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SOME PROPERTIES OF COMPLETION CLASSES FOR NORMED SPACES

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Summary. *Some properties of completion classes for normed spaces are investigated, giving a topological result similar to a theorem of V.L. Klee [13], Anderson [3], Bessaga [4]. An example of a metric space that lacks the fixed point property for the class of maps having the uniqueness property [1], is also given. In § 2 normed space completion is found by compact contractions.*

0. INTRODUCTION

The study of relationship between the fixed point property and the compactness of a topological or a metric space goes back to the fifties and the work of V. Klee [13] and E.H. Connell [6]. For example Klee gave a partial converse of Tychonoff's fixed point theorem proving that in a metrizable locally convex vector space the fixed point property for continuous functions fails for non-compact convex subsets. On the other hand the completeness of a metric space has recently been studied by various authors and different characterizations were given (see S. Park [14] and S. Park-B.E. Rhoades [15] for a comparison of these) both linking it to a variational character (see e.g. J.D. Weston [20] and particularly F. Sullivan [18] which establishes the equivalence between the completeness and I. Ekeland's variational principle [9]) and linking it also to the resolvability of fixed point equations or also to the existence of periodic points of particular functions from a space into itself (see T.K. Hu [10], P.V. Subrahmanyam [17], W.A. Kirk [11], P. Amato [1]). In particular in [1], after defining the appropriate pseudometrics called Hausdorff semidistances, a method as been established which reduces fixed point problems (or, if necessary, other type of equations) to completion problems (or, more generally, of extension) and viceversa. Moreover, this method allows the construction of the completion by means of appropriate classes of functions from a space into itself, continuous or not continuous, called thus classes of completion. Some of the properties of these have been successively studied in [2]; moreover in [16] C. Sempi has transferred the method of completion introduced in [1] to the setting of probabilistic metric spaces.

In this paper the study of the case of normed spaces is continued. After introducing the appropriate references in § 1 and establishing a proposition of general character on the classes of completion, in § 2 adapting an argument of Bessaga [4], see also Dugundji-Granas [8], it is shown that compact contractions form a completion class for normed spaces. Let us note (see ex. 3) that also classes which are generally more extensive than those of the contractions (e.g. the class of the functions verifying the uniqueness condition, see § 1) do not necessarily complete every metric space. This suggests a possible classification of metric spaces

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Work performed with the contributions of M.P.I. and of MURST 60%.

depending on their completion classes. In § 3, after noting (see ex. 4) that dense subsets can behave differently depending on the class of completion, the idea of negligible sets with respect to a class of completion Ψ is introduced. In theorem 3.3 it is proved that the compact sets of an infinite-dimensional normed space are negligible compared to the class of compact contractions, in analogy with the behaviour with the cancelling homeomorphism investigated in [13] theor. 2 (see also R.D. Anderson [3] and C. Bessaga [4]).

1. PRELIMINARY RESULTS AND DEFINITIONS

For the reader's convenience, attention is drawn to some definitions and notations introduced in [1].

Let (E, d) be a metric space. We will denote by $\sum(E)$ the set of all ordinate pairs (Y, Ψ), where Y is a subset of E and Ψ is a set of functions from Y into itself; sometimes we will write $\Psi(Y)$ instead of (Y, Ψ). A pair (Y, Ψ) will be called *a completion class of* E if the quotient space of $\Psi(Y)$ with respect to the equivalence relation

$$f \cong g \Leftrightarrow \sigma(f, g) = 0, \tag{1.1}$$

is a completion of E. Also we will say that $\Psi(Y)$ *completes* E. Recall that σ is the pseudometric introduced in [1], i.e.

$$\begin{aligned} \sigma(f, g) = \max\{ &\sup_{(x_n)\in S_f} \inf_{(y_n)\in S_g} \overline{\lim_{n\to\infty}} d(x_n, y_n), \\ &\sup_{(x_n)\in S_g} \inf_{(y_n)\in S_f} \overline{\lim_{n\to\infty}} d(x_n, y_n)\} \end{aligned} \tag{1.2}$$

S_f being the set of all sequences in $D(f)$, the domain of f, such that

$$\lim_{n\to\infty} d(x_n, f(x_n)) = |f|, \tag{1.3}$$

where $|f|$ is the minimal displacement of f, i.e.

$$|f| = \inf_{x\in D(f)} d(x, f(x)). \tag{1.4}$$

Moreover, we will denote by $U(Y)$ the set of all functions $f : Y \to Y$ which verify the following properties:

$$|f| = 0 \tag{1.5}$$

(1.6) f satisfy *the uniqueness condition* introduced in [1], i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$(d(x, f(x)) \leq \delta,\ d(y, f(y)) \leq \delta) \Rightarrow d(x, y) \leq \varepsilon$$

for any $x, y \in Y$.

