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QUASINORMABLE SPACES OF HOLOMORPHIC FUNCTIONS

SEÁN DINEEN

Quasinormable spaces were introduced by Grothendieck [34] as a collection of spaces, with good stability properties, which included all Banach spaces and all nuclear spaces (a recent article of Meise-Vogt [44] quantifies this statement in an elegant fashion for Fréchet spaces). Quasinormable spaces in which the bounded sets are precompact are called Schwartz spaces and form a more restrictive but even more stable class of spaces (they are closed under the formation of arbitrary products and quotients).

In this article we study these properties on space of holomorphic tùnctions with the three standard topologies of infinite dimensiona1 holomorphy, τ_0, τ_w and τ_δ. In the presence of one or more of a variety of countability conditions on the underlying locally convex space we found that the spaces of holomorphic functions tumed out to be either quasinormable or Schwartz spaces. The compact open topology, τ_0, and nuclear spaces both enjoy intrinsic compactess properties and in these cases (§ 1 and 2) we found it possible to study directly the Schwartz property. For the τ_0 topology our countability conditions were-each compact set is contained in the absolutely convex hull of a null sequence, sequential completeness, and each null sequence is Mackey null. We only considered the τ_w and τ_δ topologies on nuclear spaces as the result for the compact open topology is well known and we assumed the existence of a basis and a property which can be compared on the one hand to the defining property of A-nuclear spaces and on the other hand to the sequence space characterization of Schwartz spaces. To obtain general results for the τ_w and τ_δ topologies we used countable neighbourhood systems (in Fréchet spaces) and countable systems of bounded sets (in $\mathcal{DF}$ spaces) together with S-absolute Schauder decompositions of holomorphic function spaces on balanced domains.

In all cases we found new results which both simplified and contained known results and which suggest further possibilities towards the realization of a more unified theory of locally convex space structures on spaces of holomorphic functions on infinite dimensiona1 domains.

1. BASIC DEFINITIONS AND THE SCHWARTZ PROPERTY FOR THE COMPACT OPEN TOPOLOGY

We let E denote a locally convex space over the complex numbers C and let $cs(E)$ denote the set of all continuous seminorms on E. We refer to Grothendieck [34, 35] and Horvath [37] for properties of locally convex spaces and to Dineen [23] for holomorphic functions on locally convex spaces.

If U is an open subset of a locally convex space E we let H(U) denote the space of

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all C-valued holomorphic functions on U and let $\mathcal{H}_{HY}(U)$ denote the set of all C-valued Gâteaux holomorphic functions on U which are continuous on the compact subsets of U. The compact open topology on $\mathcal{H}_{HY}(U)$ and $\mathcal{H}(U)$ is denoted by τ_0. A seminorm p on $\mathcal{H}(U)$ is called τ_w continuous if there exists a compact subset K of U such that for every open subset V of U containing K there exists $c(V) > 0$ such that

$$p(f) \leq c(V)||f||_V$$

for every f in $\mathcal{H}(U)$. In this case we say that p is ported by the compact set K. The τ_w topology is the topology generated by all τ_w continuous seminorms on $\mathcal{H}(U)$. A seminorm p on $\mathcal{H}(U)$ is τ_δ continuous if for any increasing countable open cover of U, $(V_n)_{n=1}^\infty$, there exists a positive integer n_0 and $C > 0$ such that

$$p(f) \leq C||f||_{V_{n_0}}$$

for every f in $\mathcal{H}(U)$.

The τ_δ topology is the topology generated by all τ_δ continuous seminorms. We always have $\tau_0 \leq \tau_w \leq \tau_\delta$. For any positive integer n we let $\mathcal{P}(^nE)$ denote the space of (continuous) C-valued n-homogeneous polynomials on E. On $\mathcal{P}(^nE)$ the topologies τ_w and τ_δ coincide. If U is a balanced open subset of E and $f \in \mathcal{H}(U)$ we let $\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}$ denote the Taylor series expansion of f at 0. We have $\frac{\hat{d}^n f(0)}{n!} \in \mathcal{P}(^nE)$ for all n. We let $S = \{(\alpha_n)_{n=1}^\infty ; \alpha_n \in \mathbb{C} \text{ and } \limsup_{n\to\infty} |\alpha_n|^{1/n} \leq 1\}$.

Definition 1.1. *([23, p. 114]) A Schauder decomposition $\{E_n\}_n$ of a locally convex space E is called an S-absolute decomposition if the following conditiom are satisfied;*

(1.1) *for any* $\sum_{n=1}^{\infty} x_n \in E, x_n \in E_n$ *all* n, *and* $(\alpha_n)_n \in S$ *the series* $\sum_{n=1}^{\infty} \alpha_n x_n \in E$,

if $p \in cs(E)$ *and* $(\alpha_n)_n \in S$ *then the seminorm*

(1.2)
$$\tilde{p}\left(\sum_{n=1}^{\infty} x_n\right) := \sum_{n=1}^{\infty} |\alpha_n| p(x_n)$$

$x_n \in E_n$ *all n, belongs to* $cs(E)$.

Example 1.2. ([23, chapter 3]) If U is a baianced open subset of a locally convex space E then $\{(\mathcal{P}(^nE), \tau)\}_{n=0}^\infty$ is an S-absolute decomposition for $(\mathcal{H}(U), \tau)$ where $\tau = \tau_0, \tau_w$ or τ_δ. The expansion used to obtain this decomposition is the Taylor series expansion at the origin.

Example 1.3. A subset B of a domain U in a locally convex space E is said to lie strictly inside U if there exists a neighbourhood V of zero such that $B + V \subset U$. If U is an open subset of a locally convex space we let $\mathcal{H}_b(U)$ denote the subspace of $\mathcal{H}(U)$ consisting of those functions which are bounded on the bounded subsets of E which lie strictly inside U. We let β denote the topology on $\mathcal{H}_b(U)$ of uniform convergence on these sets. If U is a balanced domain in a locally convex space E then $\{(\mathcal{P}(^n E), \beta)\}_{n=0}^{\infty}$ is easily seen, by the methods employed to obtain the result given in example 1.2, to be an S-absolute decomposition for $(\mathcal{H}_b(U), \beta)$. For further information regarding this space we referto [2, 32, 33, 38, 39].

To simplify notation when considering holomorphic functions on balanced domains we shall, unless there is some possibility of confusion, write $f = \sum_{n=0}^{\infty} P_n$ as the Taylor series expansion at the origin where $P_n = \frac{\hat{d}^n f(0)}{n!} \in \mathcal{P}(^n E)$ and, we may often suppose we are dealing with seminorms on the function space $\mathcal{F}$ which satisfy

$$p\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} p(P_n)$$

forany $\sum_{n=0}^{\infty} P_n$ in $\mathcal{F}$.

We now define quasinormable spaces.

Definition *1.4. ([34,35,37]) A locally convex space E is quasinormable if for every neighbourhood U of 0 in E there exists a neighbourhood V of 0 in E such that for all $\lambda > 0$ there exists a bounded subset M_λ of E with*

$$V \subset M_\lambda + \lambda U \tag{1.3}$$

Clearly it suffices to consider U and V from a fundamental neighbourhood basis system and in terms of seminorms we may rephrase (1.3) as follows.

E, a locally convex space, is quasinormable if for every $p \in cs(E)$ there exists $q \in cs(E)$ such that for any $\lambda > 0$ there exists a bounded subset M_λ of E with

$$\{x \in E; q(x) \leq 1\} \subset M_\lambda + \{x \in E; p(x) \leq \lambda\} \tag{1.4}$$

When a seminorm q satisfies (1.4) with respect to the seminorm p then we shall say that q is *associated* with p. If p_1 and p_2 are equivalent seminorms on E then $q \in cs(E)$ is associated with p_1 if and only if it is associated with p_2.

Definition *1.5. A locally convex space E is a Schwartz space if and only if it is quasinormable and its bounded sets are precompact.*

The following proposition gives two further characterizations of Schwartz spaces (the first is comparable with (1.3)).

Proposition *1.6. [34, 35, 37] A locally convex space E is a Schwartz space if and only if either of the following equivalent conditions are satisfied.*

(a) For every $p \in cs(E)$ there exists $q \in cs(E)$, $q \geq p$, such that for every $\lambda > 0$ there exists a precompact subset K_λ of E satisfying

$$\{x \in E; q(x) \leq 1\} \subset K_\lambda + \{x \in E, p(x) \leq \lambda\} \tag{1.5}$$

(b) For every $p \in cs(E)$ there exists $q \in cs(E)$, $q \geq p$, such that if $\{x_n\}_n \subset E$, $q(x_n) \leq 1$ all n, then $\{x_n\}_n$ contains a p-Cauchy subsequence.

$\mathcal{DF}$ spaces (and in particular Banach spaces) are quasinormable and a Fréchet-Montel space is quasinormable if and only if it is a Fréchet-Schwartz space.

For the next theorem we need the following two conditions on a locally convex space *E*.

