Note di Matematica Vol. XII, 43-48 (1992)

ON THE PROJECTIVE LB-SPACES

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Dedicated to the memory of Professor Gottfried Köthe

1. INTRODUCTION

In his paper on projective spaces [7] G. Köthe described projective elements in the category of Banach spaces (these are $l_1(\Gamma)$-spaces; for the separable case see [13]). Then, he also studied projective spaces in the category of all locally convex spaces and in the category of LB-spaces but, unfortunately, the questions of the precise description of these spaces were left open. The first problem was solved by Geyler [2] and it turns out that the lc direct sums of one-dimensional spaces are the only projective spaces in the category of all lcs.

The main aim of this paper is to solve the second problem as follows:

Main Theorem. *A (strict) LB-space Z is projective in the category of all (strict) LB-spaces, i.e., for every (strict) LB-space X every quotient map $q : X \to Z$ has a right inverse («a lifting»), iff Z is isomorphic to the locally convex direct sum of a sequence of $l_1(\Gamma)$-spaces.*

It should be noted that the classes of projective spaces in the categories of strict LB-spaces, separated LB-spaces and general LB-spaces are equal (all linear continuous maps are morphisms). Indeed, each element X of one of those three categories is a quotient of $Y = \bigoplus_{n\in\mathbb{N}} l_1(\Gamma_n)$, so if X is projective in the corresponding category, then X is a complemented subspace of Y (and therefore it is a strict LB-space). On the other hand, if X is a complemented subspace of Y and $q : Z \to X$ is a quotient map, where Z is an arbitrary (non-necessarily separated) LB-space, then there is another quotient map $Q : \bigoplus_{n\in N} l_1(\Lambda_n) \to Z$. Köthe [7, 6, (6)] proved that projective locally complete separated LB-spaces are exactly complemented subspaces of lc direct sums of $l_1(\Gamma)$-spaces. Thus, there is a right inverse $s : X \to \bigoplus_{n\in N} l_1(\Lambda_n)$ of the composition $q \circ Q$ and $Q \circ s$ is a right inverse of the map q.

In particular, the above cited Köthe's result implies the «if» part of the Main Theorem above. We will prove that complemented subspaces Z of $\bigoplus_{i\in\mathbb{N}} l_1(\Gamma_i)$ are isomorphic to lc

(*) The final version of the paper was written during the author's visit at the University of Wuppertal on Prof. D. Vogt's invitation.

direct sums of $l_1(\Gamma)$-spaces. The proof consists of three parts: (i) a construction in Z of a complemented copy W of a direct sum of «as large as possible» $l_1(\Gamma)$-spaces; (ii) an embedding of Z (as a complemented subspace) into $U \simeq W$; (iii) an application of Pełczyński's decomposition procedure. The proof of the step (ii) (and, especially, the new simple idea how to control $U \simeq W$) presented here is due to Prof. D. Vogt. The original proof was much longer and the author is very indebted for this considerable improvement and consent to publish it.

The result for separable Z is due to Metafune and Moscatelli [10, Th. 2.1, Cor. 3.4] but their method is unapplicable in the general case since it goes through the dual space and it is well known that there are complemented subspaces of $\prod_{i\in\mathbf{N}} l_\infty(\Gamma_i)$, for large Γ_i, which are not isomorphic to any product of $l_\infty(\Gamma)$-spaces.

It should be pointed out that, besides of the mentioned above, there are a few other papers on liftings of linear operators and projective spaces, for instance: [1], [3], [4], [5], [8], [11], [12], [17] and the whole variety of papers on liftings of maps into nuclear spaces, for example, see the review in [18].

The following Rosenthal's criterion [15, Cor. p. 29, Th. 3.3 and Prop. 3.1] (comp. [14, Lemma 1.1]) of the existence of $l_1(\Gamma)$ complemented subspaces in Banach spaces is the main tool of our work.

