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THE AFFINE PYTHAGOREAN THEOREM OF PAPPUS

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Abstract. *The generalization of the Pythagorean theorem known as Pappus' theorem is extended to R^n.*

One version of Pappus' theorem is the following:

Pappus' Theorem. *Suppose a proper triangle ABC has parallelograms $\mathcal{A}$ and $\mathcal{B}$ given on sides AB and BC, respectively, both outside the triangle. Produce the sides of $\mathcal{A}$ and $\mathcal{B}$ opposite AB and BC until they meet in a point P. Let C be the parallelogram with vector sides $\vec{CA}$ and $\vec{PB}$. Then the sum of the areas of $\mathcal{A}$ and $\mathcal{B}$ is the area of C.*

Here this is extended to R^n; using the terminology introduced below we show

Theorem. *Let S be a proper n-simplex in R^n. Suppose $\mathbf{g}_i$ is a vector pointing out of S through the face F_i or $\mathbf{g}_i$ is parallel to $F_i, i = 1, \dots, n$. The sum of the volumes of the prisms $(F_i, \mathbf{g}_i), i = 1, \dots, n$, is the volume of $(F_0, P\vec{V}_0)$ where V_0 is the point common to all hyperplanes of the faces $F_i, i = 1, \dots, n$, and P is the point common to all parallel translates, T_i, of these hyperplanes by $\mathbf{g}_i$, respectively, $i = 1, \dots, n$.*

In particular, Kazarinoff's [2] extension to R^3 follows; in R^3, if three triangular face prisms are given on the outside of a tetrahedron, then this construction determines a triangular face prism on the remaining face of volume the sum of the three given volumes.

Cook Wilson [5] noted that there are several essentially equivalent versions of this result using signed volumes.

For clarity only one form of Pappus' Theorem in R^n is stated above, but all signed versions follow from the arguments here as well.

The n-simplex S with vertices $V_0, V_1, \dots, V_n$ is proper, if the n edges $\mathbf{a}_i = V_0\vec{V}_i$ are linearly independent vectors. The closed face F_i of S is the $(n-1)$-dimensional simplex determined by all of the V_j except V_i. A *face prism* $(F_i, \mathbf{g}_i)$ of an n-simplex is the section of the cylinder on F_i with axis parallel to $\mathbf{g}_i$ determined by the hyperplane containing F_i and by a hyperplane T_i, which is the hyperplane containing the translate of F_i by $\mathbf{g}_i$; for $i \neq 0, (F_i, \mathbf{g}_i) = \left\{ \mu \mathbf{g}_i + \sum_{j=1, j\neq i}^{n} \lambda_j \mathbf{a}_j + O\vec{V}_0 : 0 \leq \mu, \lambda_j \leq 1, \sum \lambda_j \leq 1 \right\}$. Let A_i be a point on the hyperplane determined by the face F_i of the simplex S. If $\mathbf{g}_i = kV_i\vec{A}_i, k \geq 0$,

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then $\mathbf{g}_i$ will be said to be pointing *out of* F_i, or *out of* S *through* F_i. Finally, let U be the standard simplex at the origin having the standard basis $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ of R^n as spanning edges.

Proof. Let A be the matrix with columns $\mathbf{a}_i, i = 1, \ldots, n$. Assume the points $V_0, V_1, \ldots, V_n$ are labeled so that det $A > 0$. The affine tranformation $f(\mathbf{x}) = A^{-1}(\mathbf{x} - \vec{OV_0})$ takes S onto U and multiplies all volumes by det A^{-1} [4]. Also f takes parallel hyperplanes to parallel hyperplanes and outward pointing vectors to outward pointing vectors. Thus it suffices to prove the theorem for U. Suppose then that a face prism $(U_i, \mathbf{h}_i)$ is given on the face of U in the coordinate hyperplane $\langle \mathbf{e}_i, \mathbf{x} \rangle = 0, i = 1, \ldots, n$. The volume of a standard simplex in R^k is $1/k!$ and the volume of a prism $(U_i, \mathbf{h}_i)$ is the height times the content of the base [3]. Thus, the volume of a face prism is also $1/(n-1)!$ times the volume of the parallelepiped having an edge $\mathbf{h}_i$ together with the spanning edges of U_i. If the hyperplanes T_i meet in $\vec{OP} = (x_1, \ldots, x_i, \ldots, x_n)$ then all $x_i \leq 0$, because for each $i, \mathbf{h}_i$ points out of U_i. Thus the volume of $(U_i, \mathbf{h}_i)$ is the height $|x_i|$ times the base content $1/(n-1)!$, showing the sum of the volumes of the given face prisms to be $(|x_1| + |x_2| + \ldots + |x_n|)/(n-1)!$. The remaining face U_0 is spanned by edges $\mathbf{e}_i - \mathbf{e}_1, i = 2, \ldots, n$, so, by the standard determinant expression for the volume of a parallelepiped [4], the face prism $(U_0, \vec{PO})$ has volume

$$\frac{1}{(n-1)!} \det \begin{pmatrix} |x_1| & -1 & -1 & \ldots & -1 \\ \vdots & 1 & 0 & & 0 \\ \vdots & 0 & 1 & & \\ & \vdots & \vdots & \vdots & \\ |x_n| & 0 & 0 & & 1 \end{pmatrix}$$

$$= \frac{1}{(n-1)!} \det \begin{pmatrix} |x_1| + \ldots + |x_n| & 0 & 0 & \ldots & 0 \\ |x_2| & 1 & & & \\ \vdots & 0 & & & \\ & \vdots & \vdots & & \\ |x_n| & 0 & & & 1 \end{pmatrix}$$

$$= \frac{1}{(n-1)!}(|x_1| + \ldots + |x_n|)$$

as required to complete the proof.

It is also clear that if each x_i is arbitrarily chosen above, the sign of each x_i can be related to a signed volume to get all signed versions of this theorem as well.

It may be of historical interest to interpret Pappus' Theorem as a divergence theorem. Think of the prism $(F_i, \mathbf{g}_i)$ as inducing flux of magnitude $|x_i|A_i$ through face F_i, where A_i is the content of F_i. Pappus' result essentially calculates a constant vector field $P\vec{V}_0$ on R^n which induces an inward flux through F_i equal in magnitude to that induced by $(F_i, \mathbf{g}_i)$, whenever $V_0 \in F_i$, and then notes the total flux into S at V_0 equals the flux out of the opposite side F_0. Thus Pappus' theorem is seen to be an early special case of the divergence theorem. (Other cases like the classical Pythagorean theorem seem more difficult to recognize, perhaps because of the emphasis on squares).

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