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CONVEX OPERATORS ON RIESZ SPACES AND THEIR DUALITY

SHOZO KOSHI

Dedicated to the memory of Professor Gottfried Köthe

1. In this note, we shall explain a new duality theorem concerning convex operators from a Riesz space E to another Riesz space F. Riesz space means here a vector lattice. Duality theorems on convex operators from a topological vector space to a Dedekind complete Riesz space is considered by many authors [1], [5]. But the key point in these argument is to fix the range space F, so that we can not treat dual operator as an operator from dual space F^* to E^*, but we can deal with dual operators from F^* to E^* in this new theory.

We shall explain only main idea of the theory in this note. At first we shall consider a band preserving convex operator ϕ from a Dedekind complet Riesz space E into a Dedekind complete Riesz space F.

A linear sublattice N of a Dedekind complete Riesz space E is called a (projection) band if

$$N + N^{\perp} = E$$

where $N^{\perp}$ is an orthogonal complement of N. By the above decomposition, we can define a projection operator $[N]$ from E onto the projection band N with $[N]x = x_1$ for $x \in E$ and

$$x = x_1 + x_2, \quad x_1 \in N, x_2 \in N^{\perp}.$$

We have the following property for N.

$$[N][N] = [N]$$

and

$$[N]E = N.$$

It is known that every projection operator is order continuous i.e. $[N]x_\lambda \to [N]x$ for all order convergent directed system $\{x_\lambda\}$ with $x_\lambda \uparrow x$.

It is known that all projection bands in E can be considered as a Boolean lattice and this Boolean lattice is complete since E is Dedekind complete.

Two Dedekind complete Riesz spaces E and F are called band-isomorphic if there exists an isomorphism between two Boolean lattice of projection bands of E and F as a Boolean lattice structure, i.e. there is a correspodnence N to N' by the isomorphism of Boolean lattices of projection bands for E and F.

2. An operator ϕ from E to F is called band preserving if

$$\phi[N] = [N']\phi$$

for all projection bands N and N' where N and N' are corresponding by the isomorphism, when E and F are band isomorphic to each others.

We have the following lemma.

Lemma 1. *If ϕ is band preserving from E to F, then*

$$\phi(x+y) = \phi(x) + \phi(y) \qquad if \qquad |x| \cap |y| = 0 \ (i.e. x \perp y)$$

and $\phi(x) \perp \phi(y)$. Moreover, $\phi(0) = 0$.

In the following, we assume that every Riesz space E or F in this note is a Nakano space. A Dedekind complete Riesz space E is called a Nakano space if there exists a functional m on E such that

1) $0 \leq m(x) \leq +\infty$ for $x \in E$,
2) if $m(\alpha x) = 0$ for all $\alpha \geq 0$, then $x = 0$ and $m(\alpha x)$ is a convex function of real α,
3) for any $x \in E$, there exists $\alpha > 0$ such that $m(\alpha x) < +\infty$.
4) $|x_1| \leq |x_2|$ imply $m(x_1) \leq m(x_2)$,
5) $x \perp y$ imply $m(x+y) \leq m(x_2)$,
6) $x_\lambda \uparrow$ imply $m(x) = \sup_\lambda m(x_\lambda)$,
7) if $x_\lambda \uparrow$ and $\sup m(x_\lambda) < \infty$, then $x = \sup x_\lambda$ exists in E.

Last property is called monotone complete or Lebeque property.

Also we assume the Boolean lattices of projection bands of E is non-atomic, i.e. for every non-zero projection $[N]$ is always devided into two non-zero projections orthogonal to each others with $[N] = [N_1] + [N_2]$ and $N_1 \perp N_2$.

In this situation, Shimogaki proved the following:

Lemma 2. *Let E be a Dedekind complete non-atomic Nakano space and ϕ be a band preserving operator from E to E. Then there exist an element $c \geq 0$ and a positive number γ such that*

$$|\phi(x)| \leq c + \gamma|x| \qquad for\ all \quad x \in E.$$

From this lemma, we can easily see that ϕ is order bounded i.e. for all $x \in E$, there exists $\sup\{|\phi(a)|; |a| \leq |x|\}$ in F.

For the continuity of convex band preserving operators, we have the following lemma.

Lemma 3. *Let E and F be band isomorphic Nakano spaces and let ϕ be a band preserving convex operator from E to F. Then, $x_\lambda \downarrow 0$ imply $\phi(x_\lambda) \to 0$ (in order).*

ϕ is called convex if

$$\phi(\alpha x + \beta y) \leq \alpha\phi(x) + \beta\phi(y)$$

for $x, y \in E$ and $\alpha, \beta \geq 0$ with $\alpha + \beta = 1$.

Let Λ be a set of directed system and $x_\lambda (\lambda \in \Lambda)$ is order convergent to 0 if

$$\sup_\mu \inf_{\lambda \geq \mu} x_\lambda = \inf_\mu \sup_{\lambda \geq \mu} x_\lambda = 0.$$

We write $x_\lambda \to 0$ (in order) if x_λ is order convergent to 0.

Proof. Let a be a positive element of E and I_a be an order ideal generated by a. Any element x of I_a can be represented by integral form of a continuous function on a Stonean compact space whose elements are maximal ideals consisting of Boolean algebra of projections on E. Since ϕ is an band preserving convex operator from E to F, $\phi(x)$ can be represented by integral form

$$\phi(x) = \int G(x(s), s)\, ds;$$

$G(\alpha, s)$ being a function of α and s, where $x(s)$ is a continuous function on maximal ideal space with

$$x = \int x(s)\, ds \qquad (\text{integral form of } x).$$

We can prove that $F(\alpha, s)$ is a convex function of real variable α.

