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ON HOMOMORPHISMS BETWEEN LOCALLY CONVEX SPACES

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Dedicated to the memory of Professor Gottfried Köthe

Homomorphisms $f : E \to F$ between locally convex spaces E, F, i.e. continuous linear maps which are open onto the range, occur quite often and they are nice to handle. Unfortunately, the stability properties of the class of homomorphisms are poor. For instance, a homomorphism $f : E \to F$ will in general not remain a homomorphism, if E and F are endowed with, for instance, their strong topology; the transpose $f^t : F' \to E'$ will usually not be a homomorphism, and the behaviour of the bitranspose is still worse. The investigation of homomorphisms has a good tradition, in fact, it goes back to Banach and was dealt with afterwards by Dieudonné, L. Schwartz, Grothendieck and Köthe (see for example [12]).

The purpose of this article is twofold:

first, to study the stability behaviour of the class of homomorphisms with a bit of a systematic touch (see (1.4), (1.8), (2.3), (2.5)); and second, to apply new methods and results from the recent development of the structure theory of Fréchet, $LB-$ and LF-spaces to the context of homomorphisms.

For instance, we obtain a(nother) characterization for the quasinormability of Fréchet spaces E by the property that for every monomorphism $j : E \to F$ with F Fréchet, j remains a homomorphism for the topology of uniform convergence on strongly compact sets both on E and on F (see (1.9), (1.10)).

Proposition (2.7) presents a general background for the fact that for the famous quotient map $g : E \to \ell^1$ with E Fréchet Montel, the transpose is not a monomorphism for the weak (sic!) topologies.

In section 3, where we deal with the bitranspose of homomorphisms, we give an example of a quotient map $q : E \to F$ with E Fréchet such that $q^{tt} : E'' \to F''$ is not a homomorphism between the strong biduals. Finally, we present a fairly general condition on a strict LF-space, under which its strong bidual will again be an LF-space.

Notations. Given a locally convex space $E = (E, \mathfrak{J})$ and a linear subspace $L \subset E$, we denote by $\mathfrak{J} \cap L$ the relative topology induced by $\mathfrak{J}$ on L and by $\mathfrak{J}/L$ the quotient topology on the quotient E/L. For a dual pair $\langle E, F\rangle$ we denote by $\sigma(E,F)$, $\beta(E,F)$, $\tau(E,F)$ the corresponding weak, strong and Mackey topology on E, respectively. $\sigma(F,E)$, $\beta(F,E)$, $\tau(F,E)$ are defined analogously. For a subset $A \subset E$ let $A^\circ := \{y \in F : |\langle a, y\rangle| \leq 1$ for all $a \in A\}$ be the polar of A; in order to avoid misunderstandings, we will sometimes write more precisely $A^{\circ F}$.

For a locally convex space E, let E' denote its topological dual and $E'' := (E',$

$\beta(E',E))'$ its bidual. For a linear continuous map $f : E \to F$ between locally convex spaces, we will write $f^t : F' \to E'$ for its transpose and $f^{tt} : E'' \to F''$ for its bitranspose.

For a subset $A \subset E$ let ΓA denote its absolutely convex hull and $[A]$ its linear span. If $A = \Gamma A$ we write p_A for the corresponding Minkowski functional on $[A]$.

A locally convex space E is called locally complete, if for every closed bounded subset $A = \Gamma A \subset E$, the space $([A], p_A)$ is complete.

A linear map $f : E \to F$ between two locally convex spaces is called a homomorphism, if it is continuous and open onto its range. Injective homomorphisms are called monomorphisms.

Inductive limits of a sequence of Banach resp. Fréchet spaces will be called LB-spaces resp. LF-spaces.

1. THE BEHAVIOUR OF HOMOMORPHISMS UNDER A CHANGE OF TOPOLOGIES

1.1. *Let E, F be locally convex spaces and let $f : E \to F$ be a linear map. Then f is a homomorphism if and only if in the canonical factorization of f (where $E/\ker f$ carries the quotient topology and $f(E)$ the relative topology) the linear bijection $\widehat{f}$ is a homeomorphism.*

$$\begin{array}{ccc} E & \xrightarrow{f} & F \\ \downarrow q & & \uparrow j \\ E/\ker f & \xrightarrow{\widehat{f}} & f(E) \end{array}$$

Our aim in this section is to investigate what it means that a homomorphism $f : E \to F$ remains a homomorphism if the topologies on E and F are changed in a suitable way. In this direction we have for example

1.2. *Let E, F be locally convex spaces, let $f : E \to F$ be a continuous linear map, and put $L := \ker f$, $L^o := L^{oE'}$, $f(E)^o := f(E)^{oF'}$*

(α) If f is a homomorphism as a map $(E, \beta(E, E')) \to (F, \beta(F, F'))$, then

i) on E/L the topologies $\beta(E,E')/l$ and $\beta(E/L, L^o)$ coincide and

ii) on $f(E)$ the topologies $\beta(E,E') \cap f(E)$ and $\beta(f(E), F'/f(E)^o)$ coincide.

(β) If f is a homomorphism as a map $E \to F$, and if both i) and ii) are satisfied, then f is a homomorphism as a map $(E, \beta(E, E')) \to (F, \beta(F, F'))$.

Proof. We have the following canonical factorization of f:

$$\begin{array}{ccccc} & (E/L,\beta(E/L,L^\circ)) & \xrightarrow{\widehat{f}} & (f(E),\beta(f(E),F'/f(E)^\circ)) & \\ {}^{q}\nearrow & \Big\uparrow id & & \Big\downarrow id & \searrow{}^{j} \\ (E,\beta(E,E')) & & & & (F,\beta(F,F')) \\ {}^{q}\searrow & & & & \nearrow{}^{j} \\ & (E/L,\beta(E,E')/L) & \xrightarrow{\widehat{f}} & (f(E),\beta(F,F')\cap f(E)) & \end{array}$$

where all the maps indicated by arrows are continuous.

(α) If $f:(E,\beta(E,E'))\to(F,\beta(F,F'))$ is a homomorphism, then by (1.1) the map $\widehat{f}:(E/L,\beta(E,E')/L)\to f(E),\beta(F,F')\cap f(E))$ is a linear homemomorphism, whence the two identity maps indicated in the diagram are also homeomorphisms.

