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t-SPREADS OF $PG(n,q)$ AND REGULARITY

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Abstract. *In this paper the theory of t-spreads of finite projective spaces is developed using purely geometric methods. This is achieved using the classical Segre Variety over a finite field.*

1. INTRODUCTION

We shall be concerned with t-spreads of $PG(n,q)$, that is, partitions of the points of $PG(n, q)$ into pairwise disjoint t-dimensional spaces. It is well known (see, for example, [9]) that $PG(n,q)$ admits a t-spread if and only if $t+1$ divides $n+1$ so we write $n=(s+1)(t+1)-1$. Such a t-spread has $\omega = q^{s(t+1)} + q^{(s-1)(t+1)} + \ldots + q^{t+1} + 1$ elements.

The theory of t-spreads of $PG(2t+1,q)$ has been developed from both an algebraic (or coordinate) point of view, see [1], and a geometric point of view, see [5] and [15]. However so far t-spreads of $PG((s+1)(t+1)-1,q)$ for $s \geq 2$ have only been studied from an algebraic point of view, see [2] and [11]. In this paper we present a geometric theory of t-spreads of $PG((s+1)(t+1)-1,q)$ using the classical Segre Variety over $GF(q)$.

In particular, we will be concerned with providing a geometric interpretation of the concept of regularity for $s \geq 2$. When $s=1$ regularity is defined as follows. A t-regulus in $PG(2t+1,q)$ is a set $\mathcal{R}$ of $q+1$ pairwise disjoint t-dimensional subspaces such that a line meeting three elements of $\mathcal{R}$ must meet every element of $\mathcal{R}$. Such a line l is called a *transversal line* of $\mathcal{R}$ and meets every element of $\mathcal{R}$ in a unique point.

There is a unique transversal through each point of each element of $\mathcal{R}$; in particular, the transversal lines of $\mathcal{R}$ are pairwise disjoint. The existence of t-reguli is well known; in fact the non-degenerate quadrics of index $t+1$ in $PG(2t+1,q)$ are covered by t-reguli.

If A, B and C are pairwise disjoint t-dimensional subspaces of $PG(2t+1,q)$ then there is a unique t-regulus $\mathcal{R}$ of $PG(2t+1,q)$ containing A, B and C (see [9]). A t-spread $\mathcal{W}$ of $PG(2t+1,q)$ is *regular* if for every triple A,B,C of elements of $\mathcal{W}$, the t-regulus determined by A, B and C is contained in $\mathcal{W}$. As in [9], a t-spread $\mathcal{W}$ of $PG(2t+1,q)$ is regular if and only if for each line l of $PG(2t+1,q)$ not contained in any element of $\mathcal{W}$, the elements of $\mathcal{W}$ meeting l form a t-regulus in $PG(2t+1,q)$.

The purpose of this paper is to find a geometrical generalisation of the ideas of t-reguli and regularity applicable to spreads of $PG((s+1)(t+1)-1,q)$ for $s \geq 2$. This generalisation includes the following definition of regularity of 1-spreads of $PG(2s+1,q)$ for $s \geq 1$ given in [10]. A 1-spread $\mathcal{W}$ of $PG(2s+1,q)$ is *regular* if the $q+1$ lines of $\mathcal{W}$ meeting a line l, not contained in $\mathcal{W}$, form a regulus in some 3-dimensional subspace of $PG(2s+1,q)$.

2. THE SEGRE VARIETY $\mathcal{SV}_{t+1,s+1}$

The Segre variety $\mathcal{SV}_{t+1,s+1}$ appeared first in the work of C. Segre in 1891 (see [14]), where it was studied in projective spaces over infinite fields. For a discussion of the classical Segre variety over an infinite field, see [6] or [12]. The theory is still valid over finite fields, giving the Segre variety in the finite projective space $PG((s+1)(t+1)-1,q)$.

Here we investigate the behaviour of the Segre variety and some of its subvarieties over the finite field $GF(q)$. The Segre variety is defined as follows.