Now, by $\overline{LU}(E)$ we denote the set of all $f \in U(Y)$ where Y is a closed subset of E. Moreover we will denote by $C(E)$ (respectively by $C_o(E)$) *the class of all contractions* (respectively *the class of all compact contractions*) of E into itself.

Finally ξ_a is the constant function of value a, $[f]$ is the equivalence class of f in $\overline{LU}(E)$ and $[f]_\Psi$ the equivalence class in Ψ.

In the sequel the following result, will be useful.

1.1 Proposition. *Let $\Psi(Y) \in \sum$, with $\Psi \subseteq \overline{LU}(E)$. Let us suppose that any $f \in \Psi$ is uniformly continuous and that*

$$[\xi_a] \cap \Psi \neq 0 \qquad \forall a \in E. \tag{1.5}$$

Then the following assertions are equivalents

(1.6) *Ψ is a completion class of E,*

(1.7) *For every Cauchy sequence (a_n) in E, there exists $f \in \Psi$ such that*

$$\lim_{n\to\infty} d(a_n, f(a_n)) = 0.$$

Proof. First of all, we note that for any $a \in E$ and for any $f_a \in \Psi \cap [\xi_a]$ we have

$$[f_a]_\Psi = [\xi_a] \cap \Psi \tag{1.8}$$

Moreover the mapping

$$J_\Psi : E \to \overline{\Psi} \text{such that} J_\Psi(a) = [f_a]_\Psi$$

is an isometry of E in $\overline{\Psi}$. Here $\overline{\Psi}$ is the quotient metric space of (Ψ, σ) with respect to the equivalence relation (1.1) and $J_\Psi(E)$ is a dense subspace of $\overline{\Psi}$.

$\underline{(a) \Rightarrow (b)}$. If (a_n) is a Cauchy sequence of E, then $J_\Psi(a_n) = [f_{a_n}]_\Psi$ is a Cauchy sequence in $\overline{\Psi}$. Since $\overline{\Psi}$ is complete, this sequence converges to an $[f]_\Psi \in \overline{\Psi}$. Therefore, for a fixed $f_n \in J_\Psi(a_n)$, because

$$\overline{\sigma}([f_n]_\Psi, [f]_\Psi) = \sigma(f_n, f)$$

we have $\sigma(f_n, f) \to 0$. Let us consider a sequence $(z_k) \in S_f$. Since $\sigma(f, f_n) \to 0$, as follows from (1.2) there exists $v \in \mathbb{N}$ such that

$$\inf_{(x_k)\in S_{f_n}} \overline{\lim_{k\to\infty}} d(x_k, z_k) < \varepsilon$$

for any $n \geq \nu$. Therefore if $n \geq \nu$ there is an element (x_k^n) of S_{f_n} such that

$$\overline{\lim_{k\to\infty}} d(x_k^n, z_k) < \varepsilon.$$

On the other hand $f_n \in [\xi_{a_n}]$ and hence $d(x_k^n, a_n) \to 0$, when $k \to \infty$. Then, there exists $\nu' \in \mathbb{N}$ such that

$$d(z_k, a_n) < 2\varepsilon \qquad \forall n, k \geq \nu'.$$

In particular $d(z_n, a_n) < 2\varepsilon$ for any $n \geq \nu'$, and consequently $d(a_n, z_n) \to 0$. The uniform continuity of f implies the assertion.