(1.6). *Every compact subset of E is contained in the closed absolutely convex hull of a null sequence.*

(1.7). *Every null sequence $\{x_n\}_n$ is Mackey null, i.e. there exists a sequence of positive real numbers $(\lambda_n)_n$ such that $\lambda_n \to +\infty$ and $\lambda_n x_n \to 0$ as $n \to \infty$.*

A locally convex space which satisfies (1.7) is said to satisfy the *Mackey condition.* If for every absolutely convex bounded subset *A* of a locally convex space *E* there exists another absolutely convex bounded subset *B* of *E* such that E_B, the vector subspace of *E* spanned by *B* and normed with the Minkowski functional of *B,* induces its original topology on *A* then we say that *E* satisfies the *strict Mackey condition.* If *E* satisfies the strict Mackey condition then *E* satisfies the Mackey condition. An infrabarrelled locally convex space is quasinormable if and only if its strong dual satisfies the strict Mackey condition ([34, p. 106]). It is well known that Fréchet spaces satisfy both (1.6) and (1.7).

An inductive limit $(E, \tau) = \varinjlim_{\alpha \in A} (E_\alpha, \tau_\alpha)$ is said to be *compactly regular* if for each compact subset *K* of *E* there is an α in *A* such that *K* is contained and compact in (E_α, τ_α). Compactly regular inductive limits of spaces satisfying (1.6) and (1.7) also satisfy both conditions. In particular strict and compact inductive limits of Fréchet spaces satisfy (1.6) and (1.7). From this and some known results it is easy to see that $\mathcal{DFM}$ spaces satisfy (1.6) and (1.7) if and only if they are $\mathcal{DFS}$ spaces. For further details and examples we refer to Bierstedt [7] and Floret [30].

If E is a Banach space and $P \in \mathcal{P}(^nE)$ we let $||P|| = \sup\{|P(x)|; ||x|| \leq 1)$.

Lemma 1.7. *Let $P \in \mathcal{P}(^nE)$, E a Banach space. If $||y|| \leq 1$ and $||x - y|| \leq 1$ then*

$$|P(x) - P(y)| \leq n(2e)^n ||P|| ||x - y||$$

Proof. Let A denote the unique symmetric n-linear form associated with P. It is well known that $||P|| \leq ||A|| := \sup_{||x_i||<1} |A(x_1, \ldots, x_n)| \leq e^n ||P||$. We have

$$\begin{aligned}
|P(x) - P(y)| &= |P(y + x - y) - P(y)| \\
&= \left| \sum_{j=1}^{n} \binom{n}{j} A(x - y)^j (y)^{n-j} \right| \\
&\leq ||A|| . \sum_{j=1}^{n} \binom{n}{j} ||x - y||^j ||y||^{n-j} \\
&\leq e^n ||P|| ((||y|| + ||x - y||)^n - ||y||^n) \\
&\leq e^n ||P|| ||x - y|| n \sup_{1 \leq j \leq n} (||x - y|| + ||y||)^j \\
&= n(2e)^n ||P|| ||x - y||.
\end{aligned}$$

The following theorem extends results of Nelimarkka [54, corollaries 3 and 4]. We let $\Gamma(A)$ and $\overline{\Gamma}(A)$ denote respectively the balanced convex hull and the closed absolutely convex hull of the subset A of the locally convex space E.

Theorem 1.8. *Let E denote a sequentially complete locally convex space satisfying (1.6) and (1.7). If U is an open subset of E then $(\mathcal{H}(U), \tau_0)$ is a Schwartz space.*

Proof. Since the compact open topology is a local topology and products and subspaces of Schwartz spaces are Schwartz spaces it suffices, as in [54, proposition 1], to consider the case where U is a convex balanced open subset of E.

Let K denote an arbitrary compact subset of U and let $\{x_n\}_n$ denote a null sequence in E such that $K \subset \overline{\Gamma}(\{x_n\}_n) \subset U$. By (1.7), we can choose a sequence of positive real numbers $(\lambda_n)_n$ such that $\lambda_n \to \infty$ and $\lambda_n x_n \to 0$ as $n \to \infty$. Since E is sequentially complete we may suppose without loss of generality that $\overline{\Gamma}(\{\lambda_n x_n\}_n)$ is also a compact subset of U. Choose $\alpha > 1$ such that $\tilde{K} := \alpha\overline{\Gamma}(\{\lambda_n x_n\}_n)$ is again a compact subset of U. For f in $\mathcal{H}(U)$ we let

$$|||f|||_{\tilde{K}} = \sum_{n=0}^{\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_{\tilde{K}}$$

If $f \in H(U)$ and $|||f|||_{\tilde{K}} \leq 1$ then for each positive integer m

$$\left\|\frac{\hat{d}^m f(0)}{m!}\right\|_{\tilde{K}} = \sup_{x\in\tilde{K}}\left|\frac{\hat{d}^m f(0)}{m!}(x)\right| \leq 1.$$

By restricting $\frac{\hat{d}^m f(0)}{m!}$ to the subspace of E spanned by $\tilde{K}$, $E_{\tilde{K}}$, we obtain an element of $\mathcal{P}({}^m E_{\tilde{K}})$ when $E_{\tilde{K}}$ is endowed with the norm whose unit ball is $\tilde{K}$. If $x \in \overline{\Gamma}(\{x_n\}_n)$, $y \in \tilde{K}$ and $0 \leq \delta \leq 1$ then lemma 1.7 implies

$$(1.8)\qquad \left|\frac{\hat{d}^m f(0)}{m!}(x+\delta y) - \frac{\hat{d}^m f(0)}{m!}(x)\right| \leq m(2e)^m\delta.$$

Let $(f_n)_n$ denote a sequence in H(U) satisfying $|||f_n|||_{\tilde{K}} \leq 1$ for all n.

By taking subsequences and using a diagonal process, if necessary, we may suppose that $\left(\frac{\hat{d}^m f_n(0)}{m!}\right)_n$ converges uniformly as $n \to \infty$ on the compact subsets of $sp\{x_1, \ldots, x_l\}$ for all m and l. To complete the proof, it suffices, by proposition 1.6(b) to show that

$$\|f_n - f_m\|_K \to 0 \quad \text{as } n, m \to \infty$$

Let $\epsilon > 0$ be arbitrary. For all n we have

$$\sum_{m=0}^{\infty} \alpha^m \left\|\frac{\hat{d}^m f_n(0)}{m!}\right\|_K \leq |||f_n|||_{\tilde{K}} \leq 1.$$

Hence, we can choose m_0 such that

$$(1.9)\qquad \sum_{m=m_0}^{\infty}\left\|\frac{\hat{d}^m f_n(0)}{m!}\right\|_K \leq \sum_{m=m_0}^{\infty}\frac{1}{\alpha^m} \leq \epsilon$$

for all n.

Let $\epsilon' = \frac{\epsilon}{\sum_{m=1}^{m_0-1} m(2e)^m}$. Now choose 1 such that $\lambda_n \geq \frac{1}{\epsilon'}$ for all $n > 1$. If $n > l$ then $\frac{x_n}{\epsilon'} \in \overline{\Gamma}(\{\lambda_n x_n\}_n)$ and $x_n \in \epsilon'\tilde{K}$.

If $n \leq 1$ then $x_n \in \overline{\Gamma}(\{x_n\}_{n=1}^{l})$. Hence

$$K \subset \overline{\Gamma}(\{x_n\}_n \subset \overline{\Gamma}(\{x_n\}_{n=1}^{l}) + \epsilon'\tilde{K} \tag{1.10}$$

Hence, we can choose n_0 such that

$$\left\| \frac{\hat{d}^m f_n(0)}{m!} - \frac{\hat{d}^m f_k(0)}{m!} \right\|_{\tilde{K}\cap sp\{x_1,\dots,x_l\}} \leq \frac{\epsilon}{m_0} \tag{1.11}$$

for $m = 0, 1, \dots, m_0 - 1$ and all $n, k \geq n_0$.

Let $n, k \geq n_0$, and $x \in K$ be arbitrary. By (1.10) we have $x = y + \epsilon' z$ where $y \in \overline{\Gamma}(\{x_n\}_{n=1}^{l})$ and $z \in \tilde{K}$. Then

$$|f_n(x) - f_k(x)| \leq \sum_{m=0}^{\infty} \left| \frac{\hat{d}^m f_n(0)}{m!}(x) - \frac{\hat{d}^m f_k(0)}{m!}(x) \right|$$

$$\leq \sum_{m=0}^{m_0-1} \left| \frac{\hat{d}^m f_n(0)}{m!}(y + \epsilon' z) - \frac{\hat{d}^m f_n(0)}{m!}(y) \right|$$

(by 1.9)

$$+ \sum_{m=0}^{m_0-1} \left| \frac{\hat{d}^m f_n(0)}{m!}(y) - \frac{\hat{d}^m f_k(0)}{m!}(y) \right|$$

$$+ \sum_{m=0}^{m_0-1} \left| \frac{\hat{d}^m f_k(0)}{m!}(y) - \frac{\hat{d}^m f_k(0)}{m!}(y + \epsilon' z) \right| + 2\epsilon$$

((1.8) and (1.11))

$$\leq 2 \sum_{m=1}^{m_0-1} m(2e)^m \epsilon' + \sum_{m=0}^{m_0-1} \frac{\epsilon}{m_0} + 2\epsilon$$

$$\leq 2\epsilon + \epsilon + 2\epsilon = 5\epsilon.$$

Hence $\|f_n - f_k\|_K \leq 5\epsilon$ for $n, k \geq n_0$ and this completes the proof.

