{P}_3$*, respectively.*

Furthermore, using (4.3) and Lemma 3, we get

$$\mathcal{G}_0(^5\mathcal{B}_{11111}) = \mathcal{G}_{144}; \quad \mathcal{G}_0(^5\mathcal{B}_{11112}) = \mathcal{G}_0(^5\mathcal{B}_{11122}) = \mathcal{G}_0(^5\mathcal{B}_{11232}) = \mathcal{G}^*_{48}. \tag{4.8}$$

4.4 Qbooks of type $\mathcal{T}(4,1)$

It follows from Lemma 2(ii) that there is just one GL(6, 2)-orbit of Qbooks of type $\mathcal{T}(4,1)$ which extend the 4Qbook $^4\mathcal{B}_{1111}$, and it contains $^5\mathcal{B}_{11115}$. Similarly, there is just one GL(6, 2)-orbit of Qbooks of type $\mathcal{T}(4,1)$ which extend the 4Qbook $^4\mathcal{B}_{1112}$, and it contains $^5\mathcal{B}_{11125}$. These two orbits are distinct, since $^5\mathcal{B}_{11115}$ and $^5\mathcal{B}_{11125}$ have pentads of types $\mathcal{P}'_1$ and $\mathcal{P}'_2$, respectively. The extensions $^5\mathcal{B}_{1115r}$, $r \in \{1,2,3,4\}$, of the 4Qbook $^4\mathcal{B}_{1115}$, belong to the second orbit above since the involution $\underset{\sim}{N}_{(45)}$ maps $^5\mathcal{B}_{1115r}$ to $^5\mathcal{B}_{111r'5}$ for some $r' \in \{1,2,3,4\}$.

Any remaining orbits of type $\mathcal{T}(4,1)$ extensions of $^3\mathcal{B}_{112}$ must contain at least one of the Qbooks $^5\mathcal{B}_{112r5}$, $r \in \{2,3,4\}$. From (4.2) we see that $^5\mathcal{B}_{11225}$ and $^5\mathcal{B}_{11245}$ have pentads of type $\mathcal{P}'_2$ and $^5\mathcal{B}_{11235}$ has a pentad of type $\mathcal{P}'_3$. Arguing as in Lemma 11, we see that $^5\mathcal{B}_{112r5}$, $r \in \{2,4\}$, is isomorphic to $^5\mathcal{B}_{11125}$. For Qbooks of type $\mathcal{T}(4,1)$ our results are thus as in the next theorem.

Theorem 3. *There exist in* PG(5, 2) *just three* GL(6, 2)*-orbits of full Qbooks of type* $\mathcal{T}(4,1)$*, with representatives:*

$$^5\mathcal{B}_{11115} \in \mathcal{P}'_1; \quad ^5\mathcal{B}_{11125} \in \mathcal{P}'_2; \quad ^5\mathcal{B}_{11235} \in \mathcal{P}'_3. \tag{4.9}$$

Furthermore, using (4.3) and Lemma 3, we get

$$\mathcal{G}_0(^5\mathcal{B}_{11115}) = \mathcal{G}^{(5)}_{36}; \quad \mathcal{G}_0(^5\mathcal{B}_{11125}) = \mathcal{G}_0(^5\mathcal{B}_{11235}) = \mathcal{A}^{(5)}_{12}. \tag{4.10}$$

Remark 5. The book spread $\mathcal{S}_{11115} := \mathcal{S}_{21}(^5\mathcal{B}_{11115})$ is unusual in that it can be viewed as a quatrain book in two different ways! To see this, recall from Remark 2 that the book spread $\mathcal{S}_{11111} := \mathcal{S}_{21}(^5\mathcal{B}_{11111})$ can be viewed as a quatrain book in 21 ways. In particular it is a quatrain book with spine the line $\{a_5, b_5, c_5\} \in Q^{(5)}_1$. Now if we replace the regulus $\begin{pmatrix} a_5+u & b_5+v & c_5+w \\ a_5+v & b_5+w & c_5+u \\ a_5+w & b_5+u & c_5+v \end{pmatrix}$ in

σ_5 by its opposite $\begin{pmatrix} a_5+u & b_5+w & c_5+v \\ a_5+w & b_5+v & c_5+u \\ a_5+v & b_5+u & c_5+w \end{pmatrix}$ we thereby convert $Q_1^{(5)}$ to $Q_5^{(5)}$ and ${}^5\mathcal{B}_{11111}$ to ${}^5\mathcal{B}_{11115}$. So $\mathcal{S}_{11115}$ is a quatrain book with spine $\{a_5, b_5, c_5\}$ as well as a quatrain book with spine $\mu = \{u, v, w\}$.