(β) If $f:E\to F$ is a homomorphism, then $\widehat{f}:E/L\to f(E)$ is a linear homomorphism, whence $\widehat{f}:(E/L,\beta(E/L,L^\circ))\to(F(E),\beta(f(E),F'/f(E)^\circ))$ is also a homeomorphism. If in addition both i) and ii) are satisfied, we obtain that

$\widehat{f}:(E/L,\beta(E,E')/L)\to(f(E),\beta(F,F')\cap f(E))$ is again a homeomorphism, which implies that $f:(E,\beta(E,E'))\to(F,\beta(F,F'))$ is a homomorphism. ■

1.3. *Statement (1.2) can of course be considered as a special case of the following general concept (cf. [6; p. 128-131]): let $\mathscr{Z}$ be a (covariant) functional topology, i.e. a functor in the category LCS of locally convex spaces and linear continuous maps of the following form*

$(E,\mathfrak{J}) \rightsquigarrow^{\mathscr{Z}} (E,\mathscr{Z}(\mathfrak{J}))$ *for every locally convex space* $(E,\mathfrak{J})$
$\mathscr{Z}(f)=f$ *for every continuous linear map* f.

In other words, $\mathscr{Z}$ changes only the topology of a locally convex space in such a way that the continuity of linear maps is preserved. There is a multitude of well-known examples of such functorial topologies, such as the formation of the associated (quasi) barrelled, (ultra) bornological, nuclear, or Schwartz space topology, and, of course, the formation of the corresponding weak, strong, Mackey topology etc.

Proposition 1.4. *Let $\mathscr{Z}$ be a (covariant) functorial topology in LCS, let $E=(E,\mathfrak{J})$, $F=(F,\mathfrak{G})$ be locally convex spaces, let $f:E\to F$ be a linear continuous map, and denote* $L:=\ker f$.
(α) *If f is a homomorphism as a map $(E,\mathscr{Z}(\mathfrak{J}))\to(F,\mathscr{Z}(\mathfrak{G}))$, then*
i) on E/L the topologies $\mathscr{Z}(\mathfrak{J})/L$ and $\mathscr{Z}(\mathfrak{J}/L)$ coincide and
ii) on $f(E)$ the topologies $\mathscr{Z}(\mathfrak{G})\cap f(E)$ and $\mathscr{Z}(\mathfrak{G}\cap f(E))$ coincide.
(β) *If f is a homomorphism as a map $(E,\mathfrak{J})\to(F,\mathfrak{G})$ and if both i) and ii) are satisfied, then f is a homomorphism as a map $(E,\mathscr{Z}(\mathfrak{J}))\to(F,\mathscr{Z}(\mathfrak{G}))$.*

The proof is completely analogous to that of 1.2 making use of the following factorization of f

$$
\begin{array}{ccccccc}
 & & (E/L, \mathscr{Z}(\mathscr{Z}/L)) & \stackrel{\widehat{f}}{\longrightarrow} & (f(E), \mathscr{Z}(\mathscr{S} \cap f(E))) & & \\
 & {}^{q}\nearrow & \uparrow & & \Big| & \searrow^{j} & \\
(E, \mathscr{Z}(\mathfrak{T})) & & \Big| \; id & & id \; \Big| & & (F, \mathscr{Z}(\mathscr{S})) \\
 & {}_{q}\searrow & \Big| & & \downarrow & \nearrow_{j} & \\
 & & (E/L, \mathscr{Z}(\mathfrak{T})/L) & \stackrel{\widehat{f}}{\longrightarrow} & (f(E), \mathscr{Z}(\mathscr{S}) \cap f(E)) & &
\end{array}
$$

Remarks 1.5. (a) Obviously, the same as in (1.4) can be done for functorial topologies in the category TVS of topological vector spaces.
(b) The two «commutativity properties» i) and ii) in (1.4) are valid for the formation of the weak topology (Köthe [13; § 22, 2, (1) and (3)]) as well as the associated Schwartz-space-topology (Swart [19; 3.4 and 3.7]) and the associated strongly nuclear topology (Jarchow [11; 21.9.4.b] and [7; p. 28-29]). On the other hand, for the majority of functorial topologies $\mathscr{Z}$ the commutativity properties i) and ii) fail. For instance, ii) will necessarily fail if the fix class of $\mathscr{Z}$ is not stable under closed subspaces. Moreover, even if fix $\mathscr{Z}$ is stable under quotients, i) may fail as the example in [7; p. 36-38] show (cf. also Jarchow, loc. cit.).

A somewhat weaker statement is the following

Corollary 1.6. *Let $\mathscr{Z}$ be a (covariant) functorial topology in LCS which is refining (i.e. $\mathscr{Z}(\mathfrak{T}) \supset \mathfrak{T}$ for every locally convex space $(E, \mathfrak{T})$). Let $E = (E, \mathfrak{T})$, $F = (F, \mathscr{S})$ be locally convex spaces such that $E \in$ fix $\mathscr{Z}$ and let $f : E \to F$ be a homomorphism. Then $f : (E, \mathfrak{T}) \to (F, \mathscr{Z}(\mathscr{S}))$ is again a homomorphism.*

Remark. The hypothesis that $\mathscr{Z}$ is refining is essential as Jarchow [11; 21.9.4, end of proof] shows - for $\mathscr{Z}$ to be the associated nuclear topology.

1.7. *In order to have a closer look at a special subclass of functorial topologies, consider a functor $\mathscr{A}$ from LCS to the category SET of sets and maps of the following shape.*

For every locally convex space $E = (E, \mathfrak{T})$ let $\mathscr{A}(E)$ be a set of $\sigma(E', E)$-bounded subsets of E' such that - whenever E, F are locally convex spaces and $f : E \to F$ is linear and continuous - we have $f^t(\mathscr{A}(F)) \subset \mathscr{A}(E)$, and $\mathscr{A}$ assigns to f the map $f^t|\mathscr{A}(F) : \mathscr{A}(F) \to \mathscr{A}(E)$. Then for every locally convex space $(E, \mathfrak{T})$ the topology $\mathfrak{T}_{\mathscr{A}(E)}$ of uniform convergence on all sets in $\mathscr{A}(E)$ is a locally convex topology on E, and clearly

$$\mathscr{Z}_{\mathscr{A}} : (E, \mathfrak{T}) \rightsquigarrow \left(E, \mathfrak{T}_{\mathscr{A}(E)}\right)$$

defines a functorial topology in LCS in the sense of (1.3).