Let S_t and S_s be projective spaces of order q and of dimensions t and s respectively, and suppose that they have as systems of homogeneous coordinates respectively $(y_0, y_1, \ldots, y_t)$ and $(z_0, z_1, \ldots, z_s)$. Consider the projective space $PG((s+1)(t+1)-1,q)$ with homogeneous coordinates $(x_{00}, x_{01}, \ldots, x_{ts})$. The set of points of $PG((s+1)(t+1)-1,q)$ with $x_{ij} = y_i z_j$ for all $i = 0, 1, \ldots, t$ and $j = 0, 1, \ldots, s$ is a variety called the *Segre variety* $\mathcal{SV}_{t+1,s+1}$ in $PG((s+1)(t+1)-1,q)$.

In the following, we will occasionally use semicolons in place of commas to break up the coordinate $(s+1)(t+1)$-tuple $(x_{00}, x_{01}, \ldots, x_{ts})$ into $t+1$ blocks of $s+1$ coordinates each:

$$(x_{00}, x_{01}, \ldots, x_{0s}; x_{10}, x_{11}, \ldots, x_{1s}; \ldots; x_{t0}, x_{t1}, \ldots, x_{ts}).$$

This has no formal significance, it is just done for ease of notation.

Lemma 2.1. *(1) The Segre variety $\mathcal{SV}_{t+1,s+1}$ has two systems of linear subspaces of order q lying on it. There are $q^s + q^{s-1} + \ldots + q + 1$ spaces of dimension t, each in projective correspondence with S_t and each determined by one point $(z_0, z_1, \ldots, z_s)$ of S_s. There are $q^t + q^{t-1} + \ldots + q + 1$ spaces of dimension s, each in projective correspondence with S_s and each determined by one point $(y_0, y_1, \ldots, y_t)$ of S_t.*

(2) The spaces of each system are pairwise disjoint and there is one space of each system through any given point of $\mathcal{SV}_{t+1,s+1}$. Therefore a space of one system meets each space of the other system in a unique point.

(3) The Segre variety $\mathcal{SV}_{t+1,s+1}$ has exactly $(q^t + q^{t-1} + \ldots + q + 1)(q^s + q^{s-1} + \ldots + q + 1)$ points.

Proof. (1) Fix a point $(z'_0, z'_1, \ldots, z'_s)$ of S_s, and consider the set of points of $\mathcal{SV}_{t+1,s+1}$ given by

$$\{(y_0 z'_0, y_0 z'_1, \ldots, y_0 z'_s; y_1 z'_0, y_1 z'_1, \ldots, y_1 z'_s; \ldots; y_t z'_0, y_t z'_1, \ldots, y_t z'_s)\}$$

for $y_0, y_1, \ldots, y_t \in GF(q)$, not all zero. This set of points is a t-dimensional subspace of $PG((s+1)(t+1)-1,q)$ since it is spanned by the $t+1$ linearly independent points

$$(z_0', z_1', \dots, z_s'; 0, 0, \dots, 0; 0, 0, \dots, 0; \dots; 0, 0, \dots, 0),$$
$$(0, 0, \dots, 0; z_0', z_1', \dots, z_s'; 0, 0, \dots, 0; \dots; 0, 0, \dots, 0),$$
$$\vdots$$
$$(0, 0, \dots, 0; 0, 0, \dots, 0; \dots; 0, 0, \dots, 0; z_0', z_1', \dots, z_s').$$

It is in projective correspondence with the t-dimensional space S_t with homogeneous coordinates $(y_0, y_1, \dots, y_t)$. There are $q^s + q^{s-1} + \dots + q + 1$ choices for the point $(z_0', z_1', \dots, z_s')$ of S_s, so there are $q^s + q^{s-1} + \dots + q + 1$ such t-dimensional spaces on $S\mathcal{V}_{t+1,s+1}$. In an analogous way we fix a point $(y_0', y_1', \dots, y_t')$ of S_t, then the set of points of $S\mathcal{V}_{t+1,s+1}$ given by

$$\{(y_0' z_0, y_0' z_1, \dots, y_0' z_s; y_1' z_0, y_1' z_1, \dots, y_1' z_s; \dots; y_t' z_0, y_t' z_1, \dots, y_t' z_s)\}$$

for $z_0, z_1, \dots, z_s \in GF(q)$, not all zero, forms an s-dimensional subspace of the space $PG((s+1)(t+1)-1, q)$. Each such s-dimensional space is in projective correspondence with the space S_s with homogeneous coordinates $(z_0, z_1, \dots, z_s)$.

