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GEODESICS OF ASSOCIATIVE FUNCTIONS

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Abstract. *In this paper we will consider associative functions in* $\mathbb{R}^+$ *and we will characterize some of the most important of them using the determination of the geodesics of this kind of functions.*

A common problem shared by all surface theories lies in the characterization of a certain kind of them through properties they hold or by the determination of some interesting geometrical parameters.

The case of associative functions has been previously considered by Schweizer and Sklar [3]. Some of the most interesting contributions are those of Alsina [1] and Alsina and Sklar [2]. In the first one, developable associative surfaces are studied. In the second one, continuous and ruled associative surfaces are characterized.

A wide study on the geometrical properties determined by continuous and associative functions has been reported in [5]. The coefficients corresponding to the first and second fondamental form and the curvature of some sections of differentiable associative functions were also obtained and applications of those calculations reported.

In this work associative functions verifying some differential conditions are studied. The cases where some sections are the geodesics of the considered functions are also considered. This give rise to new characterizations of some of the most relevant associative operations.

Proposition 1. *Let* $T(x,y) = t^{-1}(t(x) + t(y))$ *be an associative function in* $\mathbb{R}^+$ *where* t *is a strictly increasing, twice differentiable function having a non-zero derivative holding* $t(0) = 0$. *Then* $T(x,y) = x + y$ *for all* x, y *in* $\mathbb{R}^+$ *if and only if all the sections* $T(x, k - x)$ *are geodesics of* T.

Proof. To compute the geodesics of T it is useful to consider T as $\vec{x}(u,v) = (g(u), g(v), g(u+v))$ with $g = t^{-1}$. Then, for any curve $v = f(u)$, $\vec{x}(u) = (g(u), g(f(u)), g(u + f(u)))$ and if $\vec{t} = \dfrac{\vec{x}'}{|\vec{x}'|}$, it can be shown that

$$\begin{aligned}\mathcal{K} = \frac{\vec{t}'}{|\vec{x}'|} = \frac{1}{2A(u)^2}&(2g''(u)A(u) - g'(u)A'(u),\\ &2(g''(f(u))f'(u)^2 + g'(f(u))f''(u))A(u) - g'(f(u))f'(u)A'(u),\\ &2(g''(u+f(u))(1+f'(u))^2 + g'(u+f(u))f''(u))A(u) - g'(u+f(u))\\ &(1+f'(u))A'(u))\end{aligned}$$

and

$$U = N \times \tilde{t} = \frac{1}{\sqrt{A(u)B(u)}}(-g'(u)g'(u+f(u))^2(1+f'(u)) - g'(u)g'(f(u))^2 f'(u),$$
$$g'(u)^2 g'(f(u)) + g'(f(u))g'(u+f(u))^2(1+f'(u)),$$
$$-g'(f(u))^2 f'(u)g'(u+f(u)) + g'(u)^2 g'(u+f(u)),$$

where

$$A(u) = g'(u)^2 + g'(f(u))^2 f'(u)^2 + g'(u+f(u))^2(1+f'(u))^2,$$
$$B(u) = g'(f(u))^2 g'(u+f(u))^2 + g'(u)^2 g'(u+f(u))^2 + g'(u)^2 g'(f(u))^2,$$

and

$$N = (\vec{x}_u \wedge \vec{x}_v)/\sqrt{B(u)}$$

By definition geodesic curves have a zero geodesic curvature i.e. $\mathcal{K} \cdot U = 0$. In the present case simple manipulations of the equations lead to

$$g'(u)g''(u)[g'(u+f(u))^2(1+f'(u)) + g'(f(u))^2 f'(u)] =$$
$$g'(f(u))[g'(u)^2 + g'(u+f(u))^2(1+f'(u))] + [g''(f(u))f'(u)^2 + g'(f(u))f''(u)]$$
$$+g'(u+f(u))[g'(u)^2 - g'(f(u))^2 f'(u)][g''(u+f(u))(1+f'(u))^2$$
$$+g'(u+f(u))f''(u)].$$

If the sections $T(x, k-x)$ are geodesics of T, then

$$f(u) = g^{-1}(k - g(u))$$

verifies the above equation which upon substitution and simplification permits to write

$$g'(u+f(u))\left(\frac{g''(u)}{g'(u)} + \frac{g'(u)^2 g''(f(u))}{g'(f(u))^3}\right) = g''(u+f(u))\left(1 - \frac{g'(u)}{g'(f(u))}\right)^2$$

After regrouping some terms we may write the equation in the form

$$(\ln(g'(u+f(u))(1+f'(u))))' = (\ln\ g'(u))'$$

so, a constant a such that

$$g'(u+f(u))(1+f'(u)) = ag'(u)$$

does exist.