Examples of functorial topologies $\mathfrak{T}_{\mathscr{B}}$ are, of course, the formation of the weak, the strong, the Mackey topology, the topology $\beta^*(E,E')$ of uniform convergence on all $\beta(E',E)$-bounded subsets of E', and - among others - the topology $\gamma(E,E')$ of uniform convergence on all $\beta(E',E)$-precompact subsets of E'. By Köthe [13; § 21, 7, (2) and (3)] the topology $\gamma(E,E')$ is the strongest «polar» topology on E (i.e. has a 0-basis of $\sigma(E,E')$-closed sets) which coincides with $\sigma(E,E')$ on all bounded subsets of E.

For the formations of the Mackey topology $E \rightsquigarrow (E,\tau(E,E'))$ and of the weak topology $E \rightsquigarrow (E,\sigma(E,E'))$, respectively, property i) in (1.4) is always satisfied (see [13; § 22, 2, (3)]), but not for the formation of the strong topology $E \rightsquigarrow (E, \beta(E,E'))$ (see [8; example]). The next statement will characterize condition ii) in (1.4) for functorial topologies $\mathfrak{T}_{\mathscr{B}}$.

Proposition 1.8. *Let $\mathfrak{T}_{\mathscr{B}}$ be a functorial topology as described in (1.7), let E,F be locally convex spaces and let $f: E \to F$ be an injective and continuous linear map. Tfae*

(α) *f is a monomorphism as a map $(E,\mathfrak{T}_{\mathscr{B}(E)}) \to (F,\mathfrak{T}_{\mathscr{B}(F)})$.*

(β) *For every $B \in \mathscr{B}(E)$ there is $C \in \mathscr{B}(F)$ such that the $\sigma(E',E)$-closed absolutely convex hull of $f^t(C)$ contains B.*

Proof.

$$(\beta) \Longleftrightarrow \forall_{B\in\mathscr{B}(E)}\exists_{C\in\mathscr{B}(F)} B \subset f^t(C)^{\circ\circ} \Longleftrightarrow \forall_{B\in\mathscr{B}(E)}\exists_{C\in\mathscr{B}(F)} B^\circ \supset f^t(C)^\circ =$$

$$= \overline{f}^1\left(C^\circ\right) \underset{f\ inj}{\Longleftrightarrow} \forall_{B\in\mathscr{B}(E)}\exists_{C\in\mathscr{B}(F)} C^\circ \cap f(E) \subset f\left(B^\circ\right) \Longleftrightarrow (\alpha). \qquad \blacksquare$$

It is well known that (α) holds for the formation of the weak topology and fails for many other functorial topologies of type $\mathfrak{T}_{\mathscr{B}}$ such as β, β^*, τ, even if $f(E)$ is closed in F. We will now investigate this question for the above introducted topology $E \rightsquigarrow (E,\gamma(E,E'))$ in the context of Fréchet spaces.

Proposition 1.9. *Let E,F be Fréchet spaces such that E is quasinormable, and let $f: E \to F$ be a monomorphism. Then f is a monomorphism as a map $(E,\gamma(E,E')) \to (F,\gamma(F,F'))$.*

Proof. We will show that (1.8) (β) is satisfied. Let $B \subset (E',\beta(E',E))$ be compact. As $(E',\beta(E',E))$ is a boundedly retractive LB-space, there is an absolutely convex $0-$*nbhd* U in E such that B is a compact subset of the Banach space E_{U° generated by $U^\circ \subset E'$. There is an absolutely convex $0-$*nbhd* V in F such that $f(E) \cap V = U$. By the Hahn-Banach Theorem f^t generates a continuous surjection from the Banach space F_{V° generated by $V^\circ \subset F'$ onto E_{U°, hence a quotient map. By the theorem of Banach-Dieudonné there is a compact set A in F_{V° such that $f^t(A) = B$. As F_{V° embeds continuously into $(F', \beta(F', F))$, A is a compact subset of $(F', \beta(F', F))$ satisfying $f^t(A) = B$. ∎

In contrast to (1.9) we have

Proposition 1.10. *Let E, F be Fréchet spaces such that F is quasinormable and E is not quasinormable, and let $f : E \to F$ be a monomorphism. Then f is not a monomorphism as a map $(E, \gamma(E, E')) \to (F, \gamma(F, F'))$.*

Proof. We will show that (1.8) (β) is violated. We have the continuous surjective transpose $f^t : (F', \beta(F', F)) = (F', \beta(F', F'')) \to (E', \beta(E', E''))$. By Bonet [1; Theorem] the LB-space $(E', \beta(E', E''))$ is not boundedly reretractive. According to Neus [15], $(E', \beta(E', E''))$ is not compactly retractive. Thus there is a compact subset B of $(E', \beta(E', E''))$, which is in particular a compact subset of $(E', \beta(E', E))$ such that - whenever X is a Banach space continuously included in $(E', \beta(E', E''))$ - the set B is not a compact subset of X.

Let A be any compact subset of $(F', \beta(F', F))$. As $(F', \beta(F', F))$ is boundedly retractive, there is a Banach space Y which is continuously included in $(F', \beta(F', F))$ such that A is a compact subset of Y. Then $f^t(A)$ is a compact subset of the Banach space $X := Y/(Y \cap \ker f^t)$ and X is continuously included in $(E', \beta(E', E''))$. Therefore $\overline{f^t(A)} = f^t(A)$ does not contain B. ∎

For surjective homomorphisms we only have the following negative result:

Example 1.11. If $(E, \mathfrak{I})$ is a Montel space, then clearly $\mathfrak{I} = \gamma(E, E')$. Moreover, for a normed space $(F, \mathscr{S})$, we have that $\mathscr{S} = \gamma(F, F')$ if and only if F is finite dimensional (if $\dim F = \infty$, the norm and the weak topology do not coincide on the unit ball).

Therefore, let e.g. E be Grothendieck-Köthe's Fréchet Montel space admitting $F := (\ell^1, \|\cdot\|_1)$ as a quotient. Then the corresponding quotient map $q : E \to F$ is not a homomorphism as a map $(E, \gamma(E, E')) \to (F, \gamma(F, F'))$.

Curiously enough, we could not find out whether for any Banach space E and a quotient map $q : E \to E/L =: F, q$ remains open for $\gamma(E, E')$ and $\gamma(F, F')$, respectively.

2. ABOUT THE TRANSPOSE OF A HOMOMORPHISM

2.1. *Let E, F be locally convex spaces and let $f : E \to F$ be a continuous linear map.*
(a) $f^t : (F', \sigma(F', F)) \to (E', \sigma(E', E))$ is a homomorphism if and only if $f(E)$ is closed in F. In fact, f^t is a weak homomorphism if and only if on $F'/f(E)^\circ$ the topologies $\sigma(F', F)/f(E)^\circ$ and $\sigma(F'/f(E)^\circ, f(E))$ coincide.