2. NUCLEAR SPACES

In this section we consider entire functions on certain fully nuclear spaces with basis. If U is an open subset of a dual nuclear space then a result of Boland [10] and Waelbroeck [56] says that $(\mathcal{H}(U), \tau_0)$ is nuclear and hence Schwartz. For this reason we only consider the τ_ω and τ_δ topologies in this section. A locally convex space which is reflexive, nuclear and dual nuclear is called fully nuclear ([11], [23, chapter 5]). If E is a fully nuclear space with basis then $E \approx \Lambda(P)$ where P is a set of weights. The Grothendieck-Pietsch criterion for nuclearity then says that for all $(\alpha_n)_n \in P$, there exists $(\beta_n)_n \in P$, and $(u_n)_n \in l_1$ such that

$$\alpha_n \leq |u_n|\beta_n \quad \text{for all } n. \tag{2.1}$$

A sequence space $E \approx \Lambda(P)$ is called Λ-nuclear ([24]) if there exists $(\delta_n)_n$, $\delta_n > 1$ and $\sum_n \frac{1}{\delta_n} < \infty$ such that

$$(\alpha_n)_n \in P \quad \text{implies } (\alpha_n \delta_n)_n \in P. \tag{2.2}$$

Comparing (2.1) and (2.2) we see that Λ-nuclear spaces are nuclear and satisfy a «uniform» Grothendieck-Pietsch criterion. By [24], $(\mathcal{H}(E), \tau_w)$ and $(\mathcal{H}(E), \tau_\delta)$ are nuclear spaces when E is a reflexive Λ-nuclear space (reflexive Λ-nuclear space are fully nuclear). Hence we obtain the Schwartz property in these two cases.

To obtain new results we define a class of spaces satisfying a property intermediate between (2.1) and (2.2) and which may also be compared with the following characterization of Fréchet-Schwartz spaces; a Fréchet space $E \approx \Lambda(P)$ is a Schwartz space if and only if for all $(\alpha_n)_n \in P$ there exists $(\beta_n)_n \in P$ and $(\gamma_n)_n \in c_0$ such that

$$\alpha_n \leq |\gamma_n|\beta_n \quad \text{for all } n. \tag{2.3}$$

Definition 2.1. *A C-nuclear space E is a fully nuclear space with basis $\Lambda(P)$ which satisfies the following condition; there exists $\delta = (\delta_n)_n$, $\delta_n > 1$ and $\left(\frac{1}{\delta_n}\right)_n \in c_0$ such that $(\alpha_n)_n \in P$ implies $(\delta_n \alpha_n)_n \in P$.*

Our methods are based on monomial expansion and we now briefly recall some definitions and results in this direction. If $m = (m_n)_{n=1}^{\infty} \in N^{(N)}$ (the set of sequences of non-negative integers which are eventually zero) and $(z_n)_n \in \Lambda(P)$ we let $z^m = \prod_{n=1}^{\infty} z_n^{m_n}$. We also denote by z^m the mapping

$$(z_m)_n \in \Lambda(P) \rightarrow z^m.$$

Since z^m is a product of continuous coefficient functionals it is a homogeneous polynomial of degree $|m| = \sum_{n=1}^{\infty} m_n$. If $\Lambda(P)$ is a fully nuclear space with basis then $\{z^m\}_{m \in N^{(N)}}$ forms an absolute basis for $(\mathcal{H}(\Lambda(P)), \tau_w)$ and an unconditional equicontinuous basis for $(\mathcal{H}(\Lambda(P)), \tau_\delta)$. If $f \in \mathcal{H}(\Lambda(P))$ we write

$$f(z) = \sum_{m \in N^{(N)}} a_m z^m$$

and call this the monomial expansion of f. The coefficients $a_m, m \in N^{(N)}$ are given by th Cauchy integral formula over finite dimensiona1 polydiscs.

Proposition 2.2. *Let $E = \wedge(P)$ denote a C-nuclear space with $\delta = (\delta_n)_n$ dejìning C nuclearity. If*

$$f(z) = \sum_{m\in N^{(N)}} a_m z^m \in \mathcal{H}(E)$$

then

$$g(z) = \sum_{m\in N^{(N)}} a_m \delta^m z^m \in H(E)$$

If p is a τ_w continuous seminorm on $\mathcal{H}(E)$ then $\tilde{p}$ defined by

$$\tilde{p}\left(\sum_{m\in N^{(N)}} a_m z^m\right) := \sum_{m\in N^{(N)}} |a_m|\delta^m p(z^m)$$

is a τ_w continuous seminorm on H(E) .

Proof. The mapping (z,), $\in$ A(P) $\rightarrow (\delta_n z_n)_n \in \wedge(P)$ is a linear topologica1 isomorphisn Hence its transpose

$$f \in \mathcal{H}(E) \rightarrow f \circ \delta \in \mathcal{H}(E)$$

maps $\mathcal{H}(E)$ onto X(E) .

If

$$f(z) = \sum_{m\in N^{(N)}} a_m z^m \in H(E)$$

then

$$f \ 0 \ \delta(z) = \sum_{m\in N^{(N)}} a_m(\delta z)^m = \sum_{m\in N^{(N)}} a_m \delta^m z^m = g(z)$$

belongs to li(E) .

Since the monomials form an absolute basis for H(E) the seminorm

$$p_1\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sum_{m\in N^{(N)}} |a_m| p(z^m)$$

is τ_w continuous. Since the linear topologica1 isomorphism mentioned above maps compac sets onto compact sets and neighbourhood systems (of compact sets) onto neighbourhoo systems (of compact sets) it follows that the mapping $f \rightarrow$ p, (f o 6) defines a τ_w continuou seminorm on 'H(E) . Since

$$p_1(f\circ\delta) = \sum_{m\in N^{(N)}} |a_m| p((\delta z)^m) = \sum_{m\in N^{(N)}} |a_m|\delta^m p(z^m) = \tilde{p}(f)$$

this completes the proof.

Theorem 2.3. *If E is a C-nuclear space then $(\mathcal{H}(E), \tau_w)$ und $(\mathcal{H}(E), \tau_\delta)$ are Schwartz spaces*

Proof. We first consider the τ_w case. Let p denote a τ_w continuous seminorm with closed unit ball U. We may suppose without loss of generality that

$$p\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sum_{m\in N^{(N)}} |a_m| p(z^m).$$

Let

$$q\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sum_{m\in N^{(N)}} |a_m| \delta^m p(z^m)$$

and let

$$V = \{f \in \mathcal{H}(E); q(f) \leq 1\}.$$

If $\lambda > 0$ we can choose J finite in $N^{(N)}$ such that $\frac{1}{\delta^m} < \lambda$ if $m \notin J$.

Let

$$B_\lambda = \left\{ \sum_{\substack{m\in J\, and\\ p(z^m)\neq 0}} a_m z^m ; q\left(\sum_{\substack{m\in J\, and\\ p(z^m)\neq 0}} a_m z^m\right) \leq 1 \right\}.$$

Since $q \geq p$ it follows that B_λ is a compact subset of $(\mathcal{H}(E), \tau_w)$.

If $f = \sum_{m\in N^{(N)}} a_m z^m \in H(E)$ and $q(f) \leq 1$ then

$$f = \sum_{\substack{m\in J\, and\\ p(z^m)\neq 0}} a_m z^m + \sum_{\substack{m\notin J\, or\\ p(z^m)=0}} a_m z^m.$$

We have $q\left(\sum_{\substack{m\in J\, or\\ p(z^m)=0}} a_m z^m\right) \leq 1$ and

$$p\left(\sum_{\substack{m\notin J\, or\\ p(z^m)=0}} a_m z^m\right) = \sum_{m\notin J} |a_m| p(z^m) = \sum_{m\notin J} \frac{1}{\delta^m} |a_m| \delta^m p(z^m)$$

$$\leq \lambda \sum_{m\notin J} |a_m| \delta^m p(z^m) \leq \lambda.$$

Hence

$$\{f \in \mathcal{H}(E); q(f) \leq 1\} \subset B_\lambda + \{f \in \mathcal{H}(E); p(f) \leq \lambda\}$$

and proposition 1.6(a) implies that $(\mathcal{H}(E), \tau_w)$ is a Schwartz space.

We now consider the τ_δ topology. Let U denote the closed unit ball of the τ_δ continuous seminorm p which we may suppose has the form

$$p\left(\sum_{m\in N^{(N)}} a_m z^m\right) = \sup_{\substack{\lambda_m \in \mathbf{C}, |\lambda_m|\leq 1 \\ J\subset N^{(N)} \\ |J|<\infty}} p\left(\sum_{m\in J} \lambda_m a_m z^m\right)$$

This implies, in particular, that

$$p\left(\sum_{m\in N^{(N)}} b_m z^m\right) \leq p\left(\sum_{m\in N^{(N)}} a_m z^m\right)$$

if $|b_m| \leq |a_m|$ for all m in $N^{(N)}$.