Theorem 1. *Let K be a bounded subset of cardinality m contained in a Banach space X. If $T : X \to l_1(\Gamma)$ is an operator satisfying the following condition: there exists $\delta > 0$ such that*

$$\|Tk_1 - Tk_2\| \geq \delta \qquad \text{for all } k_1, k_2 \in K, k_1 \neq k_2.$$

then K contains a subset K_1 of the same cardinality which forms an unconditional basis equivalent to the unit basis of $l_1(m)$ and such that for $Y := \overline{\text{lin}}\, K_1$:

(i) $T|_Y$ is an isomorphism;

(ii) Y and $T(Y)$ are complemented in X and $l_1(\Gamma)$, resp.

We also apply the following form of Pełczyński's decomposition which is a particular case of Vogt's [19, Lemma 1.1]:

Theorem 2. *Let $1 \leq p \leq \infty$ and let X be a countable direct sum of infinite-dimensional $l_p(\Gamma)$-spaces (or $c_0(\Gamma)$-spaces). If a complemented subspace Y of X contains a further complemented subspace isomorphic to X, then X is isomorphic to Y.*

Proof. Let (Z_i) be a sequence of stepspaces of an arbitrary strict LB-space Z. If Y_i is the Banach direct sum in the sense of l_p of countably many copies of Z_i, then Y_i is embedded into Y_{i+1} in an obvious way. We define $\left(\sum_{i\in\mathbf{N}} \oplus Z\right)_{l_p}$ to be the strict inductive limit of (Y_i)

(the defined space does not depend on the choice of the sequence (Z_i)).

Finally, let us consider $X \simeq \bigoplus_{i\in\mathbb{N}} l_p(\Gamma_i)$. Then $\left(\sum_{j\in N} \oplus X\right)_{l_p} \simeq X$ and Pełczyński's decomposition as described in [9, p. 54] applies whenever we take the defined above «l_p-sum» instead of the Banach direct sum in the sense of l_p.

For the functional analytic terminology see [6] or [16].

2. THE PROOF OF THE MAIN RESULT

From now on we assume that Z is a complemented subspace of $Y := \bigoplus_{i=1}^{\infty} l_1(\Gamma_1)$. If $Y_k := = \bigoplus_{i=1}^{k} l_1(\Gamma_i)$ and $Z_k := Z \cap Y_k$, $Z_0 := \{0\}$, then Z is a strict inductive limit of Banach spaces Z_k. It might be that for $k \geq j$ $\dim Z_{k+1}/Z_k < \infty$, but obviously then either $Z \simeq Z_j \oplus \varphi$ or Z is a Banach space and our result follows easily from [7].

Therefore, we may assume without loss of generality that

$$\forall k \in \mathbb{N} \ \dim Z_{k+1}/Z_k = \infty \qquad \text{and} \qquad P(Y_k) \subseteq Z_{k+1},$$

where $P : \bigoplus_{i=1}^{\infty} l_1(\Gamma_i) \to Z$ is a continuous projection.

Let us denote by d_k the density character of Z_{k+1}/Z_k. There are three possibilities:

1) (d_k) does not attain its supremum;
2) (d_k) attains its supremum infinitely many times;
3) (d_k) attains its supremum finitely many times.

In order to simplify the situation we need a lemma with an obvious proof:

Lemma 3. *If X is a Banach space and Y is its closed subspace, then the density character of X is equal to the maximum of the density characters of Y and X/Y.*

Now, if 1) holds, we may assume, by omitting some steps (Z_k) and applying Lemma 3, that (d_k) strictly increases. If 2) holds, then, by the same arguments, we may assume that (d_k) is constant. If 3) holds, then we may assume that the supremum is attained only on d_0 and we apply the above procedure once more to $(d_k)_{k>0}$. Nevertheless, the procedure must finish after finitely many steps since otherwise (d_k) would have not attained its infimum. Therefore, in the case 3) we may assume that $(d_k)_{k>0}$ is non-decreasing and d_0 is the supremum of $(d_k)_{k\geq 0}$.