5 The complete classification of Qbooks in $\mathrm{PG}(5,2)$

In Table 1, we list data relating to the nine GL(6, 2)-orbits of Qbooks found in Theorems 1, 2, and 3. The second column lists a representative for each orbit. Since $|\mathcal{G}(\mathcal{B}) : \mathcal{G}({}^5\mathcal{B})|$ is the size of the $\mathcal{G}(\mathcal{B})$-orbit of the Qbook ${}^5\mathcal{B}$, we determine the entries in the third column using Lemma 10. The groups $\mathcal{G}_0(\mathcal{B})$ for the various representatives have been found in (4.8), (4.10) and (4.4), and are listed in the fourth column. From the third and fourth column, we calculate the order of $\mathcal{G}({}^5\mathcal{B})/\mathcal{G}_0({}^5\mathcal{B})$, which gives the action of $\mathcal{G}({}^5\mathcal{B})$ on the pages of the Qbook, and we list it in the fifth column. In each case, we then find a subgroup of $\mathcal{G}({}^5\mathcal{B})$ of this order modulo $\mathcal{G}_0({}^5\mathcal{B})$. This subgroup, together with $\mathcal{G}_0({}^5\mathcal{B})$, generates $\mathcal{G}({}^5\mathcal{B})$, which we describe in Section 5.1. In Section 5.2, we comment briefly on the full automorphism groups of these spreads, and we list their orders in the sixth column.

Type	Content of ${}^5\mathcal{B}$	$\lvert\mathcal{G}({}^5\mathcal{B})\rvert$	$\mathcal{G}_0({}^5\mathcal{B})$	$\lvert\mathcal{G}({}^5\mathcal{B})/\mathcal{G}_0({}^5\mathcal{B})\rvert$	$\lvert\mathcal{G}(\mathcal{S}_{21})\rvert$
$\mathcal{T}(5,0)$	$Q_{11111} \in \mathcal{P}_0$	17280	$\mathcal{G}_{144}$	120	$362880 = 7.5.3^4.2^7$
	$Q_{11112} \in \mathcal{P}_1$	1152	$\mathcal{G}^*_{48}$	24	$1152 = 3^2.2^7$
	$Q_{11122} \in \mathcal{P}_2$	384	$\mathcal{G}^*_{48}$	8	$384 = 3.2^7$
	$Q_{11232} \in \mathcal{P}_3$	5760	$\mathcal{G}^*_{48}$	120	$5760 = 5.3^2.2^7$
$\mathcal{T}(4,1)$	$Q_{11115} \in \mathcal{P}'_1$	864	$\mathcal{G}^{(5)}_{36}$	24	$1728 = 3^3.2^6$
	$Q_{11125} \in \mathcal{P}'_2$	72	$\mathcal{G}^{(5)}_{12}$	6	$72 = 3^2.2^3$
	$Q_{11235} \in \mathcal{P}'_3$	288	$\mathcal{A}^{(5)}_{12}$	24	$288 = 3^2.2^5$
$\mathcal{T}(3,2)$	$Q_{11155} \in \mathcal{P}''_2$	108	$\langle W, W^*\rangle$	12	$108 = 3^3.2^2$
	$Q_{11255} \in \mathcal{P}''_3$	36	$\langle W^*\rangle$	12	$36 = 3^2.2^2$