Since the mapping

$$(z_n)_n \in E \to (\delta_n z_n)_n$$

maps increasing countable open covers of E onto increasing countable open covers of E it follows that

$$q\left(\sum_{m\in N^{(N)}} a_m z^m\right) := p\left(\sum_{m\in N^{(N)}} a_m \delta^m z^m\right)$$

also defines a τ_δ continuous seminorm on $\mathcal{H}(E)$. With the same choice of B_λ and q as in the τ_w case the proof is completed by noting that, if $q\left(\sum_{m\in N^{(N)}} a_m z^m\right) \leq 1$ then

$$p\left(\sum_{\substack{m\notin J \text{ or} \\ p(z^m)=0}} a_m z^m\right) = p\left(\sum_{m\notin J} \frac{1}{\delta^m} a_m \delta^m z^m\right) \leq \lambda p\left(\sum_{m\notin J} a_m \delta^m z^m\right)$$

$$\leq \lambda p\left(\sum_{m\in N^{(N)}} a_m \delta^m z^m\right) = \lambda q\left(\sum_{m\in N^{(N)}} a_m z^m\right) \leq \lambda.$$

3. FRÉCHET SPACES

To prove quasinormability results on Fréchet spaces we use the following abstract results. A minor variation will also be used in the next section.

Theorem 3.1. *Let $\{E_n\}_{n=1}^{\infty}$ denote an S-absolute decomposition of the locally convex space E. If*

(3.1) $\qquad E_n$ *is quasinormable for all n,*

and

(3.2) $\qquad$ *for any sequence $(p_n)_n$, $p_n \in cs(E_n)$, there exists a sequence*

of positive real numbers $(\alpha_n)_n$ such that $p := \sum_{n=1}^{\infty} \alpha_n p_n \in cs(E)$

then E is quasinormable.

Proof. Let $p = \sum_{n=1}^{\infty} p_n \in cs(E)$. For each positive integer n there exists, by (3.1), a $q_n \in cs(E_n)$ associated with p_n. By (3.2) there exists a sequence of positive real numbers $(\alpha_n)_n$ such that $q := \sum_{n=1}^{\infty} \alpha_n q_n \in cs(E)$.

Let

$$\tilde{q}\left(\sum_{n=1}^{\infty} x_n\right) = \sum_{n=1}^{\infty} \alpha_n q_n(x_n) + \sum_{n=1}^{\infty} n^2 p_n(x_n)$$

for $\sum_{n=1}^{\infty} x_n \in E$. Since $\{E_n\}_{n=1}^{\infty}$ is an S-absolute decomposition it follows that $\tilde{q} \in cs(E)$. We now show that $\tilde{q}$ is associated with p. Let $\lambda > 0$ be arbitrary. Choose n_0, a positive integer, such that $\frac{1}{n_0^2} \leq \frac{\lambda}{2}$. If $x = \sum_{n=1}^{\infty} x_n \in E$ and $\tilde{q}(x) \leq 1$ then

$$n_0^2 \sum_{n=n_0}^{\infty} p(x_n) \leq \sum_{n=n_0}^{\infty} n^2 p(x_n) \leq \tilde{q}\left(\sum_{n=n_0}^{\infty} x_n\right) \leq 1$$

and

$$p\left(\sum_{n=n_0}^{\infty} x_n\right) \leq \frac{1}{n_0^2} \leq \frac{\lambda}{2}.$$

By (1.4) there exists, for each n a bounded subset M_n of E_n such that

$$\{x_n \in E_n : q_n(x_n) \leq 1\} \subset M_n + \left\{x_n \in E_n : p_n(x_n) \leq \frac{\alpha_n \lambda}{2^{n+1}}\right\}.$$

Let $M = \sum_{n=1}^{n_0-1} \frac{M_n}{\alpha_n}$. Then M is a bounded subset of E. Moreover, if $x_n \in E_n$ and, $\tilde{q}\left(\sum_{n=1}^{n_0-1} x_n\right) \leq 1$, then $q_n(x_n) \leq \frac{1}{\alpha_n}$ and $x_n \in \frac{M_n}{\alpha_n} + \left\{y_n \in E_n; p_n(y_n) \leq \frac{\lambda}{2^{n+1}}\right\}$.
Hence

$$\sum_{n=1}^{n_0-1} x_n \in \sum_{n=1}^{n_0-1} \frac{M_n}{\alpha_n} + \left\{\sum_{n=0}^{n_0-1} y_n \in E; y_n \in E_n \text{ and} \sum_{n=1}^{n_0-1} p_n(y_n) \leq \frac{\lambda}{2} \cdot \sum_{n=1}^{n_0-1} \frac{1}{2^n}\right\}$$
$$\subset M + \left\{y \in E; p(y) \leq \frac{\lambda}{2}\right\}.$$

Hence $\{x \in E; \tilde{q}(x) \leq 1\} \subset M + \{x \in E; p(x) \leq \lambda\}$ and E is quasinormable. This completes the proof.

Remarks 3.2. (a) If E is quasinormable then (3.1) is satisfied.

(b) Condition (3.2) is similar to the countable neighbourhood property (c.n.p.) introduced by Floret [30]. A locally convex space E has c.n.p. if for every sequence $(p_n)_n, p_n \in cs(E)$, there exists a sequence of positive real numbers $(\alpha_n)_n$ and $p \in cs(E)$ such that $\alpha_n p_n \leq p$ for all n. If E has c.n.p. then it satisfies (3.2) for any S-absolute decomposition. Futther properties of locally convex spaces with c.n.p. may be found in Bonet [13] and Dierolf [20] and for applications to infinite dimensiona1 holomorphy we refer to Colombeau-Mujica [17] and [23, corollary 2.301.

(c) Let τ_1 and τ_2 be two locally convex topologies on E and suppose $\{(E_n, \tau_i)\}_{n=1}^{\infty}$ is an S-absolute decomposition for (E, τ_i), $i = 1,2$. If $\tau_1 \leq \tau_2$ and $\tau_1|_{E_n} = \tau_2|_{E_n}$ for all n then (E, τ_2) satisfies (3.2) if (E, τ_1) satisfies it. This is the case for the τ_w and τ_δ topologies on $\mathcal{H}(U)$, U a balanced domain in a locally convex space.

(d) If $(K_n)_n$ is a sequence of compact subsets of a Fréchet space E then there exists a sequence of positive real numbers $(\alpha_n)_n$ such that $\overline{\bigcup_n \alpha_n K_n}$ is a compact subset of E (see for instance [35, p. 156]). This fact and theorem 3.1 can be combined to reduce the proof of theorem 1.8 for Fréchet spaces to the homogeneous polynomial case.

The following theorem is due to Bierstedt-Meise [8, proposition 16] for $\tau = \tau_w$ and E a Fréchet-Schwartz space (see also Nelimarkka [53, corollary 4.3] for a proof using operator ideals) and to J.M. Isidro [40] fora Banach space and $\tau = \tau_w$ or τ_δ.

Theorem 3.3. *If U is a balanced open subset of a Fréchet space then $(\mathcal{H}(U), \tau)$ is quasinormablefor $\tau = \tau_w$ or τ_δ.*

Proof. Since $(\mathcal{P}(^nE), \tau_w) = (\mathcal{P}(^nE), \tau_\delta)$ is a countable inductive limit of Banach spaces it is quasinormable [35, p. 177] and condition (3.1) is satisfied by the S-absolute decomposition $\{(\mathcal{P}(^nE), \tau_w)\}_{n=0}^{\infty}$ of $(\mathcal{H}(U), \tau), \tau = \tau_w$ or τ_δ. By remark 3.2(c) it suffices to complete the proof for $\tau = \tau_w$.

Let $(V_n)_{n=1}^{\infty}$ denote a decreasing fundamental neighbourhood system at the origin in E. Let $p_n \in cs((\mathcal{P}(^nE), \tau_w))$ for each non-negative integer n. For any pair of positive integers n and m there exists $c_n(V_m) > 0$ such that

$$p_n(P_n) \leq c_n(V_m)\|P_n\|_{V_m}$$

forall $P_n \in \mathcal{P}(^nE)$.

Without loss of generality we may suppose that the sequence $\{c_n(V_m)\}_{m=1}^{\infty}$ is an increasing sequence for each n. Let $\alpha_n = \dfrac{1}{c_n(V_n)}$ for all n. Then

$$\alpha_n p_n(P_n) \leq \frac{c_n(V_m)}{c_n(V_n)}\|P_n\|_{V_m} \leq \|P_n\|_{V_m}$$

for all $n \geq m$ and all $P_n \in \mathcal{P}(^nE)$.