(2) Given any two points $(y_0', y_1', \dots, y_t')$ and $(y_0'', y_1'', \dots, y_t'')$ of S_t, the s-dimensional spaces that they detemine (as in (1)) are disjoint, and similarly any two elements of the system of t-dimensional spaces on $S\mathcal{V}_{t+1,s+1}$ are disjoint. The point

$$(y_0' z_0', y_0' z_1', \dots, y_0' z_s'; y_1' z_0', y_1' z_1', \dots, y_1' z_s'; \dots; y_t' z_0', y_t' z_1', \dots, y_t' z_s')$$

lies on the t-dimensional space of $S\mathcal{V}_{t+1,s+1}$ which is determined by the point $(z_0', z_1', \dots, z_s')$ of S_s and the s-dimensional space determined by the point $(y_0', y_1', \dots, y_t')$ of S_t, and these spaces are unique. Conversely the t-dimensional space of $S\mathcal{V}_{t+1,s+1}$ determined by the point $(z_0', z_1', \dots, z_s')$ of S_s meets the s-dimensional space determined by the point $(y_0', y_1', \dots, y_t')$ of S_t in the unique point

$$(y_0' z_0', y_0' z_1', \dots, y_0' z_s'; y_1' z_0', y_1' z_1', \dots, y_1' z_s'; \dots; y_t' z_0', y_t' z_1', \dots, y_t' z_s')$$

of $S\mathcal{V}_{t+1,s+1}$.

(3) The number of points of the Segre variety $S\mathcal{V}_{t+1,s+1}$ is the number of elements of the system of t-dimensional spaces multiplied by the number of points in such a t-dimensional space. Alternatively, it is the number of elements in the system of s-dimensional spaces multiplied by the number of points in such an s-dimensional space. ∎

The system of t-dimensional subspaces of $S\mathcal{V}_{t-1,s+1}$ will be called the *first system* of subspaces and the system of s-dimensional spaces will be called the *second system* of subspaces.

Lemma 2.2. *(1) The first system of $S\mathcal{V}_{t+1,s+1}$ can be obtained by joining corresponding points of $t+1$ pairwise disjoint, projectively related s-dimensional subspaces in a space $PG((s+1)(t+1)-1,q)$. The second system of subspace of $S\mathcal{V}_{t+1,s+1}$ is obtained similarly by joining corresponding points of $s+1$ pairwise disjoint, projectively related t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$.*

(2) There is a unique Segre variety $S\mathcal{V}_{t+1,s+1}$ containing any $t+2$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $t+1$ in a hyperplane. Similarly, there is a unique Segre variety $S\mathcal{V}_{t+1,s+1}$ containing $s+2$ t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $s+1$ in a hyperplane.

Proof. (1) We choose a convenient system of homogeneous coordinates for the space $PG((s+1)(t+1)-1,q)$ so that the $t+1$ s-dimensional spaces are:

$$\{(x_0,x_1,\ldots,x_s;0,\ldots,0;\ldots;0,\ldots,0) : x_i \in GF(q)\}$$
$$\{(0,\ldots,0;x_0,x_1,\ldots,x_s;0,\ldots,0;\ldots;0,\ldots,0) : x_i \in GF(q)\}$$
$$\vdots$$
$$\{(0,\ldots,0;\ldots;0,\ldots,0;x_0,x_1,\ldots,x_s) : x_i \in GF(q)\}.$$

For $x'_0,x'_1,\ldots,x'_s \in GF(q)$, not all zero, construct the t-dimensional space spanned by the points

$$(x'_0,x'_1,\ldots,x'_s;0,\ldots,0;\ldots;0,\ldots,0),$$
$$(0,\ldots,0;x'_0,x'_1,\ldots,x'_s;0,\ldots,0;\ldots;0,\ldots,0),$$
$$\vdots$$
$$(0,\ldots,0;\ldots;0,\ldots,0;x'_0,x'_1,\ldots,x'_s).$$