$\underline{(b) \Rightarrow (a)}$ It is necessary to prove that $(\overline{\Psi}, \overline{\sigma})$ is complete. In order to do this, considering a Cauchy sequence $([f_n])$ of elements of $\overline{\Psi}$, $J_\Psi(E)$ being dense in $\overline{\Psi}$ it is possible, for every $n \in \mathbb{N}$, to fix $[f_{a_n}] \in J_\Psi(E)$ with $f_{a_n} \in [\xi_{a_n}]$ such that

$$\overline{\sigma}([f_n], [f_a]) \leq 2^{-n} \qquad \forall n \in \mathbb{N}$$

and it is obviously possible to replace f_{a_n} by ξ_{a_n}. Since J_Ψ is an isometry, it follows that (a_n) is a Cauchy sequence in E, and then, by (b), there is a function $f \in \Psi$ such that $(a_n) \in S_f$. This implies $[\xi_{a_n}]$ converges to $[f]$, and thus, since $[\xi_{a_n}] = [f_{a_n}]$, the assertion is proved.

2. NORMED SPACES

Given E a normed space, we are going to prove that $C_o(E)$, the class of compact contractions of E, is a completion class of E, i.e. the quotient space of $(C_o(E), \sigma)$ with respect to the equivalence relation (1.1) is a completion of E. This result has already been stated in [1].

2.1. *Theorem. Compact contractions are a completion class for normed spaces.*

Proof. Let (x_n) be a Cauchy sequence of E which does not converge (the other case is trivial). The function

$$c(x) = \lim_n ||x - x_n|| \qquad x \in E \tag{2.1}$$

is such that

$$(2.2) \qquad \|c(x) - c(y)\| \leq \|x - y\| \qquad \forall x, y \in E$$

First of all let us note that it is possible to extract from (x_n) a subsequence (x_{n_k}), with $x_{n_o} := 0$, such that

$$(2.3) \qquad \|x_{n_{k+1}} - x_{n_k}\| \leq \frac{1}{2^{k+2}} \text{ and } c(x_{n_k}) < 1 \; \forall k \in \mathbb{N}$$

so that the series $\|x_{n_1}\| + \ldots + \|x_{n_{k+1}} - x_{n_K}\| + \ldots$ converges to a positive number $a < \frac{1}{2}$.
Assuming now for $k, n \in \mathbb{N}$

$$z_k = \frac{1}{2a} x_{n_k}, S_o = 0 \text{ and } S_{n+1} = 2 \sum_{k=0}^{n} \|z_{k+1} - z_k\|$$

so that $\lim_{n \to \infty} S_n = 1$, we have

$$[o, 1[= \bigcup_{n=0}^{\infty} [S_n, S_{n+1}[.$$

Every $x \in [0, 1[$ can be written in the form $x = \alpha S_k + (1 - \alpha) S_{k+1}$ for a suitable $k \in \{0, 1, \ldots\}$ and a suitable $\alpha \in [0, 1]$. Now we define $\gamma : [0, 1[\to E$ via

$$\gamma(x) = \alpha x_{n_k} + (1 - \alpha) x_{n_{k+1}}$$

Let us note that $\gamma([0, 1[)$ is the union of the segment joining the point x_{n_k} with $x_{n_{k+1}}, k \in \mathbb{N}$, and also that

$$(2.4) \qquad \|\gamma(x) - \gamma(y)\| \leq a|x - y| \qquad \forall x, y \in [0, 1[.$$

In fact since $S_{k+1} - S_k = 2\|z_{k+1} - z_k\| \; k \in \mathbb{N}$, if

$$x = \alpha 2 S_k + (1 - \alpha) 2 S_{k+1} \text{ and } y = \beta 2 S_k + (1 - \beta) 2 S_{k+1}$$

for $k \in \mathbb{N}$ and $\alpha, \beta \in [0, 1[$, then

$$(2.5) \qquad \|\gamma(x) - \gamma(y)\| = \|(\alpha - \beta) x_{n_k} + (\beta - \alpha) x_{n_{k+1}}\|$$
$$= a(S_{k-1} - S_k)|\alpha - \beta| = a|x - y|,$$

while if

$$S_k \leq x \leq S_{k+1} \leq y \leq S_{k+2},$$

because of (2.5), it follows that

$$||\gamma(x) - \gamma(y)|| \leq a|x - 2S_{k+1}| + a|2S_{k+1} - y| = a|x - y|$$

and by this, reasoning by induction, we verify (2.4).