Let $p = \sum_{n=0}^{\infty} \alpha_n p_n$ and suppose $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U)$. If m is any positive integer then

$$p\left(\sum_{n=0}^{\infty} P_n\right) \leq \sum_{n=0}^{m-1} \frac{c_n(V_m)}{c_n(V_n)}\|P_n\|_{V_m} + \sum_{n=m}^{\infty}\|P_n\|_{V_m}$$
$$\leq \sup_{1 \leq n \leq m}\left(\frac{c_n(V_m)}{c_n(V_n)}\right)\sum_{n=0}^{\infty}\|P_n\|_{V_m}.$$

A simple application of the Cauchy inequalities shows that p is ported by $\{0\}$ and in particular it is τ_w continuous and (3.2) is satisfied. This completes the proof.

To give a corollary to the above theorem we need the following lemma. This lemma could also be used to shorten the proof of theorem 1.8 and it also clear that a general theorem of the same kind is true for any S-absolute composition.

Lemma 3.4. *If U is a balanced open subset of a locally convex space E and $\tau = \tau_0, \tau_w$ or τ_δ then the following are equivaient*
(a) the bounded subsets of $(\mathcal{H}(U), \tau)$ are precompact,
(b) the bounded subsets of $(\mathcal{P}(^nE), \tau)$ are precompact for every non-negative integer n.

Proof. Since $(\mathcal{P}(^nE), \tau)$ is a closed complemented subspace of $(\mathcal{H}(U), \tau)$ it follows that $(a) \Rightarrow (b)$.

Now suppose (b) is true. Let B denote a bounded subset of $(\mathcal{H}(U), \tau)$ and let p denote a r-continuous seminorm on $\mathcal{H}(U)$. We may suppose without loss of generality that

$$p\left(\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) \text{ for all} \sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U).$$

Let $B_n = \left\{\frac{\hat{d}^n f(0)}{n!}; f \in B\right\}$. It is easily seen that B_n is a bounded subsets of $(\mathcal{P}(^nE), \tau)$ for all n.

Let $\epsilon > 0$ be arbitrary. For each n there exists a finite subset F_n of $\mathcal{P}(^nE)$ such that B_n c $F_n + \{P \in \mathcal{P}(^nE); p(P) \leq \frac{\epsilon}{2^n}\}$.

Let

$$\tilde{p}\left(\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} n^2 p\left(\frac{\hat{d}^n f(0)}{n!}\right) \text{ for all} \sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!} \text{ in } \mathcal{H}(U).$$

Since $\{($ 'P("E) , $\tau\}_{n=0}^{\infty}$ is an S-absolute decomposition for ('H(U) , τ) it follows that $\tilde{p}$ is r-continuous. Now choose n_0 , a non-negative integer, such that

$$\sum_{n=n_0}^{\infty}\frac{M}{n^2} < \epsilon \text{ where } M = \sup_{f\in B}\tilde{p}(f)$$

Let

$$F = \left\{\sum_{n=0}^{n_0-1} P_n; P_n \in F_n \text{ all } n\right\}.$$

Since each F_n is finite F is also finite. If $f \in B$ then p $\left(\frac{\hat{d}f(0)}{n!}\right) \leq \frac{M}{n^2}$ and hence

$$\sup_{f\in B}\left\{\sum_{n=n_0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right)\right\} \leq \sum_{n=n_0}^{\infty}\frac{M}{n^2} \leq \epsilon.$$

Hence

$$B \text{ c } F + \left\{f \in \mathcal{H}(U); p(f) \leq \sum_{n=0}^{n_0-1}\frac{\epsilon}{2^n} + \epsilon\right\} \subset F + \{f \in \mathcal{H}(U); p(f) \leq 3\epsilon\}$$

and this complete the proof.

Corollary *3.5. If U is a balanced open subset of a Fréchet space then the following are equivalent*

(a) $\tau_0 = \tau_w$ *on* $\mathcal{P}({}^nE)$ *for all n,*

(b) $(\mathcal{P}({}^nE), \tau_w)$ *is a* $\mathcal{DFM}$ *space for all n,*

(c) $\tau_0 = \tau_w$ *on* $H(U)$,

(d) $(\mathcal{H}(U), \tau_w)$ *is a Schwartz space,*

(e) $(\mathcal{H}(U), \tau_\delta)$ *is a reflexive Schwartz space.*

Proof. The equivalence of (a) and (c) is given in Ansemil-Ponte [4]. For any positive integer n, $(\mathcal{P}({}^nE), \tau_w)$ is an infrabarrelled $\mathcal{DF}$ space and $(\mathcal{P}({}^nE), \tau_0)$ is a semi-Montel space. Hence $(a) \Rightarrow (b)$. Since $(\mathcal{H}(V), \tau_\delta)$ is a complete barrelled space and $(\mathcal{H}(U), \tau_w)$ is complete, theorem 3.3 and lemma 3.4 show that (d) and (e) are satisfied if and only if $(\mathcal{P}({}^nE), \tau_w)$ is semi-Montel for all n. Hence (b), (d) and (e) are all equivalent. To complete the proof we show that $(b) \Rightarrow (a)$. Let $(B_m)_{m=1}^\infty$ denote a fundamental sequence of convex balanced bounded closed subsets of $(\mathcal{P}({}^nE), \tau_w)$, n a non-negative integer. If (b) is satisfied then each B_m is τ_w-compact and since $\tau_w \geq \tau_0$ it follows that $\tau_w|_{B_m} = \tau_0|_{B_m}$ for all m. Now $(\mathcal{P}({}^nE), \tau_0) = \varinjlim_m B_m$ in the category of topological spaces and continuous mappings [50]. Hence $\tau_0 \geq \tau_w$ on $\mathcal{P}({}^nE)$ and since we always have $\tau_w \geq \tau_0$ this implies $\tau_w = \tau_0$ on $\mathcal{P}({}^nE)$. Hence $(b) \Rightarrow (a)$ and this completes the proof.

Any Fréchet-Schwartz space satisfies the conditions of the above corollary [11, 42, 50, 51] and if the conditions are satisfied by a Fréchet space then this space must be Fréchet-Montel. For Fréchet-Montel spaces we have negative [5] and positive results [4, 26, 31]. A survey of results conceming the coincidence of τ_0 and τ_w on spaces of holomorphic functions is given in [1].

We now consider holomorphic functions of bounded type on a balanced domain in a Fréchet space (example 1.3).

Proposition *3.6. If U is a balanced open subset of a Fréchet space E then $(\mathcal{H}_b(U), \beta)$ is quasinormable if and only if $(\mathcal{P}({}^nE), \beta)$ is quasinormable for each integer n.*

Proof. By theorem 3.1 it suffices to show that (3.2) is satisfied by the S-absolute decomposition $\{(\mathcal{P}({}^nE), \beta)\}_{n=0}^\infty$ of $(\mathcal{H}_b(U), \beta)$. Let $p_n \in cs((\mathcal{P}({}^nE), \beta))$ for each positive integer n. For each n there exists a bounded subset B_n of E such that

$$p_n(P) \leq ||P||_{B_n} \qquad \text{for all } P \in \mathcal{P}({}^nE)$$

By [35, p. 156] there exists a sequence of positive real numbers $(\lambda_n)_n$ such that $B := \bigcup_n \lambda_n B_n$ is a bounded subset of E. On multiplying by a scalar if necessary we may suppose B is a strictly bounded subset of U.

Then

$$p\left(\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!}\right) := \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_B$$

defines a β continuous seminorm on $\mathcal{H}_b(U)$. Moreover,

$$\sum_{n=0}^{\infty}\lambda_n^n p_n\left(\frac{\hat{d}^n f(0)}{n!}\right) \leq \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_{\lambda_n B_n} \leq \sum_{n=0}^{\infty}\left\|\frac{\hat{d}^n f(0)}{n!}\right\|_B$$

for all $\sum_{n=0}^{\infty}\frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}_b(U)$ and hence $\sum_{n=0}^{\infty}\lambda_n^n p_n$ is a β-continuous seminorm on $\mathcal{H}_b(U)$.

Hence (3.2) is satisfied and the proof is complete.

Example 3.7. If E is a normed linear space then $(\mathcal{P}(^nE), \beta)$ is a Banach space and hence is quasinormable. Proposition 3.6 implies that $(\mathcal{H}_b(U), \beta)$ is quasinormable for any balanced open subset of E. This result is due to Ansemil-Ponte [2] for convex balanced domains and to Isidro [39] for balanced domains. If E is a distinguished Fréchet space with absolute basis and U is a balanced open subset of E then ($\mathcal{H}_b(U), \beta)$ is quasinormable. This result and further examples arose in a different context and will appear in [27].

4. $\mathcal{LB}$ AND $\mathcal{LF}$ SPACES

In this section we consider holomorphic functions on various countable inductive limits of Fréchet spaces ($\mathcal{LF}$ spaces) and Banach spaces (CZ3 spaces). The situation for compact inductive limits is relatively straightforward and well known. If E is a compact inductive limit of Banach spaces then E is a $\mathcal{DFM}$ space and $\tau_0 = \tau_w = \tau_\delta$ on $\mathcal{H}(U)$ for any open subset U of E. Moreover, $(\mathcal{H}(U), \tau_0)$ is a Fréchet-Montel space and $(\mathcal{H}(U), \tau_0)$ is quasinormable if and only of E is a $\mathcal{DFS}$ space ([22, 53])-this also follows from theorem 1.8 and the remarks before lemma 1.7.