This last equation is equivalent to

$$(g(u+f(u)))' = (ag(u))',$$

leading to the existence at a constant b such that

$$g(u+f(u)) = ag(u)+b$$

If k is allowed to vary, we obtain two functions $a(k)$ and $b(k)$ such that, for all k, u with $k \geq g(u)$

$$g(u+f(u)) = a(k)g(u)+b(k)$$

and

$$f(u) = g^{-1}(a(k)g(u)+b(k)) - u,$$

and then

$$g^{-1}(k-g(u)) = g^{-1}(a(k)g(u)+b(k)) - u.$$

If $u = 0$ then $b = Id$ and for $k = g(u)$ it is shown that $a = 0$, or

$$g^{-1}(k-g(u)) = g^{-1}(k) - u.$$

If $x = g(u)$, then for all x, k with $k \geq x, g^{-1}(k) - g^{-1}(x) = g^{-1}(k-x)$, whence $g^{-1} = t$ has the form $g^{-1}(x) = cx$ for some constant c and $T(x,y) = x+y$ for all x, y in $\mathbb{R}^+$.

The reciprocal is proven trivially.

Using similar techniques and analogous proofs proposition 2 may be suggested. In this proposition some important associative operations on the real line are characterized from the fact that some sections are geodesics.

Proposition 2. *Let $T(x,y) = t^{-1}(t(x)+t(y))$ be an associative function in $\mathbb{R}^+$, where t is a bijective, twice differentiable function having a non-zero derivative with $t(0) = 0$.*
(1) T has the form $T(x,y) = x+y+axy$ for a given constant a if and only if the sections $T(k,y)$ with k constant (or $T(x,k)$ with k constant) are geodesics of T.
(2) $T(x,y) = x+y$ for all x, y in $\mathbb{R}^+$ if and only if the sections $T(x, t^{-1}(k-t(x)))$ with k constant, are geodesics of T.
(3) T has the form $T(x,y) = x+y+axy$ for a given constant a if and only if the sections $T(x, t^{-1}(t(x)+k))$ with k constant, are geodesics of T.

(4) If for a fixed $k, t'(x+k) \neq t'(x)$ *for all* x *in* $\mathbb{R}^+$ *and the section* $T(x, x+k)$ *is a geodesic of* T, *then* T *is the sum in* $y = x + k$.

Note 1. The point (3) of the above proposition may be generalized if the restriction $t(0) = 0$ is lifted. In this case one gets that two constants $\alpha, \beta \neq 0$ exist such that $T(x,y) = \alpha + \frac{1}{\beta}(x-\alpha)(x-\beta)$ if and only if the sections $T(x, t^{-1}(t(x)+k))$ are geodesics of T.

Note 2. If instead of using $t(0) = 0$, the restrictions $t(1) = 0$ and $t(0) = +\infty$ are applied, T is the product if and only if exist $a, b > 0$ with $\frac{\ln a}{\ln b}$ irrational and such that the sections $T\left(x, \frac{1}{a}\right)$ and $T\left(x, \frac{1}{b}\right)$ are geodesics of T.

Special attention should be devoted to the case where the diagonals of the unit square or a given level curve are geodesics.

Proposition 3. *Let* $f, g, h : [0, +\infty) \to [0, +\infty)$ *be bijective, twice differentiable functions and with non-zero derivative.*
(1) Let $T(x,y) = f(g(x) + g(y))$. *The section* $T(x,x)$ *is always a geodesic of* T.
(2) Let $T(x,y) = f^{-1}(f(x) + f(y))$. *The section* $T(x, 1-x)$ *is a geodesic of* T *if and only if* $T(x, 1-x)$ *is a straight line.*
(3) Let $T(x,y) = f(g(x) + h(y))$. *The section* $T(x, t^{-1}(1 - t(x)))$ *is a geodesic of* T *if and only if two constants* a, b *such that* $g^{-1}(x) = ah^{-1}(1-x) + b$ *exist.*

Note 3. Similar results are found if the associative operations are defined on the unit interval or, more general, in closed and bounded intervals of the real line.

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