(b) If f is even a monomorphism, then $f^t : (F', \tau(F', F)) \to (E', \tau(E', E))$ is open if and only if $f(E)$ is closed in F. See Köthe [13; § 22.2, (4)].

2.2. *Symmetrically to (1.4) let $\mathscr{X}$ be a contravariant functorial topology in LCS, i.e. a functor in LCS of the following form $E \sim \overset{\mathscr{X}}{\sim} \sim> (E', \mathscr{X}(E))$ for every locally convex space E and $\mathscr{X}(f) = f^t$ for every continuous linear map $f : E \to F$.*

In other words, $\mathscr{X}$ assigns a locally convex topology to the dual E' of a locally convex space E in such a way that the transpose of a continuous linear map will be continuous with respect to the assigned topologies on the duals. Examples of contravariant functorial topologies $\mathscr{X}(E)$ are, for example, the formation of $\beta(E', E)$, $\beta(E, E'')$, $\sigma(E', E'')$, or the bornological topology associated to $\beta(E', E)$. Other examples, which do not only depend on the dual pair $\langle E, E' \rangle$, would be the topology of uniform convergence on all precompact subsets of E, and also the so-called inductive topology on E', which is by definition the strongest locally convex topology on E' such that for every $0 - nbhd\ U$ in E the inclusion $([U^\circ], p_{U^\circ}) \to E'$ is continuous. We abbreviate $E \sim\sim\sim> E'_{\text{ind}} = \text{ind}_{U\to}([U^\circ], p_{U^\circ})$.

Proposition 2.3. *Let $\mathscr{X}$ be a contravariant functorial topology in LCS, let E, F be locally convex spaces and let $f : E \to F$ be a homomorphism. Then $f^t : (F', \mathscr{X}(F)) \to$ $\to (E', \mathscr{X}(E))$ is a homomorphism if and only if*
i) on $(\ker f)^\circ$ the topologies $\mathscr{X}(E/\ker f)$ and $\mathscr{X}(E) \cap (\ker f)^\circ$ coincide and
ii) on $F'/f(E)^\circ$ the topologies $\mathscr{X}(f(E))$ and $\mathscr{X}(F)/f(E)^\circ$ coincide.

Proof. The canonical factorization $f = j \circ \widehat{f} \circ q$ (see (1.1)) leads to the factorization $f^t =$ $= q^t \circ \widehat{f}^t \circ j^t$, where the surjection $j^t : (F', \mathscr{X}(F)) \to (f(E)', \mathscr{X}(f(E)))$ and the injection $q^t : ((E/\ker f)', \mathscr{X}(E/\ker f)) \to (E', \mathscr{X}(E))$ are continuous, and the bijection $\widehat{f}^t :$ $(f(E)', \mathscr{X}(f(E))) \to ((E/\ker f)', \mathscr{X}(E/\ker f))$ is a topological isomorphism. This implies the assertion. ∎

As an application we would like to mention: a homomorphism $f : E \to F$ between locally convex spaces E, F leads to a homomorphism $f^t : F'_{\text{ind}} \to E'_{\text{ind}}$ if and only if the canonical inclusion $(E/\ker f)'_{\text{ind}} \to E'_{\text{ind}}$ is a monomorphism.

Proof. Condition ii) in (2.3) is satisfied, because the natural injection $j^t : F'_{\text{ind}} \to f(E)'_{\text{ind}}$ is a quotient map. In fact, by Hahn-Banach Theorem we have for every $0 - nbhd\ U = \Gamma U$ in F that $j^t(U^{\circ F'}) = (U \cap f(E))^{\circ f(E)'}$; hence the assertion follows by the transitivity of final topologies in LCS. ∎

For the Köthe-Grothendieck-Fréchet-Montel space E admitting a quotient map $q : E \to$ $\to \ell^1$ we obtain that q^t is not a monomorphism for the inductive topologies (note that for a

Fréchet space E we have $(E)'_{\text{ind}} = (E', \beta(E', E''))$).

2.4. *Analogously to (1.7) we obtain a subclass of contravariant functorial topologies $\mathscr{X}$ in the following way:*
For every locally convex space E let $\mathscr{B}(E)$ be a set of bounded subsets of E such that - whenever E, F are locally convex spaces and $f : E \to F$ is linear and continuous - we have $f(\mathscr{B}(E)) \subset \mathscr{B}(F)$. Then, for every locally convex space E the topology $\mathfrak{I}_{\mathscr{B}(E)}$ of uniform convergence on all sets in $\mathscr{B}(E)$ is a locally convex topology on E', and clearly $\mathscr{X}_{\mathscr{B}} : E \sim\sim\sim> (E', \mathfrak{I}_{\mathscr{B}(E)})$ defines a contravariant functorial topology in LCS in the sense of (2.2). Parallel to (1.8) we have

Proposition 2.5. *Let $\mathscr{X}_{\mathscr{B}}$ be a contravariant functorial topology in LCS as described in (2.4), let E, F be locally convex spaces and let $f : E \to F$ be a surjective and continuous linear map. Tfae*
(α) *f^t is a monomorphism as a map $(F', \mathfrak{I}_{\mathscr{B}(F)}) \to (E', \mathfrak{I}_{\mathscr{B}(E)})$.*
(β) *For every $B \in \mathscr{B}(F)$ there is $C \in \mathscr{B}(E)$ such that the closed absolutely convex hull of $f(C)$ contains B.*

The proof is completely symmetrical to that of (1.8).