Our general approach is to first «localize» the seminorms involved to the Fréchet and Banach spaces used to construct the inductive limits. We then apply the results of the previous section and finally use a Hahn-Banach type extension theorem for homogeneous polynomials to lift the results to the whole space. The extension method forced us to restrict ourselves to strict inductive limits-which may be considered the extreme opposite of the compact inductive limit-and this is fortunate in view of the above remarks. Recently Bonet-Peris [13] have given an example of a strict inductive limit for which the space of 2-homogeneous polynomials is not quasinormable. Their method was to show that certain tensor mappings were not monomorphisms. Because of the close connection between monomorphism and extension theorems we were able to combine our method with a modification of the Bonet-Peris method to fully characterize, with mild restrictions, the standard strict $\mathcal{LB}$ spaces of Moscatelli type on which the spaces of holomorphic functions are quasinormable. This gives simple concrete examples of quasinotmability and non-quasinormability.

We also obtain positive results for direct sums of Fréchet-Schwartz spaces with Schauder basis which admit continuous norms. The continuous norm condition is probably just a temporary convenience, although, we show by example that our approach does require this hypothesis. The basis requirement and the Schwartz with continuous norm hypothesis imply that the space of holomorphic functions contains a τ_w dense subspace of fimctions each of which is bounded on every neighbourhood of a basic neighbourhood system (this is always true for Banach spaces) and gives rise to an approximation problem which may be of independent interest.

We begin this section by stating without proof a strong version of theorem 3.1.

Theorem 4.1. *Let $\{E_n\}_n$ denote an S-absolute decomposition for the locally convex space E. Then E is quasinormable of and only if the following two conditions are satisfied.*

(4.1) *Each E_n is quasinormable*

If $\sum_{m=1}^{\infty} p_n \in cs(E)$, $p_n \in cs(E_n)$ all n, then there exists for each n a seminorm

(4.2) $q_n \in cs(E_n)$, *associated with p,, such that* $q := \sum_{n=1}^{\infty} q_n \in cs(E)$.

Comparing theorems 3.1 and 4.1 we note that conditions (3.1) and (3.2) may be checked separately while condition (4.2) only makes sense when conditions (4.1) is satisfied. Theorem 4.1 is more useful than 3.1 in the nontrivial $\mathcal{LB}$ case.

We now discuss the extension property. A continuous linear mapping $T : E \to F$ is called a *monomotphism* if $T : E \to T(E)$ is an isomorphism. If E is a locally convex space we let $\bigotimes_{n,\pi,s} E$ denote the vector space of *symmehic n-tensors* on E with the projective topology.

Lemma *4.2. If F is a closed subspace of a locally convex space E then the following are equivalent for each positive integer n.*

(1) The canonica1 mapping $J_n : \bigotimes_{n,\pi,s} F \to \bigotimes_{n,\pi,s} E$ is a monomorphism.

(2) Each locally bounded subset of $\mathcal{P}({}^nF)$ extends to a locally bounded subset of $\mathcal{P}({}^nE)$.

If, in addition, E is metrizable then the above are equivalent to
(3) Each element of $\mathcal{P}(^n F)$ extends to an element of $\mathcal{P}(^n E)$.

Proof. We have $\left(\bigotimes_{n,\pi,s} F\right)' = \mathcal{P}(^n F)$ and the equicontinuous subsets of $\left(\bigotimes_{n,\pi,s} F\right)'$ are identified with the locally bounded subsets of $\mathcal{P}(^n F)$ and similarly for E. Since the topology of a locally convex space is the topology of uniform convergence on the equicontinuous subsets of the dual it follows that (1) and (2) are equivalent.

If E is metrizable then $\bigotimes_{n,\pi,s} F$ and $\bigotimes_{n,\pi,s} E$ are metrizable and [41, corollary p. 265] implies that J_n is a monomorphism if and only if (3) is satisfied.

The mapping J_n is a monomorphism if and only if its extension to the completions is also a monomorphism. If E and F are metrizable and $i : F \to E$ is a continuous injective linear mapping and each $P \in \mathcal{P}(^n F)$ can be extended using i to an element of $\mathcal{P}(^n E)$ then [37, corollary p. 265] also implies that i is a monomorphism. For this reason we only considered subspaces in Lemma 4.2 and for the same reason our main technique (theorem 4) only applies to strict inductive limits.

Definition *4.3. If F is a subspace of E and for each n condition (2) of lemma 4.2 holds for the pair (F, E) then we say that (F, E) has the polynomial extension property.*

Lemma *4.4. If the pair of Banach spaces (F, E) has the polynomial extension property then for each n and each $\epsilon > 0$ there exists $\alpha > 0$ such that each $P \in \mathcal{P}(^n F)$ has an extension to $\tilde{P} \in \mathcal{P}(^n E)$ satisfying*

$$||\tilde{P}||_{U+\alpha V} \leq (1+\epsilon)||P||_U$$

(U is the unit ball of F and V the unit ball of E).

Proof. Since U and $V \cap F$ define equivalent norms on F there exists $\delta > 0$ such that $U \subset \delta V$. By the polynomial extension property there exists for each n, $M_n > 0$, such that each $P \in \mathcal{P}(^n F)$ satisfying $||P||_U \leq 1$ has an extension $\tilde{P}$ satisfying $||\tilde{P}||_V \leq M_n$. Hence each $P \in \mathcal{P}(^n F)$ has an extension $\tilde{P}$ satisfying

$$||\tilde{P}||_V \leq M_n ||P||_U.$$

The method of lemma 1.7 can now be used to show

$$||\tilde{P}||_{U+\alpha V} \leq ||P||_U \left(1 + \sum_{j=1}^{n-1} \alpha^j \delta^{n-j} \binom{n}{j} \cdot \frac{n^n}{n!} M_n\right)$$

and by choosing α sufficently small we obtain the required estimate.

Definition 4.5. ***The strict inductive limit,*** $\varinjlim_n E_n$, ***is said to have the extension property if each*** E_n ***has a convex balanced neighbourhood basis at the origin*** $\mathcal{V}_n$ ***such that for all*** n ***and all*** $V \in \mathcal{V}_n$ ***there exists*** $W \in \mathcal{V}_{n+1}$ ***such that the pair*** $((E_n)_V, (E_{n+1})_W)$ ***has the polynomial extension property.***

A strict inductive limit of Banach spaces has the extension property if and only if (E_n, E_{n+1}) has the polynomial extension property for all n.

Proposition 4.6. ***If the strict inductive limit*** $E = \varinjlim_n E_n$ ***has the extension property then*** (E_n, E) ***has the polynomial extension property for all n.***

Proof. If n and m are positive integers and $\mathcal{F}$ is a locally bounded subset of $\mathcal{P}(^n E_m)$ then there exists $V_m \in \mathcal{V}_m$ and $M > 0$ such that $\sup_{P\in\mathcal{F}} \|P\|_{V_m} \leq M < \infty$. We now choose inductively a sequence $(V_j)_{j>m}$, $V_j \in \mathcal{V}_j$, such that $((E_j)_{V_j}, (E_{j+1})_{V_{j+1}})$ has the polynomial extension property for all $j \geq$ m. By the remarks preceding definition 4.3, we see that $V_j \cap E_\ell$ and $V_k \cap E_\ell$ are unit balls of equivalent norms on E_ℓ for all $\ell \geq$ m, all $j \geq \ell$ and all $k \geq \ell$. If we let $\alpha_m = 1$ we can choose inductively, using lemma 4.4, a sequence of positive rea1 numbers $(\alpha_j)_{j\geq m}$ such that each $P \in \mathcal{F}$ has an extension $\tilde{P}$ to $E = \bigcup_{j\geq m} E_j$ satisfying

$$\|\tilde{P}\|_{\sum_{j\geq m} \alpha_j V_j} \leq M + 1$$

Since $\sum_{j\geq m} \alpha_j V_j$ is a neighbourhood of zero in E this completes the proof.

Theorem 4.7. ***If*** U ***is a balanced open subset of a strict*** $\mathcal{LF}$ ***space*** $E = \varinjlim_n E_n$ ***and E has the extension property then*** $(\mathcal{H}(U), \tau_w)$ ***(resp.*** $(\mathcal{H}(U), \tau_\delta)$***) is quasinormable if the following condition holds;***
for each τ_w ***(resp.*** τ_δ***) continuous seminorm p on*** $\mathcal{H}(U)$ ***there exists a positive integer m such that***

(4.3) $$\text{if } f \in \mathcal{H}(U) \text{ and } f|_{U\cap E_m} = 0 \text{ then } p(f) = 0.$$

Proof. Let p denote a τ_δ continuous seminorm on H(U) . We may suppose, without loss of generality, that

$$p\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} p(P_n) \qquad \text{for all } \sum_{n=0}^{\infty} P_n \in H(U).$$

and we also suppose that (4.3) is satisfied by the positive integer m.

