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REMARKS ON SOME BASIC PROPERTIES OF TSIRELSON'S SPACE

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Abstract. *This note presents a new approach to some known, but difficult to prove, results: it is shown how all the basic properties, and some other less well-known, of Tsirelson space and its dual follow from an inequality stated in Tsirelson's original paper [9].*

1. BACKGROUND

We base our approach to the properties of Tsirelson's space on the use of weakly-p-summable sequences. Throughout the paper p^* denotes the conjugate number of p; if $p = 1$, l_∞ plays the role of c_0.

Preliminary Definitions. *A sequence* (x_n) *in a Banach space* X *is said to be* weakly-p-summable $(p \geq 1)$ *if there is a* $C > 0$ *such that*

$$\sup_n \left\| \sum_{k=1}^n \xi_k x_k \right\| \leq C \cdot \|(\xi_n)\|_{l_{p^*}}$$

for any $(\xi_n) \in l_{p^*}$.
It is said to be p-Banach-Saks, $1 < p < +\infty$, *if*

$$\left\| \sum_{k=1}^n x_k \right\| \leq C \cdot n^{1/p}$$

for some constant $C > 0$ *and all* $n \in \mathbb{N}$.
We shall say that the sequence (x_n) *is weakly-p-convergent (resp. p-Banach-Saks convergent) to* $x \in X$ *if the sequence* $(x_n - x)$ *is weakly-p-summable (resp. p-Banach-Saks).*

These sequences allow us now to introduce two special classes of reflexive spaces, W_p and p-Banach-Saks.

Definition. *Let* $1 \leq p < +\infty$. *A subset* K *of a Banach space* X *is said to be relatively weakly-p-compact if any sequence in* K *admits a weakly* $- p -$ *convergent sub sequence. We say that* $X \in W_p$ *if its closed unit ball is weakly-p-compact.*

Example of spaces in W_p are: $l_p \in W_{p*}$, $1 < p < +\infty$; and $L_p \in W_r$ where $r = \max\{p^*, 2\}$, $1 < p < +\infty$. In general, James' characterization of super-reflexivity [4], combined with the Bessaga-Pelczynski selection principle, implies:

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Proposition 1 [1]. *Let X be an infinite-dimensional super reflexive Banach space. Then there are numbers $p > q > 1$ such that $X \in W_p$ but $X \notin W_q$.*

Definition [5]. *Let $1 < p < +\infty$. A Banach space is said to have the p-Banach-Saks property when each bounded sequence (x_m) admits a sub-sequence (x_n) and a point x such that $(x_n - x)$ is a p-Banach-Saks sequence.*

One of the main results of [1] may now be stated:

Proposition 2. *Let $1 < p < +\infty$. Every $X \in W_p$ has the p^*-Banach-Saks property. Every Banach space X with the p-Banach-Saks property belongs to W_r for all $r > p^*$.*

Many arguments in what follows will be simplified by the introduction of another class of Banach spaces, in some sense the *dual* of W_p: we say that a Banach space $X \in C_p$ if weakly-p-summable sequences are norn null. Two simple examples are: $l_p \in C_r$ for all $r < p^*$, $1 \leq p < +\infty$; and $L_p \in C_r$, for $r < \min\{p^*, 2\}$, $1 \leq p < +\infty$. Another example results from the application of Orlicz' theorem: that spaces of *cotype s* belongs to C_r for all $r < s^*$.

It is clear that an infinite-dimensional Banach space cannot simultaneously belong to C_p and W_p. It is also clear that subspaces of spaces in C_p (resp. W_p) themselves belong to C_p (resp. W_p). Quotients of spaces in W_p belong to W_p (obviously false for C_p).

2. PROPERTIES OF T AND T^*

We shall now apply the preceding ideas to obtain most of the basic properties of Tsirelson's space T^*. This is a reflexive Banach sequence space which does not contain a copy of l_p for $1 \leq p < +\infty$, or c_0. The following is Tsirelson's original definition:

Let K be a weakly compact set of c_0 such that

1) K is contained in the unit ball of c_0;

2) for any sequence $x \in K$, if y is another sequence such that $|y(n)| \leq |x(n)|$ for all $n \in \mathbb{N}$, then $y \in K$;

3) given $x_1, \ldots, x_N \in K$ such that if $x_n(i) \neq 0$ then $x_m(j) = 0$ for all $m > n$ and $j \leq i$, then the element y, defined as $y(n) = (x_1(n) + \ldots + x_N(n))/2$ for $n \geq N$ and 0 otherwise, belongs to K;

4) given $x \in K$, then there is a $k \in \mathbb{N}$ such that the element y, defined as $y(n) = 2x(n)$ for $n \geq k$ and 0 otherwise, belongs to K.

Then the space T^* is defined as the span of the absolutely convex closed hull of K in c_0 with the norm having K as the unit ball.

The first result we want to show is the following:

Proposition 3. $T^* \in W_p$ *for all* $p > 1$.

Current method of proof: Reference ([3], p.52) provides a sketch of a proof of this result that is quite difficult. In essence is as follows.

First of all, one needs a profound theorem of Krivine [6] asserting that if X satisfies a lower-p-estimate then, for all n, X contains n disjointly supported vectors equivalent to the canonical basis of l_p^n. Then one passes to T, and sees that disjointly supported vectors in T must be equivalent to the unit vector basis of l_1^n (see[3], Prop.V.8). One next shows that in T (and therefore in T^*) the canonical basis dominates, and is dominated by, its blocks (see[3], Chapter II).