Let us return to the functorial topology $E \sim\sim\sim> f(E)'_{\text{ind}}$ for a moment. Let E be a Fréchet space and let $L \subset E$ be a quasinormable closed subspace. Then the quotient map $q : E \to E/L$ lifts bounded sets (i.e. $\forall B \subset E/L$ bounded $\exists A \subset E$ bounded such that $q(A) \subset \subset B$) (see DeWilde [5]), whence $q^t : (L^\circ, \beta(L^\circ, E/L)) \to (E', \beta(E', E))$ is a monomorphism on account of (2.5). Moreover, as L is quasinormable, $(E'/L^\circ, \beta(E'/L^\circ, L))$ is an LB-space. As $\beta(E', E'')/L^\circ \supset \beta(E', E)/L^\circ \supset \beta(E'/L^\circ, L)$ and $\beta(E', E'')$ is LB, we obtain that the natural surjection $(E', \beta(E', E)) \to (E'/L^\circ, \beta(E'/L^\circ, L))$ is a quotient map. Consequently, L'' can be identified with $L^{\circ\circ} := L^{\circ E' \circ E''}$, and $(E/L)''$ corresponds to $E''/L^{\circ\circ}$. The quotient map $q^{tt} : (E'', \beta(E'', E')) \to (E''/L^{\circ\circ}, \beta(E'', E')/L^{\circ\circ}) = = (E''/L^{\circ\circ}, \beta(E''/L^{\circ\circ}, L^\circ))$ lifts bounded sets, as $\ker q^{tt} = L^{\circ\circ} = L''$ is the bidual of a quasinormable Fréchet space, hence quasinormable. From this we get that $q^t : (L^\circ, \beta(L^\circ, (E/L)'')) \to (E', \beta(E', E''))$ is again a monomorphism (via (1.8)). Since $G'_{\text{ind}} = (G', \beta(G', G''))$ for every Fréchet space G we obtain that $q^t : (E/L)'_{\text{ind}} \to E'_{\text{ind}}$ is a monomorphism.

Our next aim is to investigate the special case of the strong topology $E \sim\sim\sim> (E', \beta(E', E))$.

It should be mentioned that Palamodov [16] and also Zarinov [20] gave the following characterization in terms of duality functors: the transpose of a homomorphism $f : E \to F$ between locally convex spaces in a homomorphism for the strong topologies if and only if there is a set M such that $\mathscr{D}^1_M(\ker f) = 0$ and $\mathscr{D}^+_M(f(E)) = 0$.

We will first concentrate on the class of Fréchet spaces.

2.6. *Let E be a Fréchet space and $L \subset E$ a Fréchet subspace.*

For the inclusion $j : L \to E$ we have: if L is distinguished, then the surjection $j^t : (E', \beta(E', E)) \to (L', \beta(L', L))$ is continuous and open (note that here $\beta(L', L)$ is an LB-space topology and $\beta(E', E)$ admits the stronger LB-space topology $\beta(E', E''))$. If $(L', \beta(L', L))$ admits no discontinuous bounded linear form, then at least $j^t : (E', \sigma(E', E'')) \to (L', \sigma(L', L''))$ will be a homomorphism.

On the other hand, if L is not distinguished but E is distinguished (we may choose E to be a countable product of Banach spacce), then $j^t : (E', \beta(E', E)) \to (L', \beta(L', L))$ is clearly not open, and if $(L', \beta(L', L))$ admits a discontinuous bounded linear form and $(E', \beta(E', E))$ does not, then $j^t : (E', \sigma(E', E'')) \to (L', \sigma(L', L''))$ is not open.

For the quotient map $q : E \to E/L$ we have: if L is quasinormable or if E/L is a Montel space, then q lifts bounded sets (see DeWilde [5] and Köthe [13; § 21.10.(3)]) which implies that $q^t : (L^\circ, \beta(L^\circ, E/L)) \to (E', \beta(E', E))$ is a monomorphism. For instance, this situation is satisfied if E is a Schwarts space or if E is Banach. It also holds for the case that E is a product of the space $\omega = \mathbb{K}^{\mathbb{N}}$ and a Banach space X, as wsa shown in [21]. In [22] it was proved that for a Fréchet space E tfae

i) E is topologically isomorphic to the product $X \times Y$ where X is Banach and Y is either ω of finite dimensional.

ii) Every closed subspace of E is a quojection.

On the other hand, the often mentioned quotient map $q : E \to \ell^1$ of the Köthe-Grothendieck-Fréchet-Montel space E onto ℓ^1 has a transpose $q^t : (\ell^1, \|\cdot\|_\infty) \to (E', \beta(E', E))$ which is certainly not a monomorphism. In fact, q^t is not even a monomorphism for the weak topologies corresponding to the strong topologies, which was utilized in Bonet-Dierolf [2; Example 4]. We would like to model a general background of this phenomenon.

Proposition 2.7. *Let X be a Fréchet space and let $Y \subset Z \subset X$ be closed subspaces such that Z is reflexive but Z/Y is not reflexive. Then for the quotient map $q : X \to X/Y$, the transpose $q^t : ((X/Y)', \sigma((X/Y)', (X/Y)'')) \to (X', \sigma(X', X''))$ is not a monomorphism.*

Proof. We have the following commutative diagram

$$\begin{array}{ccccccc}
(X', \beta(X', X'')) & \xrightarrow{\text{id}} & (X', \beta(X', X)) & \xrightarrow{r} & (Z', \beta(Z', Z)) & = & (Z', \tau(Z', Z)) \\
\uparrow q^t & \nearrow & q^t & & & & \downarrow p^t \\
(Y^{\circ X'}, \beta(Y^{\circ X'}, X/Y)) & \xrightarrow{s} & (Y^{\circ Z'}, \beta(Y^{\circ Z'}, Z/Y)) & & \xrightarrow{\text{id}} & & (Y^{\circ Z'}, \tau(Y^{\circ Z'}, Z/Y))
\end{array}$$

of canonical continuous linear maps, where $p : Z \to Z/Y$ denotes the quotient map. Since Z/Y is not reflexive, there is a linear form φ on $Y^{\circ Z'} = (Z/Y)'$ which is $\beta(Y^{\circ Z'}, Z/Y)-$

continuous, but not $\tau(Y^{\circ Z'}, Z/Y)$-continuous. Then, $\varphi \circ s$ is a continuous linear form on $(Y^{\circ X'}, \beta(Y^{\circ X'}, X/Y))$. Assume, there is $\psi \in X''$ such that $\psi \circ q^t = \varphi \circ s$.

We first show that $\psi|\ker r = 0$. Let $h \in \ker r = Z^{\circ X'} \subset q^t(Y^{\circ X'})$; there is $g \in Y^{\circ X'}$ such that $h = q^t(g)$ and we obtain $\langle \psi, h \rangle = \langle \psi, q^t(g) \rangle = \langle \psi \circ q^t, g \rangle = \langle \varphi, s(g) \rangle = 0$ as $p^t(s(g)) = r(h) = 0$ and p^t is injective.