Table 1. The Qbook orbit representatives

5.1 The nine stabilizer groups $\mathcal{G}({}^5\mathcal{B})$

Qbooks of harmony type $\mathcal{T}(5,0)$

(1): $\bar{N}_{(12)}$ and $\bar{N}_{(12345)}$ act on the quatrains of the pentad Q_{11111} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(1)}, Q_1^{(2)}, Q_1^{(3)}, Q_1^{(4)}, Q_1^{(5)})$, respectively. Hence, these two elements generate a group inducing Sym(5) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11111}) = \langle \mathcal{G}_{144}, \bar{N}_{(12)}, \bar{N}_{(12345)} \rangle \cong ((Z_2^4 : Z_3) \times Z_3) : \mathrm{Sym}(5) \tag{5.1}$$

(2): $\bar{N}_{(12)}$ and $W\bar{N}_{(234)}$ act on the pentad Q_{11112} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(2)}, Q_1^{(3)}, Q_1^{(4)})$, respectively. Hence, these two elements generate a group inducing Sym(4) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11112}) = \langle \mathcal{G}_{48}^*, \bar{N}_{(12)}, W\bar{N}_{(234)} \rangle \cong (Z_2^4 : Z_3) : \mathrm{Sym}(4) \tag{5.2}$$

(3): $W^2\bar{N}_{(13)}$ and $K_1\bar{N}_{(15)(34)}K_1$ act on the pentad Q_{11122} as $(Q_1^{(1)}, Q_1^{(3)})$ and $(Q_1^{(1)}, Q_2^{(5)})(Q_1^{(3)}, Q_2^{(4)})$, respectively. Hence, these two elements generate a group inducing $\mathcal{D}_8$ on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11122}) = \langle \mathcal{G}_{48}^*, W^2\bar{N}_{(13)}, K_1\bar{N}_{(15)(34)} \rangle \cong (Z_2^4 : Z_3) : \mathcal{D}_8 \tag{5.3}$$

(4): $\bar{N}_{(12)}$ and $K_4''\bar{N}_{(12345)}K_4''$ act on the pentad Q_{11232} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(1)}, Q_1^{(2)}, Q_2^{(3)}, Q_3^{(4)}, Q_2^{(5)})$, respectively. Hence, these two elements generate a group inducing Sym(5) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11232}) = \langle \mathcal{G}_{48}^*, \bar{N}_{(12)}, K_4''\bar{N}_{(12345)}K_4'' \rangle \cong (Z_2^4 : Z_3) : \mathrm{Sym}(5) \tag{5.4}$$

Qbooks of harmony type $\mathcal{T}(4, 1)$

(5): $\bar{N}_{(12)}$ and $\bar{N}_{(234)}$ act on the pentad Q_{11115} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(2)}, Q_1^{(3)}, Q_1^{(4)})$, respectively. Hence, these two elements generate a group inducing Sym(4) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11115}) = \langle \mathcal{G}_{36}^{(5)}, \bar{N}_{(12)}, \bar{N}_{(234)} \rangle \cong ((Z_2^2 : Z_3) \times Z_3) : \mathrm{Sym}(4) \tag{5.5}$$

(6): $W\bar{N}_{(12)}$ and $\bar{N}_{(23)}$ act on the pentad Q_{11125} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(2)}, Q_1^{(3)})$, respectively. Hence, these two elements generate a group inducing Sym(3) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11125}) = \langle \mathcal{A}_{12}^{(5)}, W\bar{N}_{(12)}, \bar{N}_{(23)} \rangle \cong (Z_2^2 : Z_3) : \mathrm{Sym}(3) \tag{5.6}$$

(7): $\bar{N}_{(12)}$ and $K_1\bar{N}_{(234)}K_1$ act on the pentad Q_{11235} as $(Q_1^{(1)}, Q_1^{(2)})$ and $(Q_1^{(2)}, Q_2^{(3)}, Q_3^{(4)})$, respectively. Hence, these two elements generate a group inducing Sym(4) on the pages of the book. So,

$$\mathcal{G}(^5\mathcal{B}_{11235}) = \langle \mathcal{A}_{12}^{(5)}, \bar{N}_{(12)}, K_1\bar{N}_{(234)}K_1 \rangle \cong (Z_2^2 : Z_3) : \mathrm{Sym}(4) \tag{5.7}$$