The set of t-dimensional spaces so constructed is the set of elements of the first system of subspaces of a Segre variety $S\mathcal{V}_{t+1,s+1}$, and they can be used to find the second system as in Lemma 2.1 (1). Similarly we can choose homogeneous coordinates for the space $PG((s+1)(t+1)-1,q)$, so that the given t-dimensional spaces are:

$$\{(x_0,0,\ldots,0;x_1,0,\ldots,0;x_t,0,\ldots,0) : x_i \in GF(q)\}$$
$$\{(0,x_0,0,\ldots,0;0,x_1,0,\ldots,0;\ldots;0,x_t,0,\ldots,0) : x_i \in GF(q)\}$$
$$\vdots$$
$$\{(0,\ldots,0,x_0;0,\ldots,0,x_1;\ldots;0,\ldots,0,x_t) : x_i \in GF(q)\}.$$

For $x'_0, \ldots, x'_t \in GF(q)$, not all zero, construct the s-dimensional space spanned by the points

$$(x'_0, 0, \ldots, 0; , x'_1, 0, \ldots, 0; \ldots; x'_t, 0, 0, \ldots, 0),$$
$$(0, x'_0, 0, \ldots, 0; 0, x'_1, 0, \ldots, 0; \ldots; 0, x'_t, 0, \ldots, 0),$$
$$\vdots$$
$$(0, \ldots, 0, x'_0; 0, \ldots, 0, x'_1; \ldots; 0, \ldots, 0, x'_t).$$

The set of s-dimensional spaces so constructed is the second system of subspaces of a Segre variety $\mathcal{SV}_{t+1,s+1}$, and can be used to find the first system of subspaces as in Lemma 2.1 (1).

(2) Through a general point of $PG((s+1)(t+1)-1,q)$ there passes a unique t-dimensional space meeting each of $t+1$ pairwise disjoint s-dimensional spaces, no t in a hyperplane. This space is called a *transversal space* to the s-dimensional spaces, and meets each of the s-dimensional spaces necessarily in a unique point. So given $t+2$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$, no $t+1$ lying in a hyperplane, there are $q^s+q^{s-1}+\ldots+q+1$ t-dimensional spaces meeting all of them, each in a unique point. These t-dimensional spaces are pairwise disjoint and together with the s-dimensional spaces they are the first and second systems of subspaces, respectively, of a Segre variety $\mathcal{SV}_{t+1,s+1}$. Similarly, given $s+2$ t-dimensional spaces in $PG((s+1)(t+1)-1,q)$, no $s+1$ in a hyperplane, there are $q^t+q^{t-1}+\ldots+q+1$ s-dimensional subspaces of $PG((s+1)(t+1)-1,q)$ meeting all of them, each in a unique point. These s-dimensional spaces are pairwise disjoint and together with the t-dimensional spaces they are the second and first systems of subspaces, respectively, of a Segre variety $\mathcal{SV}_{t+1,s+1}$. ∎

We now investigate the properties of certain subvarieties of a Segre variety, an idea which will become important later.

A *subvariety* $\mathcal{SV}_{t+1,r+1}$ of a Segre variety $\mathcal{SV}_{t+1,s+1}$ is a Segre variety where every element of the first system of subspaces of the subvariety belongs to the first system of the variety, and every element of the second system of subspaces of the subvariety is an r-dimensional subspace of an element of the second system of the variety. In particular, a subvariety $\mathcal{SV}_{t+1,r+1}$ of $\mathcal{SV}_{t+1,s+1}$ lies in an $((r+1)(t+1)-1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$.

Lemma 2.3. *The Segre variety $\mathcal{SV}_{t+1,s+1}$ admits subvarieties $\mathcal{SV}_{t+1,r+1}$ for every value of r with $0 \le r \le s$.*

Proof. Let S_s be an element of the second system of subspaces of $\mathcal{SV}_{t+1,s+1}$. For any value of r, with $0 \le r \le s$, let S_r be an r-dimensional subspace of S_s. The elements of the first system of $\mathcal{SV}_{t+1,s+1}$ meeting S_s in points of S_r are the t-dimensional spaces of the first system of subspaces of a subvariety $\mathcal{SV}_{t+1,r+1}$. Each element of the second system of subspaces of $\mathcal{SV}_{t+1,r+1}$ is found either by intersecting the elements of the first system of $\mathcal{SV}_{t+1,r+1}$ with the elements of the second system of $\mathcal{SV}_{t+1,s+1}$ or alternatively by finding the r-dimensional subspace of each element of the second system of $\mathcal{SV}_{t+1,s+1}$ which is projectively equivalent to the subspace S_r of S_s under the original projectivity relating the elements of the second system. ∎

