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ON BASES IN SPACES OF INFINITELY DIFFERENTIABLE FUNCTIONS ON SPECIAL DOMAINS WITH CUSP

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Dedicated to the memory of Professor Gottfried Köthe

This paper is dedicated to the memory of Professor Gottfried Köthe, whose fundamental works had great influence on the arising and the developement of the Rostov-on-Don mathematical school on Functional analysis founded by Professor M.G. Haplanov (1902-1977); the authors of this paper belong to the 3rd and 2nd generation of this school respectively.

The problem of Grothendieck on the existence of bases in nuclear F-spaces was solved negatively by M.M. Zobin and B.S. Mityagin [1], but their construction is very artificial (as well as other results in this direction, see, for example, [2]-[6]); up to now there are no examples of concrete nuclear functional F-spaces without basis.

For a long time the space $C^\infty(\overline{\Omega})$ of all infinitely differentiable functions on a domain with a sufficiently sharp cusp was considered as a candidate for this role [3]-[7].

The interest to these spaces aroused in connection with results [8]-[10] about the dependence of their linear topological properties upon the sharpness of a cusp.

It is interesting also to consider the subspace of all functions which are flat in the point of the cusp (vanishing with all derivatives), because such a subspace is topologically as complicated as the whole space (see proposition below).

The purpose of the present article is the construction of bases in the above mentioned subspaces for domains of a special kind with an arbitrary sharpness of the cusp.

1. In the sequel, we consider domains of the form

$$\Omega_\psi = \{x = (x_1, x_2) \in \mathbf{R}^2 : |x_2| < \psi(x_1), 0 < x_1 < 1\},$$

with a non-increasing function

$$\psi(t) = \{\psi_{k-1} : t_k < t < t_{k-1}; k \in \mathbb{N}, t_0 = 0\}, \quad \lim_{k\to\infty} \psi_k = 0.$$

By $C_F^\infty(\overline{\Omega}_\psi)$ we denote the set of all infinitely differentiable functions defined on Ω_ψ such that the functions and their derivatives are uniformly continuous in the domain Ω_ψ and flat in the point $(0,0)$, i.e.

$$\lim_{x\to 0,0} D^\alpha f(x) = 0$$

for any $\alpha \in \mathbb{Z}_+^2$.

The topology is determined by the sequence of norms

$$\|f\|_r = \sup\{|D^\alpha f(x)| : x \in \Omega_\psi, \alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}_+^2, |\alpha| \le r\}, r \in \mathbb{N}_+,$$

here $|\alpha| = \alpha_1 + \alpha_2$.

Proposition 1.1. *Let ψ, ψ_1 be functions satisfying the above mentioned restrictions and $C_F^\infty(\Omega_\psi) \simeq C_F^\infty(\Omega_{\psi_1})$ ($\simeq$ isomorphic). Then there exists a constante $\varepsilon > 0$, such that*

$$[\psi(t^{\frac{1}{\varepsilon}})]^{\frac{1}{\varepsilon}} \le \psi_1(t) \le [\psi(t^\varepsilon)]^\varepsilon$$

in some neighbourhood of zero.

This statement can be proved by an unimportant modification of the consideration in [8], [9].

It is useful to compare this result with the well known fact (see, for example, [11]) that the subspace $C_0^\infty(\overline{\Omega})$ of all functions in $C^\infty(\overline{\Omega})$ which are flat on the whole boundary $\partial\Omega$ are mutually isomorphic for all bounded domains $\Omega \in \mathbb{R}^n$.

Theorem 1.2. *Let the sequence (t_k) satisfy the following condition*

(1) $$\exists \varepsilon > 0 : t_k = o((t_k - t_{k+1})^\varepsilon).$$

Then there exists a basis in the space $C_F^\infty(\Omega_\psi)$.

Some results in this direction were announced in [12], [13]. The construction of such a basis is based on the idea of Mityagin's construction of a basis in $\mathbb{C}^\infty(\mathbb{R})$ ([14], §7): the domain Ω_ψ is divided in a countable set of pieces (in our case rectangles) and the space $C_F^\infty(\overline{\Omega}_\psi)$ is decomposed in the direct topological sum.

The desired basis will consist of products of functions, which will have been built from the classical orthonormal functions on the line-segment.

2. We shall use the well-known facts (see, for example, [14]):

(1) The system of Chebyshev polynomials

(2) $$T_n(t) = \cos(n \arccos t)$$

is a basis in the space $C^\infty[-1, 1]$.

(2) The system

(3) $$H_n(x) = h_n\left(tg\frac{\pi}{2}x\right), x \in [-1, 1],$$

where

$$h_n(x) = \frac{(-1)^n}{(2^n n!)^{\frac{1}{2}}\pi^{\frac{1}{4}}} e^{\frac{x^2}{2}}(e^{-x^2})^{(n)}, x \in \mathbb{R}, n \in \mathbb{N},$$

are the Hermite functions, constitutes a basis in the subspace $C_o^\infty[-1,1]$ of all functions, which are flat in the end-points of the segment.