If we denote by ψ the extension of γ to the interval $]-\infty, 1[$ obtained by assuming $\psi(x) = 0$ for negative values of x, then obviously (2.4) extends to the functions ψ, and then

$$(2.6) \qquad ||\psi(x) - \psi(y)|| \leq a|x - y| \qquad \forall x, y \in]-\infty, 1[.$$

Now the application $f : E \to E$ defined by

$$(2.7) \qquad f(x) = \psi(1 - c(x)) \qquad \forall x \in E,$$

is a contraction of E into itself. In fact, for $x, y \in E$ such that $c(x) > 1$ and $c(y) > 1$, it follows that

$$||f(x) - f(y)|| = 0 \leq ad(x, y)$$

while if $c(x) \leq 1$ and $1 \leq c(y)$ we have

$$\begin{aligned} ||f(x) - f(y)|| &= a||\psi(1 - c(x)) - \psi(0)|| = a(1 - c(x)) \\ &\leq a[(1 - c(x)) - (1 - c(y))] \leq a||x - y|| \end{aligned}$$

for (2.1). Finally, for $c(x) < 1$ and $c(y) < 1$ again for (2.1) and (2.5), we have

$$\begin{aligned} ||f(x) - f(y)|| &= a||\gamma(1 - c(x)) - \gamma(1 - c(y))|| \\ &= a|c(x) - c(y)| \leq a||x - y||. \end{aligned}$$

Therefore

$$||f(x) - f(y)|| \leq a||x - y|| \qquad \forall x, y \in E,$$

meaning that f is a contraction of E. It is still to be proved, in virtue of the theorem 1.1 that

$$(2.8) \qquad \lim_p ||x_p - f(x_p)|| = 0$$

Let us, first of all, observe that, assuming that $f(x_{n_p})$ belongs to the segment of end points $x_{n_{k_p}}$ and $x_{n_{k_{p+1}}}$,

$$\begin{aligned} ||x_{n_p} - f(x_{n_p})|| &\leq ||x_{n_p} - x_{n_{k_p}}|| + ||x_{n_{k_p}} - f(x_{n_p})|| \\ &\leq ||x_{n_p} - x_{n_{k_p}}|| + ||x_{n_{k_p}} - x_{n_{k_{p+1}}}|| < \varepsilon \end{aligned}$$

for p large enough, (x_{n_p}) being extracted from the Cauchy sequence (x_n). Thus it follows

$$\lim_{p \to \infty} ||x_{n_p} - f(x_{n_p})|| = 0. \tag{2.9}$$

On the other hand, f being a contraction, we have

$$\begin{aligned} &||x_n - f(x_n)|| \leq ||x_n - x_{n_p}|| + ||x_{n_p} - f(x_{n_p})|| \\ &+||f(x_{n_p}) - f(x_n)|| \leq (1 + a)||x_{n_p} - x_n|| \\ &+||x_{n_p} - f(x_{n_p})|| \end{aligned}$$

from which (2.8) follows, remembering (2.9) and that (x_n) is a Cauchy sequence. The theorem is thus proved.

In his paper [5], J.M. Borwein characterized with the contraction property the completeness of the uniformly Lipchitz-connected metric spaces and in particular therefore, of the normed spaces: this property consists in the fact that every contraction of the space into itself has a fixed point. The proof of the theorem 2.1, which is an adaption of C. Bessaga's technique (see [4]), for the construction of the cancelling homeomorfism in non-complete normed spaces (and it can be extended to uniformly Lipschitz-connected metric spaces) is preferable for the construction of the completion of the space with the method introduced in [1], being intrinsic to the non-complete setting of the space. J.M. Borwein also proved that there are non uniformly Lipschitz-connected metric spaces which have the contraction property but which are not complete. Thus it is not possible to complete all metric spaces with contractions: we will see from the following example 2.2 that this occurs also with the class of the functions that satisfy the uniqueness condition, which is generally noticeably greater than that of contractions. We also note that the completion by means of local contractions (thus defined in closed subspaces and at values in the subspaces themselves) is certainly possible for every metric space as we can see from the interesting result of T.K. Hu [10]. The same is valid for the diametral-contractions, as follows from the work of M.R. Taskovic [19].

2.2. Example. Let us consider, in the plane provided with the Euclidean metric the set (cf [5], Ex. 4)

$$C = \bigcup_{k\in\mathbb{N}} L_k \tag{2.10}$$

L_k being the closed segment having as end points the origin O and the point $A_k(1,2^{-k})$, supplied by induced metric.