Qbooks of harmony type $\mathcal{T}(3,2)$

(8): $\bar{N}_{(12)}$, $\bar{N}_{(23)}$, and $\bar{N}_{(45)}$ act on the pentad Q_{11155} as $(Q_1^{(1)}, Q_1^{(2)})$, $(Q_1^{(2)}, Q_1^{(3)})$, and $(Q_5^{(4)}, Q_5^{(5)})$, respectively. Hence, these two elements generate a group inducing $\mathrm{Sym}(3) \times Z_2$ on the pages of the book. So,

$$\mathcal{G}({}^5\mathcal{B}_{11155}) = \langle W, W^*, \bar{N}_{(12)}, \bar{N}_{(23)}, \bar{N}_{(45)} \rangle \cong Z_3^2 : (\mathrm{Sym}(3) \times Z_2) \tag{5.8}$$

(9): $\bar{N}_{(12)}$, $K'_1\bar{N}_{(13)}K'_1$, and $\bar{N}_{(45)}$ act on the pentad Q_{11255} as $(Q_1^{(1)}, Q_1^{(2)})$, $(Q_1^{(1)}, Q_2^{(3)})$, and $(Q_5^{(4)}, Q_5^{(5)})$, respectively. Hence, these two elements generate a group inducing $\mathrm{Sym}(3) \times Z_2$ on the pages of the book. So,

$$\mathcal{G}({}^5\mathcal{B}_{11255}) = \langle W^*, \bar{N}_{(12)}, K'_1\bar{N}_{(13)}K'_1, \bar{N}_{(45)} \rangle \cong Z_3 : (\mathrm{Sym}(3) \times Z_2) \tag{5.9}$$

5.2 The spread groups

We have seen in Section 1.2.1 that $\mathcal{S}_{11111}$, the Desarguesian spread, has automorphism group $\Gamma\mathrm{L}(3,4)$ which acts transitively on the lines of the spread. So, every line can act as a spine.

In seven of the remaining eight spreads, we find that there is just one line which, together with each of the other twenty lines, generates exactly five solids in $\mathrm{PG}(5,2)$. As this line is the spine of the book, we see that every automorphism of the spread ${}^5\mathcal{B}$ is in $\mathcal{G}({}^5\mathcal{B})$. That is, $\mathcal{G}({}^5\mathcal{B})$ is the full automorphism group of the spread.

The remaining spread is $\mathcal{S}_{11115}$. Here, we find that there are just two lines which, together with each of the other twenty lines, generate exactly five solids in $\mathrm{PG}(5,2)$. So, there are two lines which can act as the spines of a Qbook for $\mathcal{S}_{11115}$. We observed in Remark 5 that there is an automorphism of $\mathcal{S}_{11115}$ which interchanges these two spines. This automorphism can be realized by the involution which fixes a_1 and b_1 and interchanges a_5 with u and b_5 with v.

5.3 Invariant sequences

Lemma 12. *Let $\mathcal{S}_{21}$ be a line spread in* $\mathrm{PG}(5,2)$*, and let H be any hyperplane of* $\mathrm{PG}(5,2)$*. Then precisely five lines of $\mathcal{S}_{21}$ lie inside H.*

Proof. The 32 points of H^c account for 16 lines of $\mathcal{S}_{21}$ which meet H in a point. The remaining $31 - 16 = 15$ points of H must therefore support the remaining $21 - 16 = 5$ lines of $\mathcal{S}_{21}$. $\boxed{QED}$

Thus $\mathcal{S}_{21}$ gives rise to an induced partial spread $\mathcal{S}_5(H)$ in each hyperplane H of $\mathrm{PG}(5,2)$. Now all partial line spreads in $\mathrm{PG}(4,2)$ have been classified in [5, Table B.1]. In particular there exist in $\mathrm{PG}(4,2)$ ten projectively distinct kinds of partial spreads of size 5. Of the 63 partial spreads $\mathcal{S}_5(H)$ determined by the spread $\mathcal{S}_{21}$ suppose that precisely N_x belong to class Vx.1, x = a, b, ... , j; see [5, Table B.1]. Then we will say that the sequence $(N_a, N_b, \dots, N_j)$ is the *invariant sequence* $\mathcal{I}(\mathcal{S}_{21})$ of the spread $\mathcal{S}_{21}$.