For each positive integer n, we define $p|_{E_m}$ on $\mathcal{P}({}^nE_m)$ by the formula $p|_{E_m}(P) = p(\tilde{P})$ where $\tilde{P} \in \mathcal{P}({}^nE)$ and $\tilde{P}|_{E_m} = P$. By the extension property such a $\tilde{P}$ exists for each $P \in \mathcal{P}({}^nE_m)$. Moreover, if $R, S \in \mathcal{P}({}^nE_m)$ and $R|_{E_m} = S|_{E_m}$ then (4.3) implies that $p(R) = p(S)$ and hence $p|_{E_m}$ is a well defined seminorm on $\mathcal{P}({}^nE_m)$ for all n. By proposition 4.6, $p|_{E_m}$ is bounded on locally bounded subsets of $\mathcal{P}({}^nE_m)$ and hence is τ_w continuous. Theorem 3.3 and its proof imply that for each n there exists a τ_w continuous seminorm q_n on $\mathcal{P}({}^nE_m)$ which is associated to $p|_{E_m}$ on $\mathcal{P}({}^nE_m)$ and there exists a sequence of positive real numbers $(\alpha_n)_n$ such that $q := \sum_{n=0}^{\infty} \alpha_n q_n$ is τ_w continuous on $\mathcal{H}(U \cap E_m)$ and ported by $\{0\}$.

We now define $\tilde{q}_n$ on $\mathcal{P}({}^nE)$ by the formula

$$\tilde{q}_n(P) = q_n(P|_{E_m}).$$

Since E_m is a Fréchet space the τ_w bounded subsets of $\mathcal{P}({}^nE_m)$ are locally bounded. Since E has the extension property proposition 4.6 implies that $\tilde{q}_n$ is a τ_w-continuous seminorm on $\mathcal{P}({}^nE)$. Let $\tilde{q} = \sum_{n=0}^{\infty} \alpha_n \tilde{q}_n$.

If V is a neighbourhood of zero in E then $V \cap E_m$ is a neighbourhood of 0 in E_m and hence there exists $c(V \cap E_m) > 0$ such that

$$q\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} \alpha_n q_n(P_n) \le c(V \cap E_m) \sum_{n=0}^{\infty} \|P_n\|_{V \cap E_m}$$

for all $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U \cap E_m)$. If $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U)$ then

$$\begin{aligned}\tilde{q}\left(\sum_{n=0}^{\infty} P_n\right) &= \sum_{n=0}^{\infty} \alpha_n \tilde{q}_n(P_n) = \sum_{n=0}^{\infty} \alpha_n q_n(P_n|_{E_m}) \\ &\le c(V \cap E_m) \cdot \sum_{n=0}^{\infty} \|P_n|_{E_m}\|_{V \cap E_m} \\ &\le c(V \cap E_m) \sum_{n=0}^{\infty} \|P_n\|_V\end{aligned}$$

and hence $\tilde{q}$ is a τ_w continuous seminorm on $\mathcal{H}(U)$.

Now fix n. Since q_n is associated with $p|_{E_m}$ there exists for every $\lambda > 0$ a τ_w-bounded subset B_λ of $\mathcal{P}({}^nE_m)$ such that

$$\{P \in \mathcal{P}({}^nE_m); q_n(P) \leq 1\} \subset B_\lambda + \{P \in \mathcal{P}({}^nE_m); p|_{E_m}(P) \leq \lambda\}.$$

By the extension property and since τ_w bounded sets in $\mathcal{P}({}^nE_m)$ are locally bounded there exists a locally bounded subsets $\tilde{B}_\lambda$ of $\mathcal{P}({}^nE)$ such that $B_\lambda = \{P|_{E_m}; P \in \tilde{B}_\lambda\}$.

If $P \in \mathcal{P}({}^nE)$ and $\tilde{q}_n(P) \leq 1$ then $q_n(P|_{E_m}) \leq 1$ and there exists Q $\in B_\lambda$ and $R \in \mathcal{P}({}^nE_m)$ such that $p|_{E_m}(R) \leq \lambda$ and $P|_{E_m} = Q + R$.

Let $\tilde{Q} \in \tilde{B}_\lambda$ satisfy $\tilde{Q}|_{E_m} = Q$ and suppose $\tilde{R} \in \mathcal{P}({}^nE_m)$ is chosen so that $\tilde{R}|_{E_m} = R$. Then $P = \tilde{Q} + \tilde{R} + P - \tilde{Q} - \tilde{R}$ where $\tilde{Q} \in \tilde{B}_\lambda$ and $p(\tilde{R} + P - \tilde{Q} - \tilde{R}) = p(\tilde{R}) \leq \lambda$ since $P - \tilde{Q} - \tilde{R}|_{E_m} = P|_{E_m} - Q - R = 0$ and

$$p(\tilde{R}) = p|_{E_m}(\tilde{R}|_{E_m}) = p|_{E_m}(R) \leq \lambda.$$

We have thus shown that for each τ_δ continuous seminorm $p = \sum_{n=0}^{\infty} p_n$ on $\mathcal{H}(U)$ there exists a τ_w continuous seminorm $q = \sum_{n=0}^{\infty} \alpha_n q_n$ on $\mathcal{H}(U)$ such that q_n in associated with p_n for all n. Since each τ_w continuous seminorm on $\mathcal{P}({}^nE)$ can be realised as a component of a τ_δ continuous seminorm on $\mathcal{P}({}^nE)$ it follows that $(\mathcal{P}({}^nE), \tau_w)$ is quasinormable, and (4.2) is satisfied. Since $\tau_\delta \geq \tau_w$ the above also implies that (4.1) is satisfied and this completes the proof.

Condition (4.3) is satisfied by a strict inductive limit of Banach spaces [28, proposition 4] and the same result for a direct sum of Banach spaces is given in [21]. We will prove that (4.3) is true for any CB space. To prove this we need the following result conceming holomorphicity of functions on $\mathcal{LF}$ spaces. A function $f : U \to \mathbf{C}$, U a domain in a locally convex space, is *Gâteaux holomorphic* if its restriction to the finite dimensional sections of U are holomorphic as functions of several complex variables.

Proposition 4.8. *Let* $E = \lim_{\vec{n}} E_n$ *be an inductive limit of Fréchet spaces and let* $f = \sum_{m=0}^{\infty} P_m$ *denote a Gâteaux holomorphic function on* E. *Let* $(V_n)_n$ *denote a sequence of subsets of* E, $V_n \subset E_n$, $nV_n \subset V_{n+1}$ *and suppose* V_n *is a neighbourhood of* 0 *in* E_n.
If

$$\sup_m \|P_m\|_{V_n} < \infty \quad \text{for all } n \tag{4.4}$$

then $f \in \mathcal{H}(E)$.

Proof. We first fix n and consider $f|_{E_n}$. If $\alpha > 0$ is arbitrary then there exists a positive integer m_0 such that $2\alpha V_n \subset V_{m_0}$. By (4.4),

$$\sup_{m \geq m_0} ||P_m||_{2\alpha V_n} \leq \sup_{m \geq m_0} ||P_m||_{V_{m_0}} = M < \infty.$$

Hence

$$\sum_{m \geq m_0} ||P_m||_{\alpha V_n} \leq \sum_{m \geq m_0} \frac{M}{2^m} < \infty.$$

By (4.4), we have $||P_m||_{\alpha V_n} < \infty$ for $m = 0, 1, \ldots, m_0 - 1$ and hence $\sum_{m=0}^{\infty} ||P_m||_{\alpha V_n} < \infty$ for every $\alpha > 0$. This implies that $f|_{E_n} \in \mathcal{H}(E_n)$ and that, moreover, it is of uniformly bounded type. i.e.

$$\lim_{m \to \infty} \sup ||P_m||_{V_n}^{\frac{1}{m}} = 0. \tag{4.5}$$

We now show that f is locally bounded and this will complete the proof. Let $x \in E$. Without loss of generality, we may suppose $x \in E_1$. By (4.4) and (4.5)

$$\sup_m ||P_m||_{2x+V_1} = M_1 < \infty$$

By (4.5) there exists a positive integer m_1 such that

$$||P_m||_{2x+V_1+V_2} \leq \frac{1}{2} \text{ for all } m \geq m_1.$$

Using the method of lemma 1.7 we can find $\lambda_2 > 0$ such that $||P_m||_{2x+V_1+\lambda_2 V_2} \leq M + \frac{1}{2}$ for all m.

By induction and the same method we can find a sequence $(\lambda_m)_m$, $\lambda_1 = 1$, such that $||P_m||_{2x+2W} \leq M_1 + 1$ for all m where $2W = \sum_{n=1}^{\infty} \lambda_n V_n$. The set W is convex balanced and absorbing and each V_n is a neighbourhood of 0 in E_n. Hence W is a neighbourhood of 0 in E. Since

$$||f||_{x+W} \leq \sum_{m=0}^{\infty} ||P_m||_{x+W} \leq \sum_{m=0}^{\infty} \frac{1}{2^m} ||P_m||_{2x+2W} \leq \sum_{m=0}^{\infty} \frac{M_1 + 1}{2^m} < \infty$$

f is locally bounded. This completes the proof.

Remark 4.9. If $E = \varinjlim_n E_n$ is an $\mathcal{CB}$ space then (4.4) may be replaced by

$$\limsup_{m\to\infty} \|P_m\|_{V_n} < \infty \qquad \text{for all } n$$

but this *cannot* in general be used for $\mathcal{LF}$ spaces.