We prove that *The metric space* $(U(C),\sigma)$ *is not a completion of* C. Let (P_n) a non-convergent Cauchy sequence of elements of C and let f be a continuous function of E into itself verifying the uniqueness condition and such that

$$\lim_n d(P_n, f(P_n)) = 0. \tag{2.11}$$

Obviously, (P_n) cannot be eventually included in any of the L_k segments (otherwise it would converge to an element of such segment). On the other hand, because (P_n) does not converge to O, there exists a neighbourhood of O which does not contain a subsequence of (P_n); since (P_n) is a Cauchy sequence, it follows that there exists a positive number δ such that $\delta \leq x_n \forall n \in \mathbb{N}$, x_n being the abscissa of the point P_n and therefore for every $n \in \mathbb{N}\, \delta \leq d(P_n, 0)$. Now $f(0) \neq 0$ because $f \in U(C)$. Supposing $f(0) \in L_h$, and setting

$$\sum\nolimits_+ = \{X \in L_h | d(d(X), A_h) \leq d(X, A_h)\}$$

$$\sum\nolimits_- = \{X \in L_h | d(d(X), A_h) \geq d(X, A_h)\}$$

we observe that these are non empty, closed subsets of L_h because $0 \in \sum_+$ and $A_H \in \sum_-$, and the function $\Phi : L_h \to \mathbb{R}$ defined by

$$\Phi(X) = d(f(X), A_h) - d(X, A_h)$$

is continuous. As L_h is connected, it follows that $\sum_+ \cap \sum_- \neq \phi$. Therefore f has a fixed point $P(x,y)$ in L_h and necessarily $0 < y$, as $P \neq 0$. We now observe that because the Cauchy sequence is not definitively contained in any of the segment L_k it follows that

$$\lim_n d(P_n, P) \geq y,$$

and, on the other hand, from the (2.11) for any positive r, and for n large enough one has

$$P_n \in S_{f,r} \text{ and } P \in S_{f,r}$$

so it follows that

$$\delta(S_{f,r}) \geq y,$$

against the hypothesis that $f \in U(C)$. Then the hypothesis (2.11) would be absurd, therefore $U(C)$ does not complete C.

3. NEGLIGIBLE SETS

The following example emphasizes how not all dense subsets of a metric space behave in the same way with respect to the classes of completion.

3.1. Example. If C is the space of Ex. 2.2, defined by (2.10), we consider the subspace $C^* = C - \{0\}$. Then $U(C^*)$ *completes* C^*. In fact, if $P_n = (x_n, 2^{-k_n}x_n) \in L_{k_n}$ is Cauchy sequence of elements of C^* that does not converge in C^*, the possible cases are:

(a) $x_n \to 0$, that is (P_n) converges at the origin;

(b) $\exists a > 0, \exists h \in \mathbb{N}, \exists \nu \in \mathbb{N}, \forall\, n \geq \nu : x_n \geq a$ and $P_n \in L_h$;

(c) neither (a) nor (b).

Case(a). One can consider the function $f : C^* \to C^*$ such that $f(P) = P'(x/2, y/2) \in C^*$ for any $P(x,, y) \in C^*$.

This is a contraction of C^* and satisfies the required property.

Case(b). In this case the sequence (P_n) obviously converges to a point $Q \in L_h$, therefore one can consider the constants contraction $f(P) = Q, P \in C^*$.

Case(c). Let us observe first that

$$\forall k \in \mathbb{N}, \exists n_k \in \mathbb{N}, \forall n \geq n_k : P_n \in C^* - \bigcup_{h \leq n_k} L_h$$

otherwise (P_n) would converge to an element of L_h for a suitable $h \leq n_k$ and therefore we would be still in the situation found in case (b). This implies that $2^{-k_n}x_n \to 0$ and as x_n cannot converge to zero, it follow that the sequence k_n diverges, so we can extract a strictly increasing subsequence q_n. Now let us consider $f : C^* \to C^*$ defined setting for every $n \in \mathbb{N}$

$$f(P) = p_{q_n} \qquad \forall P \in L_n.$$

We note that f is continuous, but not uniformly continuous because points which are very near on different segments are transformed into P_{q_n} at finite distance (therefore f is not a contraction). On the other hand it is clear that $d(P_n, f(P_n)) \to 0$, leaving us still to prove that $f \in U(C^*)$. Now for every positive r we have