Now, since $r : (X', \beta(X', X'')) \to (Z', \beta(Z', Z))$ is open (note that both space are LB-spaces), we find a continuous linear form η on $(Z', \beta(Z', Z))$ such that $\eta \circ r = \psi$. Since η is also $\tau(Z', Z)$-continuous, the linear form $\eta \circ p^t$ is $\tau(Y^{\circ Z'}, Z/Y)$-continuous. We will prove that $\eta \circ p^t = \varphi$, which yields a contradiction to the choice of φ. In fact, since s is surjective it suffices to verify that $\eta \circ p^t \circ s = \eta \circ r \circ q^t = \psi \circ q^t = \varphi \circ s$. ∎

So the transpose of a homomorphism between Fréchet spaces will by far not be a homomorphism for the strong topologies on the dual, even if one deals only with distinguished Fréchet spaces (in (2.7) we may choose all spaces distinguished: Z Montel, $Z/Y = \ell^1$, and X a countable product of Banach spaces). In contrast to this we have in the dual setting the following positive result (cf. also Floret, Moscatelli [9]):

Proposition 2.8. *Let E be a DF-space, F an arbitrary locally convex space, and $f : E \to F$ a homomorphism. Then $f^t : (F', \beta(F', F)) \to (E', \beta(E', E))$ is again a homomorphism.*

Proof. $q : E \to E/\ker f$ satisfies (2.5) (β), as E is a DF-space. This implies that the first of the two conditions considered in (2.3) is satisfied. Since $f(E)$ is again a DF-space we obtain from [17; (4.1)] that also the second condition in (2.3) is satisfied (see also Köthe [13; § 29.5, (1) and (2)]. ∎

We will finish this section with a few remarks about the contravariant functorial topology $E \rightsquigarrow (E', \gamma(E', E))$ (where $\gamma(E', E)$ denotes the topology of uniform convergence on all $\beta(E, E')$-precompact subsets of E).

Remarks 2.9. (a) Let F be a locally convex space which is p-complete (i.e. closed precompact subsets are compact), let $M \subset F$ be a closed linear subspace and let $j : M \to F$ denote the inclusion. Then $j^t : (F', \gamma(F', F)) \to (M', \gamma(M', M))$ is a quotient map according to Köthe [13; § 22.2 (3)].

(b) Let E, F be locally convex spaces and let $f : E \to F$ be a homomorphism. Furthermore, assume that F is p-complete, that $E/\ker f$ is complete and that for every compact subset A of $E/\ker f$ there is a compact subset B in E such that $q(B) \supset A$ (where $q : E \to E/\ker f$ denotes the quotient map). Then $f^t : (F', \gamma(F', F)) \to (E', \gamma(E', E))$ is a homomorphism.

The conditions on E and $E/\ker f$ are satisfied, if E is a Fréchet space or a DF-Montel-space (cf. also Schaefer [18; p. 22]).

3. ON THE BITRANSPOSE OF A HOMOMORPHISM

Let E be a locally convex space and let $L \subset E$ be a linear subspace.

The bidual L'' can be canonically identified with a subspace of $L^{\circ\circ} := L^{\circ E' \circ E''} \subset E''$. In general L'' is strictly contained in $L^{\circ\circ}$: let L be a Fréchet space whose strong dual admits a discontinuous bounded linear form and let E be a countable product of Banach spaces. Furthermore, the inclusion $i : (L'', \beta(L'', L')) \to (E, \beta(E'', E'))$ need not be a monomorphism: choose E reflexive and $L \subset E$ closed and non-barrelled.

We would like to mention, that in general the inclusion $i : L'' \to E''$ is a monomorphism for the so-called natural topologies on L'' and E''. In fact, it suffices to prove that for every $0 - nbhd\ U = \overline{\Gamma U}$ in E we have $\overline{U \cap L}^{\sigma(L'', L')} \subset \overline{U}^{\sigma(E'', E')} \cap L'' \subset 2\overline{U \cap L}^{\sigma(L'', L')}$. But this follows easily from the fact that $(U \cap L)^{\circ\circ} \subset U^{\circ\circ} \cap L^{\circ\circ} \subset (\frac{1}{2}\overline{U^\circ + L^\circ}^{\sigma(E', E'')})^\circ =$ $= (\frac{1}{2}\overline{U^\circ + L^\circ}^{\sigma(E', E)})^\circ \subset 2(U \cap L)^{\circ\circ}$ (where polars are formed with respect to the dual pair $\langle E'', E' \rangle$; note that U° is $\sigma(E', E)$-compact).

For the bitranspose q^{tt} of the quotient map $q : E \to E/L$ we have $\ker q^{tt} = L^{\circ\circ}$, whence $q^{tt}(E'')$ can be canonically identified with a subspace of $(E/L)''$. In general the inclusioon $q^{tt}(E'') \subset (E/L)''$ is strict: choose E reflexive such that E/L is not reflexive. In (3.4) we will see that $q^{tt} : (E'', \beta(E'', E')) \to ((E/L)'', \beta((E/L)'', (E/L)'))$ need not be open onto its range even if E is a Fréchet space.

Remarks 3.1. Let E, F be locally convex spaces and let $f : E \to F$ be a continuous linear map.

(a) $f^{tt} : (E'', \sigma(E'', E')) \to (F'', \sigma(F'', F'))$ is a homomorphism if and only if $f^t(F')$ is closed in $(E', \beta(E', E))$, as follows immediately from (2.1) (a).

Moreover, $f^{tt}(E'')$ is closed in $(F'', \sigma(F'', F'))$ if and only if $f^t : (F', \sigma(F', F'')) \to$ $\to (E', \sigma(E', E''))$ is a homomorphism, as again follows from (2.1) (a).

If f is even a homomorphism from E into F, then $f^t(F') = (\ker f)^\circ$ is closed in $(E', \sigma(E', E))$, hence $f^{tt} : (E'', \sigma(E'', E')) \to (F'', \sigma(F'', F'))$ is a homomorphism. The converse is not true as the following example shows: let F be a DF-space which is not bornological, let E be the associated bornological space to F and let: $f : E \to F$ be the identity map which is not a homomorphism. But $f^t : (F', \beta(F', F)) \to (E', \beta(E', E))$ is a monomorphism and $(F', \beta(F', F))$ is complete, whence $f^{tt} : (E'', \sigma(E'', E')) \to$ $\to (F'', \sigma(F'', F'))$ is a homomorphism.

(b) Assume that $f^t : (F', \sigma(F', F'')) \to (E', \sigma(E', E''))$ is a homomorphism. Then f^{tt} has $\sigma(F'', F')$-closed range, and we obtain $f^t(F')^\circ = \ker\ f^{tt}$ (this is always true),

$f^{tt}(E'') = (\ker f^t)^{\circ F''} = f(E)^{\circ\circ} = f(E)''$ (where for the last equality we again use the fact that $f^t : (F', \sigma(F', F'')) \to (E', \sigma(E', E''))$ is a homomorphism).