We need the following simple estimations for systems, which are obtained from (2), (3) by a linear change of the variable and which constitute the bases in the spaces $C^\infty[a,b]$, $C_o^\infty[a,b]$, because for our further consideration it is important to account the dependence of estimations upon the length of the interval.

Lemma 2.1. *Let us consider*

$$P_n(x) = T_n(\phi(x)),$$

where

$$\phi(x) = \frac{2x-(b+a)}{2(b-a)},$$

and the biorthogonal system of linear functionals P_n^* in the space $C^\infty[a,b]$. For each $p \in \in \mathbb{N}$ there exists $C(p) > 0$ (non depending on the intervals $[a,b]$) such that the following inequalities hold:

$$(4) \qquad ||P_n||_p \leq \frac{C(p)n^{2p}}{(b-a)^{\min(p,n)}}, n \in \mathbb{Z}_+$$

$$(5) \qquad |P_n^*(f)| \leq \frac{C(p)(b-a)^{\min(p,n)}}{n^{2p}} \cdot ||f||_p, f \in C[a,b], n \in \mathbb{Z}_+.$$

For the system $G_n(x) = H_n(\phi(x))$ and G_n^* we have similar inequalities but with p instead of $\min(p,n)$ *and $h^{q(p)}$ instead of n^{2p} with a suitable function $q : \mathbb{N} \to \mathbb{N}$.*

3. First we describe a simple construction of an linear extension operator

$$M : C^\infty[-1,1] \to C^\infty(\mathbb{R}),$$

which is a modification of the corresponding construction of Mityagin ([14], § 7).

We consider the function

$$g(x) := \begin{cases} 1, x \leq 0 \\ \left(1 + \exp\left(-\frac{1}{x} + \frac{1}{1-x}\right)\right)^{-1}, & 0 < x < 1 \\ 0, \quad x \geq 1. \end{cases}$$

Now we determine the desired extension operator on elements of the basis of $C^\infty[-1,1]$, for the Chebyshev polynomial $T_n(t)$ we define the function

$$\widetilde{T}_n(x) = g_n(x)T_n(x), g_n(x) = g\left(\frac{n^2 x}{\varepsilon^2}\right), x \in \mathbb{R}, 0 < \varepsilon < 1, n \in \mathbb{N}.$$

The possibility of the linear continuation on the whole space $C^\infty(-1,1)$ follows from the next

Lemma 3.1. *For any $m \in \mathbb{N}$ there exists $C_m > 0$ such that*

$$(6) \qquad \max\{|\widetilde{T}_n^{(m)}(t)| : t \in \mathbb{R}\} \le C_m \left(\frac{n}{\varepsilon}\right)^{4m}.$$

Proof. Since

$$(7) \qquad \max\{|g_n^{(m)}(t)| : t \in \mathbb{R}\} \le \left(\frac{n}{\varepsilon}\right)^{2m}$$

and the support of the function $\widetilde{T}_n(t)$ coincides with the segment $I_n = \left[-1 - \frac{\varepsilon^2}{n^2}, 1 + \frac{\varepsilon^2}{n^2}\right]$, it will be sufficient to estimate the derivatives of the polynomials T_n on this segment. From the inequality of Bernstein [15] it follows that

$$(8) \qquad \max\{|T_n^{(m)}(t)| : t \in I_n\} \le \max\{|T_n^{(m)}(t)| : t \in I\}\omega\left(1 + \frac{\varepsilon^2}{n^2}\right)^n,$$

where $\omega(R) = R + (R^2 - 1)^{0.5}, m \in \mathbb{Z}_+$. Because of

$$\omega\left(1 + \frac{\varepsilon^2}{n^2}\right)^n \le \left(1 + \frac{3\varepsilon}{n}\right)^n \le \exp 3\varepsilon,$$

taking into account (7), (8) and (5), we get the required inequality (6).

Using the estimations (6) we obtain the following

Proposition 3.2. *The formula*

$$(Mf)(x) = \sum_{n=0}^{\infty} T_n^*(f)\widetilde{T}_n(x)$$

defines a linear continuous operator $M : C^\infty(-1,1) \to C^\infty(-2,2)$ *which is a right-hand inverse for the operator* $R : C^\infty(-2,+2) \to C^\infty(-1,+1)$, *such that* $R(f) = f|_{[a,b]}$. *The operator* $R \circ M$ *is a linear continuous projection from* $C^\infty(\mathbf{R})$ *on the subspace* $X = M(C^\infty(\mathbf{R}))$, *such that for each* $f \in X$ *the equality* $f(x) \equiv 0$ *for* $|x| \geq 1 + \varepsilon$ *is fulfilled.*

4. We define

$$\Pi_k := \{(x_1, x_2) \in \mathbf{R}^2 : a_k < x_1 < b_k, |x_2| < r_k\}, k \in \mathbf{N},$$

where $a_1 = t_1, b_1 = 1, r_1 = \psi_0$ and $a_{2i+1} = t_{i+1}, b_{2i+1} = \dfrac{t_{i+1} + t_i}{2}, a_{2i} = b_{2i+1}, b_{2i} = a_{2i-1}, r_{2i} = r_{2i+1} = \psi_i$ for $i \in \mathbf{N}$.