Clearly spreads $\mathcal{S}_{21}$, $\mathcal{S}'_{21}$ in PG(5, 2) which have different invariant sequences will be non-isomorphic. In the case of a book spread determined by a Qbook ${}^5\mathcal{B}$ then the 15 hyperplanes through the spine μ of $\mathcal{B}$ contribute $(0, 0, \dots, 15)$ to the invariant sequence. In order to determine the full invariant sequence one needs to determine the contributions of the 48 hyperplanes $\mathcal{H}_{48}$ which meet μ in a point. In some unpublished research in 2004 R. Shaw succeeded in doing this by first finding the orbit structure of these 48 hyperplanes under the action of various relevant groups. His 2004 results are listed in column 3 in Table 2. These invariants are related to the invariants calculated by computer in [13], namely the number n_m of 3-flats containing m spread lines is calculated in [13] for $m = 3, 4, 5$. In particular $N_j = 3n_5$ and $N_i = 3n_4$. (To see that $N_i = 3n_4$, observe that if σ is a solid which contains 4 lines $\mathcal{S}_4$ of the spread $\mathcal{S}_{21}$ then the three hyperplanes which contain σ are $H_r := \langle \sigma, \lambda_r \rangle$, where λ_1, λ_2 and λ_3 denote the three lines of $\mathcal{S}_{21} \setminus \mathcal{S}_4$ which meet σ in a point. Since the induced partial spread $\mathcal{S}_5(H_r)$ in each of these 3 hyperplanes is thus of class Vi.1 it follows that $N_i = 3n_4$.)

Type	Content of ${}^5\mathcal{B}$	Invariant sequence
$\mathcal{T}(5,0)$	$Q_{11111} \in \mathcal{P}_0$	$(0,0,0,0,0,0,0,0,0,63)$
	$Q_{11112} \in \mathcal{P}_1$	$(0,0,0,0,0,0,0,0,48,15)$
	$Q_{11122} \in \mathcal{P}_2$	$(0,0,0,0,0,0,0,48,0,15)$
	$Q_{11232} \in \mathcal{P}_3$	$(48,0,0,0,0,0,0,0,0,15)$
$\mathcal{T}(4,1)$	$Q_{11115} \in \mathcal{P}'_1$	$(0,0,0,0,0,0,0,0,36,27)$
	$Q_{11125} \in \mathcal{P}'_2$	$(0,0,0,0,0,0,0,36,12,15)$
	$Q_{11235} \in \mathcal{P}'_3$	$(12,0,0,0,0,0,0,36,0,15)$
$\mathcal{T}(3,2)$	$Q_{11155} \in \mathcal{P}''_2$	$(0,0,0,0,0,0,0,27,18,18)$
	$Q_{11255} \in \mathcal{P}''_3$	$(3,0,0,0,0,0,0,36,9,15)$

Table 2. The invariant sequences

6 Computer-aided check of the main results

Book spreads in PG(5, 2) were first considered by R. Shaw who, in some unpublished 2004 research, classified them into nine different classes. The more detailed results in Table 1 have received two independent computer-aided checks, which we now describe.

Firstly, in 2004, T.P. McDonough verified using GAP [3] that there were exactly nine different kinds of quatrain books in $\mathrm{PG}(5,2)$. One observes initially that, since $\mathcal{G} = \mathrm{PGL}(6,2)$ acts transitively on the lines of $\mathrm{PG}(5,2)$, every such spread is equivalent to one with spine μ and, since $\mathrm{PGL}(4,2)$ is transitive on line spreads in $\mathrm{PG}(3,2)$, every such spread is equivalent to one whose pages are the pages of the standard book $\mathcal{B}$. Now observe that two quatrain books in $\mathcal{B}$ are $\mathcal{G}$-equivalent if, and only if, they are $\mathcal{G}(\mathcal{B})$-equivalent. The GAP program thus determines the $\mathcal{G}(\mathcal{B})$-orbits of quatrain books in $\mathcal{B}$. Representatives of the nine orbits are the pentads of quatrains $Q^{(1)}_{j_1}Q^{(2)}_{j_2}Q^{(3)}_{j_3}Q^{(4)}_{j_4}Q^{(5)}_{j_5}$ where the quintuples $j_1j_2j_3j_4j_5$ are listed in the first row of Table 3. The second row of the table lists the orders of the stabilizers of the quatrain books in $\mathcal{G}(\mathcal{B})$. The third row of the table lists the orders of the stabilizers of the quatrain books in $\mathcal{G}$. The GAP program is available from the authors.