3. t-REGULI OF RANK r AND REGULAR t-SPREADS

A *regulus* in $PG(3,q)$ is a set $\mathcal{R}$ of $q+1$ lines such that any line which meets three elements of $\mathcal{R}$ must meet every element of $\mathcal{R}$. It can also be viewed as a set of $q+1$ lines forming one set of generators of a hyperbolic quadric in $PG(3,q)$. It is this second characterisation of a regulus in $PG(3,q)$ which was used in [13] to generalise the idea of regulus in $PG(3,q)$ to a regulus of rank r in $PG(2s+1,q)$ where a regulus of rank r is the set of all lines of the first system of subspaces of a Segre variety $\mathcal{SV}_{2,r+1}$. This can be generalised further to t-reguli of *rank* r in $PG((s+1)(t+1)-1,q)$ as follows, making use of the Segre Variety $\mathcal{SV}_{t+1,s+1}$ in $PG((s+1)(t+1)-1,q)$.

For $0 \leq r \leq s$, let $\Gamma^0, \Gamma^1, \ldots, \Gamma^t$ be a set of $t+1$ pairwise disjoint r-dimensional subspaces which span a projective space $PG((r+1)(t+1)-1,q)$, and suppose there exist t projective correspondences $\alpha_i : \Gamma^0 \to \Gamma^i$, for $i = 1,2,\ldots,t$. The set of t-dimensional subspaces of $PG((r+1)(t+1)-1,q)$ formed by joining a point P of Γ^0 to the corresponding points $P^{\alpha_1}, P^{\alpha_2}, \ldots, P^{\alpha_t}$ of $\Gamma^1, \Gamma^2, \ldots, \Gamma^t$ respectively is called a *t-regulus of rank* r, and is denoted by $\mathcal{R}_r$.

When the space $PG((r+1)(t+1)-1,q)$ is a subspace of a projective space $PG(n,q)$, $n \geq (r+1)(s+1)-1$, we say that $\mathcal{R}_r$ is a t-regulus of *rank* r of $PG(n,q)$, implying that $\mathcal{R}_r$ lies in a $((r+1)(t+1)-1)$-dimensional subspace of $PG(n,q)$.

By Lemma 2.2 (1) we have

Theorem 3.1. *A t-regulus of rank r in $PG(n,q)$ is the first system of subspaces of a Segre variety $\mathcal{SV}_{t+1,r+1}$ in a subspace $PG((r+1)(t+1)-1,q)$ of $PG(n,q)$ and conversely.* ∎

Corresponding to the subvarieties $\mathcal{SV}_{t+1,k+1}$ of a Segre variety $\mathcal{SV}_{t+1,r+1}$, there are t-subreguli of rank k of a t-regulus $\mathcal{R}_r$ of rank r, for each $1 \leq k \leq r$. More precisely, the elements of a t-subregulus $\mathcal{R}_k$ of a rank k of $\mathcal{R}_r$ are all elements of $\mathcal{R}_r$ and each transversal k-dimensional subspace of $\mathcal{R}_k$ is a subspace of a transversal r-dimensional subspace of $\mathcal{R}_r$.

Corollary 3.2. *(1) A t-regulus of rank r has $q^r + q^{r-1} + \ldots + q + 1$ elements.*

(2) There is a unique t-regulus of rank r through any $r+2$ t-dimensional subspaces in $PG((r+1)(t+1)-1,q)$, no $r+1$ of which lie in a hyperplane.

(3) A t-regulus $\mathcal{R}_r$ of rank r has $q^t+q^{t-1}+\ldots+q+1$ transversal r-dimensional spaces, that is, r-dimensional spaces which meet every element of $\mathcal{R}_r$ in a unique point.

(4) A t-regulus of rank r admits t-subreguli of ranks $r-1$, $r-2,\ldots,1,0$. The number of t-subreguli of rank k for $0 \leq k \leq r$ in a t-regulus of rank r is just the number of k-dimensional subspaces of an r-dimensional projective space. ∎

Let $\mathcal{R}_r$ be a t-regulus of rank r in $PG((r+1)(t+1)-1,q)$. Any two t-subreguli $\mathcal{R}_k$ and $\mathcal{R}_m$ of $\mathcal{R}_r$ (of ranks say k and m respectively) are either disjoint or intersect in a t-subregulus of $\mathcal{R}_r$ (which is also a t-subregulus of $\mathcal{R}_k$ and of $\mathcal{R}_m$) of rank less than or equal to the smaller of the two ranks k and m.