The proof itself now follows: If, for some p, T does not admit a lower-p-estimate then the number $\inf\{q : T$ satisfies lower-q-estimates$\}$ is greater than 1, and thus T admits disjointly supported vectors equivalent to the unit vector basis of l_1^n for all n uniformly, which is a contradiction. So we have that T admits lower-p-estimates for all p. It is a standard duality argument that in that case T^* admits upper-p-estimates for all p. Finally, given a weakly null sequence in T^*, if it is norm null then there is nothing to prove; if not, we apply the Bessaga-Pelczynski selection principle to obtain a basic sub-sequence equivalent to certain blocks of the unit vector basis of T^*. These blocks satisfy an upper-p-estimate for all p since they are dominated by the basis, and thus a sub-sequence of our original sequence is weakly-p-summable. □

Simpler proof: Now our approach: It is easy to see that T^* has, for all p, the p-Banach-Saks property. This follows from the inequality given by Tsirelson ([9], p.140):

$$||\lambda_1 x_{N+1}, \ldots, \lambda_N x_{2N}|| \leq 2 \cdot \max_{1 \leq i \leq N} |\lambda_i|$$

which implies that

$$||x_1 + \ldots + x_N|| \leq K \cdot \log N$$

is valid for normalized blocks (x_i) of the canonical basis. This and the use of the Bessaga-Pelczynski selection principle in the form indicated above prove our assertion.

Proposition 2 then implies that T^* is of the class W_p for all $p > 1$. □

The next result was the motive for the construction of Tsirelson's space:

Proposition 4. *No subspace or quotient of* T *or* T^* *is isomorphic to* l_p *for* $1 \leq p < +\infty$, *or* c_0.

Proof. Pitt's theorem states that all operators of $\mathcal{L}(l_p, l_q)$ are compact for $q > p$. Therefore $l_p \in C_r$ for $r < p^*$ (in fact the two statements are equivalent), and thus l_p cannot be a subspace or quotient of a space in W_p for all $p > 1$. In conclusion, our assertion is true for T^*. A duality argument gives the same result for T.

A different approach to proving the assertion for T is to recognize that $X \in W_p$ implies $X^* \in C_r$ for $r < p^*$ (see [2] for details). Since $T^* \in W_p$ for all $p > 1, T \in C_r$ for all r, and thus l_p cannot be a subspace or quotient of such a space.

This last paragraph can also be expressed in the following proposition.

Proposition 5. *For any $p > 1, \mathscr{L}(L_p,T) = \mathbf{K}(L_p,T)$. For any p there is an operator $l_p \to T^*$ which is not compact.*

Proof. L_p spaces belong to some W_r, and $T \in C_p$ for all p. On the other hand, T^* does not belong to C_r for any $r > 1$. One can easily see that a Banach space $X \in C_p$ if and only if all operators of $\mathscr{L}(l_{p^*}, X)$ are compact.

Remark. Straeuli [8] has proved that any Banach-Saks operator with values in T is compact. This proves the first assertion of Proposition 5.

Proposition 6. *T and T^* admit no non-trivial type infinite-dimensional subspaces or quotients, and in particular, no super-reflexive subspaces or quotients.*

Proof. Consider first T^*. Recall that spaces of cotype s belong to C_r for all $r < s^*$. This shows that no finite cotype subspaces or quotients of T^* are allowed. The assertion for non-trivial type and super-reflexive spaces follows from this (but is also a consequence of the fact that super-reflexive spaces do not belong to some of the classes W_q).

The assertions for T are established by duality.

Remark. The special feature of T^*, that it belongs to W_p for all $p > 1$, determines all of its properties (and those of its dual), in particular that of not containing l_p. The latter not determine the former, however. Let us recall here the existence of a 2-convexfied Tsirelson space T_2 (see [3]): this space is uniformly convex (and thus super-reflexive) and does not contain any copy of l_p for $1 \leq p < +\infty$, or c_0. Since T_2 and T_2^* are super-reflexive, neither of them can belong to every W_p.

Neverthless, it is possible to clarify somewhat the structure of T_2 in terms of C_p and W_p. Since T_2 is of type 2 and cotype $q > 2$, it must belong to W_2 and to C_r for all $r < 2$. This is the best that can be archieved for r. Since T_2^* is of type $p < 2$ for all p and of cotype 2, it must belong to C_r for all $r < 2$ and to W_r for all $r > 2$. The question is whether T_2^* belongs to W_2 or to C_2.

Assume that it does not belong to C_2. Then the criterion, developed in [2], for detecting copies of l_p comes into play:

$$\textit{If } X \notin C_{p^*} \textit{ and } X^* \in W_p \textit{ then } X \textit{ contains a copy of } l_{p^*}.$$

Thus T_2^* should contain l_2, a contradiction.

If we recall that its dual space T_2 has, with respect to C_p and W_p, exactly the same behaviour as l_2, that is, $T_2 \in C_p$ for $p < 2$ and $T_2 \in W_2$, then the behaviour of T_2^* may be somewhat surprising: $T_2^* \in C_2$ and $T_2^* \in W_p$ for $p > 2$.

The reason however is precisely that T_2 cannot contain l_2.

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