Lemma 4.1. *Let the condition (1) hold. Then for every* $m \in \mathbf{N}$ *and* $\alpha = (\alpha_1, \alpha_2) \in \mathbf{Z}_+^2$ *there exists* $B > 0$ *such that the following inequality is fulfilled:*

$$\sup\{|D^\alpha f(x)| : x \in \Pi_k\} \leq B(t_{k-1} - t_k)^m.$$

Proof. From Whitney's theorem for infinitely differentiable functions it follows that a function $f \in C_F^\infty(\Omega_\psi)$ has an extension $\tilde{f} \in C^\infty(\overline{L})$, where $L = \{(x_1, x_2) : |x_1| < 1, |x_2| < 1\}$ (see, for example, [16], Theorem 4.1).

The following estimations can be obtained by means of the Taylor expansion in view of the flatness of the function $\tilde{f}$ in the origin:

$$\sup\{|D^\alpha f| : |x| < \delta, x \in \Omega\psi\} = O(\delta^m), \forall m \in N.$$

To finish the proof it is enough to use condition (1).

5. We use the following notation: X is the subspace of $C^\infty[0,1]$, which consists of all functions which are flat in the point 0; in the case $k = 2i - 1, i \in N$, we write $M_k, Q_k = R_k \circ M_k, X_k$ for the corresponding objects from Proposition 3.2 with the interval $[a_k b_k]$ instead of $[-1,+1]$ and $\varepsilon = \varepsilon_k = \dfrac{1}{2}\dfrac{b_{k+1} - a_{k+1}}{b_k - a_k}$; in the case $k = 2i, i \in \mathbf{N}$, we consider $Q_k = R_k(I - Q_{k-1} - Q_{k+1})$ and $X_k = Q_k(X) = \{f \in X : f(x) \equiv 0 \text{ for } x \notin [a_k, b_k]\} \simeq C_o^\infty[a_k, b_k]$.

Let us consider the system

$$S_n^{(k)}(x) = P_n^{(k)}\left(\frac{2x - a_k - b_k}{2(b_k - a_k)}\right),$$

where $P_n^{(k)}(u) = \tilde{T}_n(u), u \in \mathbf{R}, n \in \mathbf{Z}$, if k is odd, and $P_n^{(k)}(u) = H_n(u), u \in \mathbf{R}$, if k is even (we assume that the functions $H_n(u)$ are continued on the whole $\mathbf{R}$ by trivial means).

In this notations we have the next

Theorem 5.1. *The system* $\{S_n^{(k)}(x), n \in \mathbf{Z}, k \in \mathbf{N}\}$ *constitutes a basis in the space* X.

Proof. We take in account the following: the biorthogonal system on $S_n^{(k)}$ is defined by

$$S_n^{(k)*}(f) = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} f(x) S_n^{(k)} dx$$

for $k = 2i - 1, i \in \mathbf{N}$, and by

$$S_n^{(k)*}(f) = \frac{1}{b_k - a_k} \int_{a_k}^{b_k} Q_k(f) S_n^{(k)}(x) dx$$

for $k = 2i, i \in \mathbf{N}$.

Now we obtain from (1) and Proposition 3.2 the following estimations:

$$(9) \qquad ||S_n^{(k)}||_p \leq C(p)(nk)^{q(p)}$$

$$(10) \qquad ||S_n^{(k)*}||_p^* \leq \frac{C(p)}{(nk)^{r(p)}}$$

for suitable functions $C(p), q(p), r(p)$; here $||x^*||_p^*$ is the polar norm for $||x||_p$.

Using the Dynin-Mityagin Theorem we obtain from (9) and (10) that the system $\{S_n^{(k)}\}$ is a basis in X.

6. Now we can prove the following statement, which contains the theorem.

Proposition 6.1. *The system*

$$(11) \qquad l_{n_1,n_2}^{(k)}(x_1, x_2) = S_{n_1}^{(k)}(x_1) T_{n_2}\left(\frac{x}{r_k}\right), n_1 n_2 \in \mathbf{Z}, n \in \mathbf{N}$$

is a basis in the space $C_F(\Omega_\psi)$.

Proof. Using the estimations (9), (10) and (4), (5) we get the next estimation for (11):

$$||l_{n_1,n_2}^{(k)}||_p \leq C(p) n^{q(p)} k^{q(p)} \left(\frac{1}{r_k}\right)^{\min(n_2,p)}$$

For the biorthogonal system:

$$l_{n_1,n_2}^{(k)*} = S_n^{(k)*} \otimes T_n^{(k)*}),$$

where $T_n^{(k)}(x_2) = T_n\left(\frac{x_2}{r_k}\right)$, we obtain the following estimations for the polar norms:

$$||l_{n_1,n_2}^{(k)*}||_p^* \leq ||S_n^{(k)*}||_p^* \cdot ||T_n^{(k)*}||_p^* \leq C(p)\frac{r_k^{\min(n_2,p)}}{(nk)^{r(p)}}.$$

From this estimation it follows by using the Dynin-Mityagin Theorem [14] that the system (11) is a basis in $C_F(\Omega_\psi)$.

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