As we now have a definition for a t-regulus of rank r, we can use it to introduce the idea of different sorts of regularity of a t-spread corresponding to the different sorts of t-regulus which it may contain.

A t-spread $\mathcal{W}$ of $PG((s+1)(t+1)-1,q)$ is *t-regular of rank* r for $0 \leq r \leq s$ if given an r-dimensional subspace S_r of the space $PG((s+1)(t+1)-1,q)$ not meeting any element of $\mathcal{W}$ in more than one point, then the $q^r+q^{r-1}+\ldots+q+1$ t-dimensional spaces of $\mathcal{W}$ meeting it form a t-regulus of rank r. If there is no confusion then we say that $\mathcal{W}$ is *regular of rank* r. In particular, the $q^r+q^{r-1}+\ldots+q+1$ t-dimensional spaces in the t-regulus of rank r lie in an $((r+1)(t+1)-1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$.

Examples 3.3. (1) Every t-spread of $PG((s+1)(t+1)-1,q)$ is regular of rank 0. This is because given any point (or 0-dimensional subspace) of $PG((s+1)(t+1)-1,q)$, there is a unique element of the t-spread through it, and this t-dimensional space is a t-regulus of rank 0.

(2) For a t-spread of $PG(2t+1,q)$ and for a 1-spread of $PG(2s+1,q)$, regularity of rank 1 coincides with the usual definition of regularity.

We now investigate the relationship between regularity of various ranks. First we need to define *geometric*, an idea which appeared in [2] and [15]. A set $\mathcal{W}$ of pairwise disjoint t-dimensional subspaces of $PG(n,q)$ is *geometric* if for every pair of distinct elements X,Y of $\mathcal{W}$ the elements of $\mathcal{W}$ are either contained in or are disjoint from the join $\langle X,Y\rangle$ of the spaces X and Y. If $\mathcal{W}$ is a geometric t-spread then the elements of $\mathcal{W}$ contained in $\langle X,Y\rangle$ form a t-spread of $\langle X,Y\rangle$, called the t-spread *induced* on $\langle X,Y\rangle$ by $\mathcal{W}$. It is known (see [15]) that if the space $PG(n,q)$ contains a t-spread then it must contain a geometric t-spread, which occurs if and only if $t+1$ divides $n+1$.

Note that the first system of subspaces of $\mathcal{SV}_{t+1,s+1}$ is geometric. To see this, let S_t, S_t' and S_t'' be elements of the first system of subspaces of $\mathcal{SV}_{t+1,s+1}$ and suppose that $P \in S_t'' \cap \langle S_t, S_t'\rangle$. Let S_s be the element of the second system of subspaces of $\mathcal{SV}_{t+1,s+1}$ which contains the point P of S_t'' (see Lemma 2.1 (2)). Let $Q = S_t \cap S_s$ and let $R = S_t' \cap S_s$.

It follows that, in S_s, P lies on the line QR and by the projective relationship between the elements of the second system of $\mathcal{SV}_{t+1,s+1}$ it follows that the points of intersection of S_t, S_t' and S_t'' with any element of the second system of $\mathcal{SV}_{t+1,s+1}$ are collinear. Thus S_t, S_t' and S_t'' are elements of the first system of subspaces of a subvariety $\mathcal{SV}_{t+1,2}$ and $S_t'' \subseteq \langle S_t, S_t' \rangle$ showing that the first system of $\mathcal{SV}_{t+1,s+1}$ is geometric.