We know that every convex function of real variables is always continuous.

It is easy to see that $x_\lambda \downarrow 0$ then $\phi(x_\lambda) = \int G(x(s), s)\, ds \to 0$, by the methode used in [2], since ϕ transforms an order bounded set of E to an order bounded set of F by Lemma 2.

We say that x_λ is order convergent to x if $x_\lambda - x$ is order convergent to 0.

3. In the following we assume further that Dedekind complete Nakano spaces are always regular i.e. there are sufficiently many order continuous linear functionals in these spaces. Let E and F be two Dedekind complete Nakano spaces and E^* and F^* are the totality of order continuous linear functionals on E and F respectively. We assume more E and F are band isomorphic to each others. We know that E^* and F^* are also Dedekind complete Nakano spaces.

Let $L_b(E, F)$ are the totality of order continuous band preserving linear operators from E to F. We see easily that $L_b(E, F)$ is a Dedekind complete Riesz space.

Lemma 4. *Let E and F be Dedekind complete regular Nakano spaces. Then, there exists a Dedekind complete Riesz space D such that D is band isomorphic to E, F, E^*, F^*, $L_b(E,F), L_b(F^*,E^*)$ and isomorphic to dense ideal of each of $E, F, E^*, F^*, L_b(E,F)$, $L_b(F^*,E^*)$ respectively.*

Proof. Since every Riesz spaces $E, F, E^*, F^*, L_b(E,F)$ and $L_b(F^*,E^*)$ are band isomorphic, we can consider these spaces are imbedded in universal space of almost continuous functions in a locally compact Stonean spaces.

Let D be the intersection of these 6 spaces up to isomorphism. Then, D has the desired property.

Now we shall define the dual of the convex operator ϕ from E to F.

$$\phi^*(A) = \sup_{x\in E}\{A(x) - \phi(x)\} \qquad \text{for } A \in L_b(E,F)$$

So, if we restrict ϕ^* in D, we can consider ϕ^* is considered as a convex operator from a dense ideal of F^* to E^*. But, ϕ^* is not necessary band preserving in general. But, we have

Lemma 5. *Let ϕ^* be the dual of the convex operator ϕ from E to F. Then, $\phi^* - \phi^*(0)$ is a band preserving convex operator on D.*

In the same way, we can defined ϕ^{**} as the dual operator of ϕ^*. Then we have the following duality theorem.

Theorem 6. *Let ϕ be a convex operator from E to F. Then we can define the dual operator ϕ^* as the convex operator from F^* to E^* whose domain is a dense ideal in F^*. Moreover, we have*

$$\phi^{**}(x) = \phi(x) \qquad \text{for all } x \in E.$$

Proof of this theorem is followed by the usual argument in convex analysis.

4. To illustrate the theorem, we shall state here some example of usual function spaces so that the convex operator ϕ and its dual convex operator ϕ^* are defined on whole spaces E or F^*.

We shall consider $L_{p_1}(0,1)$ space and $L_{p_2}(0,1)$ space with $1 < p_1, p_2 < +\infty$. Let ϕ be a convex operator from L_{p_1} to L_{p_2}. The, there exists a function $\phi(u,t)$ defined on reals u and $t \in (0,1)$ such that

$$\phi(f)(t) = \phi(f(t),t) \qquad \text{for } f \in L_{p_1}(0,1).$$

If ϕ is band preserving, $\phi(0,t) = 0$ for all $t \in (0,1)$.

For the dual operator ϕ^* of ϕ, we can find a function $\phi^*(u,t)$ of two variables of real u and $t \in (0,1)$ such that

$$\phi^*(g)(t) = \phi^*(g(t),t).$$

Let q_1 and q_2 be $1/p_1 + 1/q_1$ and $1/p_2 + 1/q_2$. Then $L_{p_1}^* = L_{q_1}$ and $L_{p_2}^* = L_{q_2}$.

By Lemma 2, we have the following proposition.

Proposition 7. *Let ϕ be a band preserving convex operator from L_{p_1} to L_{p_2} and $\phi(u,t)$ be a function with $\phi(f)(t) = \phi(f(t),t)$. Then, we have the following condition: there exist $a \in L_{p_1}$ and a positive number γ_1 with*

$$|\phi(u,t)| \le |a(t)| + \gamma_1 |u|^{p_1/p_2} \qquad \text{for all } u \text{ and } t$$

If the domain of ϕ^* is L_{q_2}, then we have some $b \in L_{q_2}$ and a positive number γ_2 with

$$|\phi^*(v,t)| \le |b(t)| + \gamma_2 |v|^{q_2/q_1}$$

where $\phi^*(v,t)$ is a function with $\phi^*(g)(t) = \phi^*(g(t),t)$.

Moreover, we must have

$$p_1^2 + p_2^2 - p_1 p_2 (p_2 + 1) \ge 0 \tag{3}$$

(or equivalently $q_1^2 + q_2^2 - q_1 q_2 (q_1 + 1) \ge 0$).

Conversely, if (1), (2) and (3) are satisfied, then ϕ and ϕ^* are defined on whole spaces on L_{p_1} and L_{p_2} respectively.

Proof is only an elementary calculation, so it is omitted.

Finally, we remark that we can extend this theory to more general setting. I will write the extended theory in another paper with more precise proofs.

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S. Koshi
Department of Mathematics
Hokkaido University
Sapporo 060, Japan