Now the results of section 2 can be applied to the homomorphism $f^t : (F', \sigma(F', F'')) \to$ $\to (E', \sigma(E', E''))$.

We will formulate the special case of the strong bidual.

3.2. *Let E, F be locally convex spaces and let $f : E \to F$ be a homomorphism. Tfae*

i) $f^t : (F', \beta(F', F)) \to (E', \beta(E', E))$ and $f^{tt} : (E'', \beta(E'', E')) \to (F'', \beta(F'', F'))$ are both homomorphisms.

ii) $\beta((\ker f)^\circ, E/\ker f) = \beta(E', E) \cap (\ker f)^\circ$

$\beta((F', F)/f(E)^\circ = \beta(F'/f(E)^\circ, f(E))$

$\beta((\ker f^t)^\circ, F'/\ker f^t) = \beta(F'', F') \cap (\ker f^t)^\circ$

$\beta((E'', E')/f^t(F')^\circ = \beta(E''/f^t(F')^\circ, f^t(F'))$.

3.3. *Let E, F be Fréchet spaces and let $f : E \to F$ be a homomorphism.*

i) If f is a monomorphism, the continuous surjection $f^t : (F', \beta(F', F)) \to (E', \beta(E', E))$ need not be a homomorphism (see (2.6)), but it lifts bounded sets, whence by (2.5) the bitranspose $f^{tt} : (E'', \beta(E'', E')) \to (F'', \beta(F'', F'))$ is a monomorphism.

ii) If f is surjective and $f^t : (F', \beta(F', F)) \to (E', \beta(E', E))$ is a monomorphism (i.e. f lifts bounded sets with closure), then $f^{tt} : (E'', \beta(E'', E')) \to (F'', \beta(F'', F'))$ is a continuous linear surjection between Fréchet spaces, hence a quotient map.

The following example shows that the hypothesis about f^t is essential.

Example 3.4. Let $(E_n)_{n\in\mathbb{N}}$ be a projective sequence of reflexive and separable Banach spaces such that its projective limit $E :=$ proj E_n is not quasinormable (e.g. $E = \Pi \ell^3 \cap$ $\cap \ell^2(\ell^4)$, see [3]), and let $j : E \to \Pi_{n\in\mathbb{N}} E_n =: F$ be the natural embedding. According to [4; Prop. 1 and its proof] the pair $E \subset F$ does not have the bounded decomposition property, whence the induced map $\widehat{q} : \ell^\infty(F) \to \ell^\infty(F/E)$, $(x_n)_{n\in\mathbb{N}} \mapsto (x_n + E)_{n\in\mathbb{N}}$ is not open onto its range, i.e. not a homomorphism. On the other hand, $p : c_0(F) \to c_0(F/E)$, $(x_n)_{n\in\mathbb{N}} \mapsto (x_n + E)_{n\in\mathbb{N}}$ is continuous and open, and $\widehat{q} = p^{tt}$ (note that F and F/E are reflexive Fréchet spaces whence $c_0(F)'' = \ell_\infty(F)$ and $c_0(F/E)'' = \ell_\infty(F/E)$.

3.5. *Let E, F be DF-spaces and let $f : E \to F$ be a homomorphism.*

i) If f is a monomorphism and E is quasibarrelled, then $f^t : (F', \beta(F', F) \to (E', \beta(E', E))$ is a quotient map and for every bounded subset A of $(E', \beta(E', E))$ there is an equicontinuous, hence bounded subset B of $(F', \beta(F', F))$ such tht $f^t(B) \supset A$. Now (2.5) implies that $f^{tt} : (E'', \beta(E'', E')) \to (F'', \beta(F'', F'))$ is a monomorphism.

ii) If f is surjective and if $(F', \beta(F', F))$ is distinguished (e.g. if F is a retractive $LB-$ space), then $f^{tt} : (E'', \beta(E'', E')) \to (F'', \beta(F'', F'))$ is open (see (2.8) and (2.6)).

In contrast to this statement, let $F = \operatorname{ind}_{n\to} F_n$ be Köthe's incomplete LB-space whose strond dual is not distinguished, and let $q : \oplus F_n \to F$ denote the canonical quotient map. Then $q^{tt} : \oplus(F_n{}'', \beta(F''_n, F'_n)) \to (F'', \beta(F'', F'))$ is surjective but not a homomorphism as its domain is an LB-space and its range is not.

In particular, the bidual of an LB-space need not be an LB-space, whereas the bidual of a retractive resp. strict LB-space is a retractive resp. strict LB-space.

Our next aim is a similar investigation of strict LF-spaces.

3.6. *Let $(F_n)_{n\in\mathbf{N}}$ be an increasing sequence of Fréchet spaces such that for every $n \in \mathbf{N}$ the inclusion $j_n : F_n \to F_{n+1}$ is a monomorphism, and let $F := \operatorname{ind}_{n\to} F_n$ be the correspsonding strict LF-space.*

We recall a few items from [2; p. 24-25], which are due to Grothendieck: $(F', \beta(F', F))$ can naturally be identified with $\operatorname{proj}_{\leftarrow}(F'_n, \beta(F'_n, F_n))$ (w.r. to $(j_n^t : F'_{n+1} \to F'_n)_{n\in\mathbf{N}}$). The biduals $(F''_n, \beta(F''_n, F'_n))_{n\in\mathbf{N}}$ form an inductive sequence w.r. to the monomorphisms $j_n^{tt}(F''_n, \beta(F''_n, F'_n)) \to (F''_{n+1}, \beta(F''_{n+1}, F'_{n+1}))$, and their union is the space F''.

Thus we have a continuous identity id: $F''_{\text{ind}} := \operatorname{ind}_{n\to}(F''_n, \beta(F''_n, F'_n)) \to (F'', \beta(F'', F')) =: F''_b$ (where, by the way, $F''_{\text{ind}} = (F', \beta(F', F))'_{\text{ind}}$ in the sense of the inductive topology treated in § 2).

(a) The natural inclusions $i_n : F''_{n,b} := (F''_n, \beta(F''_n, F'_n)) \to F''_b$ are all monomorphisms, whence F''_{ind} is a strict LF-space.

(b) If $j_n^t : F'_{n+1,s} := (F'_{n+1}, \sigma(F'_{n+1}, F''_{n+1})) \to (F'_n, \sigma(F'_n, F''_n)) =: F'_{n,s}$ is open for all $n \in \mathbf{N}$, then F''_{ind} and F''_b have the same bounded sets.