Lemma 3.4. *Let S_r be an r-dimensional subspace of $PG((s+1)(t+1)-1,q)$ for some integer r with $1 \leq r \leq s-1$. Let $\mathcal{R}_r$ be a set of $q^r + q^{r-1} + \ldots + q + 1$ pairwise disjoint t-dimensional subspaces of $PG((s+1)(t+1)-1,q)$ each meeting S_r in a unique point. Then $\mathcal{R}_r$ is a t-regulus of rank r if and only if it is geometric and, for each line l of S_r, the set of t-dimensional spaces of $\mathcal{R}_r$ meeting l is a t-regulus of rank 1.*

Proof. The result is true if $r = 1$ so suppose that $r \geq 2$. Suppose first that $\mathcal{R}_r$ is a t-regulus of rank r. Then it is the first system of subspaces of a Segre variety $\mathcal{SV}_{t+1,r+1}$ contained in a given subspace $PG((r+1)(t+1)-1,q)$ of $PG((s+1)(t+1)-1,q)$ and hence is geometric. Now S_r belongs to the second system of $\mathcal{SV}_{t+1,r+1}$, by Lemma 2.3 and the set of elements of the first system of $\mathcal{SV}_{t+1,r+1}$ meeting a line l of S_r is the first system of a Segre subvariety $\mathcal{SV}_{t+1,2}$ contained in a $(2t+1)$-dimensional subspace of $PG((r+1)(t+1)-1,q)$, that is, a t-regulus of rank 1.

Conversely, since $\mathcal{R}_r$ is geometric, there exists a set of $r+2$ pairwise disjoint t-dimensional spaces, no $r+1$ of which lie in a hyperplane (equivalently, any $r+1$ of which span a space of dimension $((r+1)(t+1)-1))$. These determine a unique Segre variety with the chosen t-dimensional spaces being elements of its first system of subspaces and S_r being an element of its second system of subspaces. Let S_t be one of the chosen t-dimensional subspaces and let $P = S_t \cap S_r$. By hypothesis, any line l of S_r through P determines a Segre variety $\mathcal{SV}_{t+1,2}$ where the elements of the first system of subspaces of such a variety are all elements of $\mathcal{R}_r$. But such a variety is a subvariety of $\mathcal{SV}_{t+1,r+1}$, so the elements of $\mathcal{R}_r$ meeting S_r in points of l are all elements of the first system of $\mathcal{SV}_{t+1,r+1}$. Since the points of S_r can be exhausted by lines through P, the elements of the first system of subspaces of $\mathcal{SV}_{t+1,r+1}$ are all elements of $\mathcal{R}_r$ and the lemma is proved.

Lemma 3.5. *Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$ which is regular of rank r, for some r with $1 \leq r \leq s$. Then $\mathcal{W}$ is regular of each rank $r-1$, $r-2, \ldots, 1, 0$, and if it is geometric then it is also regular of each rank $r+1$, $r+2, \ldots, s$.*

Proof. For some value of k with $1 \leq k \leq r-1$, let S_{r-k} be an $(r-k)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$, not meeting any element of $\mathcal{W}$ in more than one point. This lies in an r-dimensional subspace S_r of the space $PG((s+1)(t+1)-1,q)$ not meeting any element of $\mathcal{W}$ in more than one point. The $q^r + q^{r-1} + \ldots + q + 1$ t-dimensional spaces of $\mathcal{W}$ meeting S_r a t-regulus of rank r by assumption, and the $q^{r-k} + q^{r-k-1} + \ldots + q + 1$

t-dimensional spaces of $\mathcal{W}$ meeting S_{r-k} are a subregulus of rank $r-k$ by Lemma 2.3. This shows that $\mathcal{W}$ is regular of each rank $r-1, r-2, \ldots, 1$ and it is regular of rank 0 since every t-spread is regular of rank 0 (see Examples 3.3).

Now since $\mathcal{W}$ is regular of rank r for some r with $1 \leq r \leq s-1$, then by the first part of the lemma $\mathcal{W}$ is regular of rank 1. Let S_{r+k}, for some $1 \leq k \leq s-r$, be an $(r+k)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$, not meeting any element of $\mathcal{W}$ in more than one point. There is exactly one element of $\mathcal{W}$ through each point of S_{r+k}, and since $\mathcal{W}$ is regular of rank 1, the set of $q+1$ t-dimensional spaces of $\mathcal{W}$ meeting any line of S_{r+k} is a t-regulus of rank 1. By Lemma 3.4, as $\mathcal{W}$ is geometric by assumption, the set of t-dimensional spaces of $\mathcal{W}$ meeting S_{r+k} is a t-regulus of rank $(r+k)$ and so $\mathcal{W}$ is regular of rank $(r+k)$. This shows that $\mathcal{W}$ is regular of each rank $r+1, r+2, \ldots, s$. ∎

This lemma suggests the following definition. A t-spread $\mathcal{W}$ in $PG((s+1)(t+1)-1,q)$ is called *regular* if it is regular of rank s, so it is necessarily regular of each rank $0, 1, \ldots, s$.