11111	11112	11115	11122	11125	11155	11242	11245	11255
17280	1152	864	384	72	108	5760	288	36
362880	1152	1728	384	72	108	5760	288	36

Table 3. Stabilizer orders

Secondly, the classification of all line spreads in $\mathrm{PG}(5,2)$ [13] shows that up to isomorphism there are only 9 spreads for which there are at least five 3-dimensional subspaces containing 5 spread lines. However, it is not clear that they are all book spreads. So S. Topalova recently carried through some new computer-aided work, briefly described below, which concentrated solely on book spreads. Moreover, some intermediate theoretical results, proved in the present paper, were also checked. This was done by our own software written in C++.

Without loss of generality we may fix the first 6 lines of the spread, namely the spine, the lines of the first quatrain, and one line of the second quatrain. We choose the remaining 15 spread lines from a set D of 102 lines (out of all 651 lines of the projective space) that are skew to each of the fixed 6 ones. We construct the spread by backtrack search adding to the set of these 6 lines the remaining 3 lines of the second quatrain, then the 4 lines of the third quatrain, of the fourth and of the fifth one. The lines of D are ordered lexicographically, and each line we choose is greater than the previous line of the same quatrain or if it is the first line of a quatrain, it is greater than the first line of the previous quatrain. We obtain $2048 = (2.1.1).(8.2.1.1).(4.2.1.1).(4.2.1.1)$ spreads, and from them we see that:

1. There are 2 choices for the second line of the second quatrain and only

one choice for each of the other two lines of the second quatrain — this is in agreement with the fact that each skew pair of lines (namely the spine and the first line of the second quatrain) is in two spreads in $PG(3,2)$ — section 1.2.2 and with the fact that the spine and one more line fix exactly 2 quatrains (one from $\mathcal{Q}_+$ and one from $\mathcal{Q}_-$ (Section 2).

2. There are 8 choices for the first line of the third quatrain. This is in agreement with the following facts: There are 2 possibilities for choosing $\{a_3, b_3, c_3\}$ because each skew pair of lines (namely $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$) is in two spreads in $PG(3,2)$ — Section 1.2.2 . Then there are 4 ways of choosing the smallest line of a quatrain because there are 8 quatrains, but one and the same line is in one quatrain from (2.2) and in one quatrain from (2.3).

3. Similar to 1.: there are 2 different choices for the second line of the third quatrain and a unique choice for the other two lines.

4. There are 4 choices for the first line of the fourth (fifth) quatrain — $\{a_i, b_i, c_i\}$ are already fixed so these are the 4 ways of choosing the smallest line of a quatrain (see 2.) and then there are 2 different choices for the second line and a unique choice for the other two lines.

There are 18 automorphisms which fix the first six lines of the spread. To check for isomorphism we use the same technique as in [13], i.e. apply automorphisms of $PG(5,2)$ which map the spread lines to the fixed six lines in all possible ways, and then these 18 automorphisms. We find out that there are 9 nonisomorphic book spreads.

During the isomorphism check we also determine the automorphism groups which stabilize the spreads and their subgroups which preserve the spine. The orders of these groups are the same as those which are obtained theoretically and presented in columns 6 and 3 of Table 1.

Concerning the possibility of a future computer-aided study of book spreads in $PG(7,2)$, it would seem, after some initial investigations, that although they are only a small part of all line spreads, their number is too big for a full computer-aided classification to be possible. Thus book spreads in $PG(7,2)$ with certain additional properties ought to be considered, and the knowledge of the structure of $PG(5,2)$ book spreads and their stabilizers gained in the present paper will presumably be very helpful.

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