(c) If $F'_{n,b} := (F'_n, \beta(F'_n, F_n))$ is bornological for all $n \in \mathbf{N}$, then $F'_b := (F', \beta(F', F))$ is bornological, hence F''_b is complete.

The following statement generalizes [2; Prop. 3 (i)] to the case of an arbitrary strict LF-space, cf. also the examples 2 and 4 in [2].

Proposition 3.7. *Let $F = \operatorname{ind} F_n$ be a strict LF-space as introduced in (3.6) (we keep all the notations of (3.6)). Assume that the following two conditions are satisfied*

i) $j_n^t : F'_{n+1,s} \to F'_{n,s}$ is open for all $n \in \mathbf{N}$.

ii) For all $n \in \mathbf{N}$ the space $F_n^\circ := \ker j_n^t \subset F'_{n+1}$ provided with the relative topology $\beta_n := \beta(F'_{n+1}, F_{n+1}) \cap F_n^\circ$ is distinguished.

Then id : $F''_{\text{ind}} \to F''_b$ is a topological isomorphism, i.e. F''_b is a strict LF-space.

Proof. We first remark that for every $n \in \mathbf{N}$ one has $j_n^t : F'_{n+1,s} \to F'_{n,s}$ open $\iff F''_n =$ $= (\ker j_n^t)^{\circ F''_{n+1}} \iff F''_n$ is closed in $(F''_{n+1}, \sigma(F''_{n+1}, F'_{n+1}) \iff (F_n^\circ, \beta_n)' =$ $= F''_{n+1}/F''_n$. Next we note that for every $n \in \mathbf{N}$ the quotient map $q_n : F''_{n+1,b} \to$ $\to (F''_{n+1}/F''_n, \beta(F''_{n+1}/F''_n, F_n^\circ))$ is a continuous linear surjection from a Fréchet space onto a metrizable locally convex space. In fact, because of i) we have $(F_n^\circ, \beta_n) = F''_{n+1}/F''_n$ and (F_n°, β_n) has a fundamental sequence of bounded sets.

Clearly, q_n open $\iff$ the strong dual of (F_n°, β_n) is barrelled $\iff$ the strong dual of (F_n°, β_n) is complete.

Thus i) and ii) imply that for all $n \in \mathbb{N}$,

(*) q_n is a quotient map from $F''_{n+1,b}$ onto the strong dual of (F_n°, β_n).

Now, let $U = \Gamma U$ be a $0-nbhd$ in F''_{ind}. We will show that U is a $0-nbhd$ in F''_b. There exists a sequence $(U_n)_{n\in\mathbb{N}}$ of $0-nbhds$ U_n in $F''_{n,b}$ such that $\sum_{n\in\mathbb{N}} U_n \subset U$.

We will show by induction: there is a sequence $(B_n)_{n\in\mathbb{N}}$ of bounded sets $B_n = \Gamma B_n$ in $F'_{n,b}$ such that $j_n^t(B_{n+1}) = B_n$ and $B_n^{\circ F''_n} \subset (1 - \frac{1}{2^n}) \sum_{m=1}^n U_m$ for all $n \in \mathbb{N}$. In fact: there is $B_1 \subset F'_{1,b}$ bounded such that $B_1^{\circ F''_1} \subset \frac{1}{2} U_1$. Now assume, that $B_1, \ldots, B_n$ have already been constructed. As $j_n^t : F'_{n+1,b} \to F'_{n,b}$ lifts bounded sets, there is $A \subset F'_{n+1,b}$ bounded such that $j_n^t(A) = B_n$. $V := A^{\circ F''_{n+1}} \cap U_{n+1}$ is a $0-nbhd$ in $F''_{n+1,b}$. Because of (*) there is a bounded subset $C \subset (F_n^\circ, \beta_n)$ such that $C^{\circ F''_{n+1}} \subset \frac{1}{2(2^n-1)} V + F''_n$.

$B_{n+1} := \Gamma(A \cup C)$ is an absolutely convex bounded subset of $F'_{n+1,b}$, and because of $C \subset \ker j_n^t$ we have $j_n^t(B_{n+1}) = j_n^t(A) = B_n$.

Furthermore, $B_{n+1}^{\circ F''_{n+1}} \subset A^{\circ F''_{n+1}} \cap C^{\circ F''_{n+1}} \subset A^{\circ F''_{n+1}} \cap (\frac{1}{2(2^n-1)} V + F''_n)$.

Let $\varphi \in B_{n+1}^{\circ F''_{n+1}}$. Then there are $\psi \in \frac{1}{2(2^n-1)} V$ and $\eta \in F''_n$ such that $\varphi = \psi + \eta$. Therefore

$$\eta = \varphi - \psi \in F''_n \cap \left(\frac{1}{2(2^n-1)} V + A^{\circ F''_{n+1}} \right) \subset F''_n \cap \left(1 + \frac{1}{2(2^n-1)} \right) A^{\circ F''_{n+1}} =$$

$$= \left(1 + \frac{1}{2(2^n-1)} \right) j_n^t(A)^{\circ F''_n} = \left(1 + \frac{1}{2(2^n-1)} \right) B_n^{\circ F''_n} \subset$$

$$\subset \left(1 + \frac{1}{2(2^n-1)} \right) \left(1 - \frac{1}{2^n} \right) \sum_{m=1}^n U_m = \left(1 - \frac{1}{2^{n+1}} \right) \sum_{m=1}^n U_m,$$

where the last inclusion follows by induction hypothesis. Consequently, $\varphi = \eta + \psi \in$ $\in (1 - \frac{1}{2^{n+1}}) \sum_{m=1}^n U_m + \frac{1}{2(2^n-1)} U_{n+1} \subset (1 - \frac{1}{2^{n+1}}) \sum_{m=1}^{n+1} U_m$. This finishes the «proof by induction».

As $F'_b = \text{proj}_{\leftarrow} F'_{n,b}$ there is a bounded subset B in F'_b such that $p_n(B) = B_n (n \in \mathbb{N})$, where $p_n : F' \to F'_n$ is the transpose of the inclusion $F_n \to F$. $B^{\circ F''}$ is a $0-nbhd$ in F''_b and $B^{\circ F''} = \bigcup_{n\in\mathbb{N}} p_n(B)^{\circ F''_n} = \bigcup_{n\in\mathbb{N}} B_n^{\circ F''_n} \subset \bigcup_{n\in\mathbb{N}} \sum_{m=1}^n U_m \subset U$. ∎

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