We now show that, for a t-spread of $PG((s+1)(t+1)-1,q)$, the definition of regular coincides with that of geometric.

Theorem 3.6. *Let $\mathcal{W}$ be a t-spread of $PG((s+1)(t+1)-1,q)$, $s \geq 2$. Then $\mathcal{W}$ is geometric if and only if it is regular.*

Proof. Let $\mathcal{W}$ be a geometric t-spread of $PG((s+1)(t+1)-1,q)$. Then $\mathcal{W}$ induces a regular t-spread on any $(2t+1)$-dimensional subspace $\langle W_i, W_j\rangle$ for distinct elements W_i and W_j of $\mathcal{W}$ ([15], see [3], Result 6). Let l be a line of $PG((s+1)(t+1)-1,q)$, not contained in any element of $\mathcal{W}$. Without loss of generality, suppose l meets the elements $\{W_1, W_2, \ldots, W_{q+1}\}$ of $\mathcal{W}$. Then l is contained in $\langle W_1, W_2\rangle$ and since $\{W_3, W_4, \ldots, W_{q+1}\}$ all have a point in common with $\langle W_1, W_2\rangle$ (which is their point of intersection with l) then $\{W_3, W_4, \ldots, W_{q+1}\}$ are all contained in $\langle W_1, W_2\rangle$ as $\mathcal{W}$ is geometric. As the t-spread induced on $\langle W_1, W_2\rangle$ is regular, the set of spaces $\{W_1, W_2, \ldots, W_{q+1}\}$ form a t-regulus which is a t-regulus of rank 1. Thus $\mathcal{W}$ is regular of rank 1 and geometric so by Lemma 3.5, $\mathcal{W}$ is regular of rank s and hence regular.

Conversely suppose that $\mathcal{W}$ is a regular t-spread, then it is regular of rank 1. Choose $W_i, W_j \in \mathcal{W}$, with $W_i \neq W_j$, and consider the $(2t+1)$-dimensional space $\langle W_i, W_j\rangle$. Any line l of $\langle W_i, W_j\rangle$ meets $q+1$ elements of the t-spread $\mathcal{W}$, which form a t-regulus of rank 1 in some $(2t+1)$-dimensional subspace of $PG((s+1)(t+1)-1,q)$. Thus if l meets both W_i and W_j then the elements of the t-regulus of rank 1 defined by l all lie in $\langle W_i, W_j\rangle$, since it has dimension $(2t+1)$. Now let P be any point of $\langle W_i, W_j\rangle$, and suppose that $P \in W_k$, where $k \neq i, j$. There is a line l of $PG((s+1)(t+1)-1,q)$ through P which meets both W_i and W_j. This is because the space $\langle W_i, P\rangle$ is contained in $\langle W_i, W_j\rangle$ and has dimension $t+1$. Thus it meets W_j in a point say Q, and the line $l = PQ$ passes through P

and meets both W_i and W_j. In this way we can see that every point $P \in \langle W_i, W_j \rangle$ lies on some element W_k of the t-spread $\mathcal{W}$, and this element must be contained in $\langle W_i, W_j \rangle$, and $\mathcal{W}$ is geometric. ∎

There are many applications of the theory of regularity of t-spreads of projective spaces $PG((s+1)(t+1)-1,q)$, some of which are suggested by the literature in the case of $s = 1$ or $t = 1$. For example, in [8] a well known result representation due to Bruck [4] of 1-spreads of $PG(3,q)$ is extended to regular t-spreads of $PG((s+1)(t+1)-1,q)$. More precisely, it is shown that a t-spread of $PG((s+1)(t+1)-1,q)$ is regular if and only if there is a certain s-dimensional subspace (to be called *imaginary*) of $PG((s+1)(t+1)-1,q^{t+1})$ meeting every element of the t-spread in a unique point. The proof of this result depends on the theory of projective t-spread sets developed in [7].

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