Finally, we define I_k as the maximal subset of elements form the unit ball in Z_{k+1}/Z_k such that

$$\|x-y\| > 1/2 \qquad \text{for all } x, y \in I_k, x \neq y,$$

and, as easily seen, card $I_k = d_k$ (I_k is infinite!).

Proposition 4. *Z Z is isomorphic to a complemented subspace of* $\bigoplus_{i=0}^{\infty} l_1(I_i)$.

Proof. There is a quotient map $q_k : l_1(I_k) \to Z_{k+1}/Z_k$ and we can lift it to a continuous map $T_k : l_1(I_k) \to Z_{k+1}$ for $k = 0, 1, \ldots$. The direct sum of maps T_k is a quotient map from $\bigoplus_{i=0}^{\infty} l_1(I_i)$ onto Z. This completes the proof because Z is projective for LB-spaces.

Proposition 5. *Z contains a complemented isomorphic copy of* $\bigoplus_{i=0}^{\infty} l_1(I_{2i})$.

Proof. The embedding $Z_{k+2} \to Z_{k+2} + Y_k$ and the projection P restricted to $Z_{k+2} + Y_k$ induce (respectively) the following continuous maps onto:

$$r : Z_{k+2} \to Z_{k+2} + Y_k/Y_k, \quad s : Z_{k+2} + Y_k/Y_k \to Z_{k+2}/Z_{k+1}.$$

Let us take one element from each preimage of elements of I_{k+1} with respect to the quotient map $q : Z_{k+2} \to Z_{k+2}/Z_{k+1}$ and form the set J_{k+1} from them. Since $q = s \circ r$, we have

$$\|r(x) - r(y)\| > 1/2\|s\| \quad \text{for} \quad x \neq y, x, y \in J_{k+1}.$$

Moreover, $Z_{k+2} + Y_k/Y_k \subseteq Y_{k+2}/Y_k \simeq l_1(\Gamma_{k+1}) \oplus l_1(\Gamma_{k+2})$ and Th. 1 applies to the map r. Thus there is a subspace $W_{k+2} \subseteq Z_{k+2}$, $W_{k+2} \simeq l_1(I_{k+1})$, which is mapped isomorphically by r onto a complemented subspace in Y_{k+2}/Y_k (and in Y/Y_k, with a projection p). In a standard way we obtain a projection $p_{k+2} : Z \to W_{k+2}$ which vanishes on Z_k. We can repeat the whole procedure also for $k = -1$ if we define $Z_{-1} = Y_{-1} = Y_0 = \{0\}$.

We define $R_k : Z \to W_{2k+1}$ as follows:

$$R_k(z) = p_{2k+1}(z) \qquad \text{for} \quad z \in Z_{2k+1}$$

and

$$R_k(z) = p_{2k+1} \circ (id - p_{2k+3}) \circ \ldots \circ (id - p_{2n+1})(z)$$

for $z \in Z_{2n+1} \setminus Z_{2n-1}$, $n > k$. It is easily seen that for every $z \in Z$ the sequence $(R_k(z))$ for $k \geq 0$ has only finitely many non-zero entries. Thus we can define a continuous projection $R : Z \to W$,

$$R(z) := \sum_{k \in \mathbb{N}} R_k(z),$$

where $W := \text{lin}\{W_{2k+1} : k \geq 0\}$. Therefore W is bornological (as a complemented subspace of Z) and $W \simeq \bigoplus_{k\geq 0} l_1(I_{2k})$.

Now, the main theorem is implied by Prop. 4, 5 and Th. 2 because, by the above assumptions on the sequence (d_k), $d_k = \text{card } I_k$, we have $\bigoplus_{k\geq 0} l_1(I_{2k}) \simeq \bigoplus_{k\geq 0} l_1(I_k)$.

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Received July 6, 1990 and in revised form February 21, 1991
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