$$S_{f,r} = \bigcup_{n \in \mathbb{N}} (\overline{B}(P_{q_n}, r) \cap L_n)$$

where, with $\overline{B}(x, r)$ we denote the closed ball with the centre at x and radius r. Since the sequence does not tend to 0, there exists $\gamma > 0$ such that for every $n \in \mathbb{N}$ $x_n \geq \gamma$, and so considering the point $P_\gamma^{q_n}(\gamma, 2^{-q_n}\gamma)$ of the segment L_{q_n}, to which P_{q_n} belongs,

$$\begin{aligned} d(P_{q_n}, L_n) \geq d(P_\gamma^{q_n}, L_n) &= \gamma|1 - 2^{h-k_{q_n}}|(1 + 2^{2h})^{-\frac{1}{2}} \\ &= c_{n,h}^\gamma > 0. \end{aligned}$$

We note that $c^{\gamma}_{n,h}$ is the distance of the segments L^{γ}_n and L^{γ}_h, L^{γ}_k being the part of L_k made up of points having abscissa $x \geq \gamma$. In particular for $h = n$, setting $c^{\gamma}_n = c^{\gamma}_{n,h}$, we have

$$C^{\gamma}_n \leq d(P_{q_n}, L_n) \qquad \forall n \in \mathbb{N}$$

Therefore if $r > 0$ is taken small enough

$$S_{f,r} \subseteq \bigcup_{c^{\gamma}_n \leq r} (\overline{B}(P_{q_n}, r) \cap L_n).$$

It follows that $\delta(S_{f,r}) \to 0$ when $r \to 0$, also considering that

$$\delta(\overline{B}(P_{q_n}, r) \cap L_n) \leq 2r \qquad \forall n \in \mathbb{N}.$$

Therefore $U(C^*)$ completes C^*.

3.2 Definition. *Let $\Psi(Y)$ be a completion class of (E, d) and K a subset of Y. K is said to be Ψ-negligible if $\Psi(Y - K)$ is a completion of (E, d).*

In the following theorem it appears that compact subsets of an infinite-dimensional normed space, with respect to the completion through compact contractions, have a role similar to that of the «negligible» sets introduced by C. Bessaga in [4], in order to generalize a result established by V. Klee [13], R.D. Anderson [3].

3.3 Theorem. *If K is a compact set of an infinite-dimensional normed space E, then $C_o(E - K)$ is a completion class of E.*

Proof. According to the Mazur theorem, $co(K)$, the convex hull of K, is compact and E being infinite-dimensional, $co(K)$ has an empty interior (see Riesz's Theorem). Then if we take a Cauchy sequence (x_n) of elements of K, we can find a Cauchy sequence (y_n) of elements of $E - co(K)$ such that $d(x_n, y_n) \to 0$. In fact for every n the ball $B(x_n, 1/n)$ contains points of $E - co(K)$; therefore we can consider $y_n \in (E - co(K))$ such that $d(x_n, y_n) \leq (1/n)$. Having fixed such a y_n and denoted by α Kuratowski's measure of non-compactness (see K. Deimling [7]) it follows that

$$\begin{aligned}\alpha(co(\{y_n\} \cup co(K))) &= \alpha(\{y_n\} \cup co(K)) \\ &= \max\{\alpha(\{y_n\}), \alpha(K)\} = 0,\end{aligned}$$

that is the cone K_n which projects $co(K)$ from y_n is also itself compact. Therefore $B(x_{n+1}, 1/n) - K_n \neq \emptyset$ and taking one of its elements y_{n+1} it follows that the segment of the end points y_n and y_{n+1} does not intersect K because $K \subseteq K_n$. Clearly, in this way we construct a Cauchy sequence (y_n) of elements of $E - K$ such that

1) $d(x_n, y_n) \to 0$
2) $co(\{y_n, y_{n+1}\}) \subseteq E - K \qquad \forall n \in \mathbb{N}$.

Proceeding as in the proof on the theorem 2.2, we can construct a contraction $f : E - K \to E - K$ such that $d(y_n, f(y_n)) \to 0$, as claimed.

3.4 Corollary. *Compact sets of an infinite-dimensional normed space are C_o-negligibles.*

3.5 Corollary. *Finite-dimensional flats of any infinite-dimensional normed space are C_o-negligibles.*

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