Example 4.10. (a) Let $E = \varinjlim_n E_n$ denote an inductive limit of Banach spaces. Then (4.4) holds for $f = \sum_{m=0}^{\infty} P_m : E \to \mathbb{C}$ if and only if $f \in \mathcal{H}_b(E_n)$ for all n. By Grothendieck [35, chapter 4,§3, proposition 5] each bounded subset of E is contained in the closure of a bounded subset of some E_n. Hence $f \in \mathcal{H}_b(E)$. A recent result in [28] implies, for strict inductive limits of Banach spaces, that (4.4) is only satisfied by *all* entire functions in the trivial case $E \approx \mathbb{C}^{(N)}$. In the CB space $E = \varinjlim_n E_n$ is a $\mathcal{DFS}$ space with compact linking maps then it is easily seen that f, Gateaux holomorphic on E, is holomorphic if and only if (4.4) is satisfied.

(b) Let $E = \varinjlim_n E_n$ denote an C3 space and let $f = \sum_{m=0}^{\infty} P_m : E \to \mathbb{C}$ be Gateaux holomorphic. If for each positive integer n there exists a positive integer m_n such that $P_m|_{E_n} = 0$ for all m $\geq m_n$ then (4.4) reduces to $\|P_m\|_{V_m} < \infty$ for all m and n. Since Banach spaces have bounded neighbourhoods of the origin we see that the condition is always satisfied by *CB* spaces when $P_n|_{E_m}$ is continuous for all m and n.

(c) In this example we give a new proof of a known result ([40, example 1.3], [3], [15, proposition 4.1], [33]). We include the proof as we refer to it later. Let $E = \varinjlim_n E_n$ be an *CB* space and let B_n denote the unit ball of E_n. We may suppose, without loss of generality, that $nB_n \subset B_{n+1}$ for all n. By the result of Grothendieck quoted in (a), $\{\overline{B_n}\}_{n=1}^{\infty}$ forms a fundamental sequence of bounded subsets of E. Let $(P_j)_{j=1}^{\infty}$ denote a β-bounded sequence in $\mathcal{P}(^n E)$. Since $nB_n \subset B_{n+}$, it follows that

$$\limsup_{j\to\infty} \|P_j\|_B = \limsup_{j\to\infty} \|P_j^j\|_B^{1/j} = 0$$

for every bounded subset B of E.

Hence $\sup_j \|P_j^j\|_{B_n} < \infty$ for all n. By proposition 4.8, $\sum_{j=1}^{\infty} P_j^j \in H(E)$. Hence for any x in E, there exists a neighbourhood of 0, V_x, such that

$$\limsup_{j\to\infty} \|P_j^j\|_{x+V_x}^{1/j} = \limsup_{j\to\infty} \|P_j\|_{x+V_x} < \infty$$

This implies that the sequence $\{P_j\}_j$ is locally bounded and hence τ_w bounded. Since $\tau_w \geq \beta$ and $(\mathcal{P}(^mE), \beta)$ is metrizable we conclude that $\tau_w = \beta$ on $\mathcal{P}(^mE)$ for all m.

We now show that (4.3) is satisfied by *CB* spaces.

Proposition 4.11. *Let U denote a balanced open subset of the CB space $E = \varinjlim_n E_n$ and let p denote a τ_δ continuous seminorm on $\mathcal{H}(U)$. There exists a bounded subset B of some E_m such that for all n, $p(P) \leq ||P||_B$ for all P in $\mathcal{P}(^nE)$. Moreover, there exists a positive integer m such that $p(f) = 0$ for all $f \in \mathcal{H}(U)$ satisfying $f|_{U\cap E_m} = 0$.*

Proof. We may suppose that $p\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} p(P_n)$ for all $\sum_{n=0}^{\infty} P_n \in \mathcal{H}(U)$. Let B_n denote the unit ball of E_n and suppose $nB_n \subset B_{n+1}$ for all n. Suppose the result is not true. Then for each positive integer n there exists a homogeneous polynomial P_n such that

$$p(P_n) > ||P_n||_{B_{n+1}} \geq n^{(\deg P_n)}||P_n||_{B_n}$$

for all n.

We first suppose that there exists a positive integer n_0 (the case $n_0 = 0$ hivially leads to a contradiction) such that $\deg(P_n) = n_0$ for an infinite number of positive integers. If $A := \{n; \deg(P_n) = n_0 \text{ and } ||P_n||_{B_n} = 0\}$ is infinite then $\left\{\frac{nP_n}{p(P_n)}\right\}_{n\in A}$ is a bounded subset of $(\mathcal{P}(^{n_0}E), \beta)$. Since $p\left(\frac{nP_n}{p(P_n)}\right) = n$ for all n this contradicts the conclusion of example 4.10(c). If A is not infinite then we may suppose $||P_n||_{B_n} \neq 0$ for all n. The sequence $\left\{\frac{P_n}{||P_n||_{B_n}}\right\}_{n,\deg(P_n)=n_0}$ is a bounded subset of $(\mathcal{P}(^{n_0}E, \beta))$. Since $p\left(\frac{P_n}{||P_n||_{B_n}}\right) \geq n^{n_0}$ this again contradicts the conclusion in example 4.10(c).

Hence, by taking a subsequence if necessary, we may suppose that $\deg(P_n)$ is strictly increasing.

Let

$$Q_n = \begin{cases} \dfrac{n^{(\deg P_n)}P_n}{p(P_n)} & \text{if} \quad ||P_n||_{B_n} = 0 \\ \dfrac{P_n}{n^2||P_n||_{B_n}} & \text{if} \quad ||P_n||_{B_n} \neq 0 \end{cases}$$

Since $||Q_n||_{B_n} \leq \frac{1}{n^2}$ for all n we have

$$\sum_{n=0}^{\infty} ||Q_n||_B < \infty$$

for any B contained and bounded in some E_m. By example 4.10(a) this implies that $\sum_{n=0}^{\infty} Q_n \in H(E)$. Since

$$p(Q_n) \geq n^{\deg(P_n)-2} \geq n$$

for all $n \geq 3$ this is impossible.

Hence there exists a positive integer m and B a bounded subset of E_m such that for all n

$$p(P) \leq \|P\|_B \qquad \text{for all } P \in \mathcal{P}(^nE).$$

If $f \in \mathcal{H}(U)$ and $f|_{U \cap E_m} = 0$ then $\left.\frac{\hat{d}^n f(0)}{n!}\right|_{E_m} = 0$ for all n and

$$p(f) = \sum_{n=0}^{\infty} p\left(\frac{\hat{d}^n f(0)}{n!}\right) = 0$$

This completes the proof.

If U is a balanced open subset of an Ct3 space with the extension property then proposition 4.11 and the proof of theorem 4.7 show that for each τ_0 bounded subset $\mathcal{F}$ of $\mathcal{H}(U)$ and each τ_0 continuous seminorm p on $\mathcal{H}(U)$ there exists a locally bounded subset of $\mathcal{H}(U)$, $\tilde{\mathcal{F}}$, such that

$$\sup_{f \in \mathcal{F}} p(f) = \sup_{f \in \tilde{\mathcal{F}}} p(f) < \infty.$$

Hence τ_0 and τ_δ define the same bounded subsets of $\mathcal{H}(U)$. This result for direct sums of Banach spaces is given in [2 1, proposition 3.1] and example 4.22 (below) provides new examples.

We now show how (4.3) can be combined with theorem 4.1 to show that the quasinormability problem is equivalent for the different topologies and different function spaces on an inductive limit of Banach spaces.

Proposition 4.12. *If U is a balanced open subset of the $\mathcal{LB}$ space $E = \lim_{\overrightarrow{n}} E_n$ then the following are equivalent;*

(β) $(\mathcal{H}_b(U), \beta)$ is quasinormable,

(τ_w) $(H(U), \tau_w)$ is quasinormable,

(τ_δ) $(\mathcal{H}(U), \tau_\delta)$ is quasinormable.

Proof. By example 4. 10(c), we have $\tau_w = \tau_\delta = \beta$ on $\mathcal{P}(^nE)$ for any positive integer n. Hence if any of the three conditions is satisfied then $(\mathcal{P}(^nE), \beta)$ is quasinormable and it suffices to show that if one of the conditions is satisfied then (4.2) is satisfied by any one of the others.

Let $(\mathcal{F}(U), \tau_1)$ denote one of the spaces and $(\mathcal{G}(U), \tau_2)$ another. We suppose that (τ_1) is satisfied. Let p_2 denote a τ_2-continuous seminorm on $\mathcal{G}(U)$. We suppose that

$$p_2\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} p(P_n) \text{ for all} \sum_{n=0}^{\infty} P_n \in \mathcal{G}(U)$$

By proposition 4.11, if $\tau_2 = \tau_w$ or τ_δ, and by definition if $\tau_2 = \beta$, there exists a bounded subset B of E such that for all n

$$p_2(P) \leq ||P||_B \quad \text{for all } P \in \mathcal{P}(^nE).$$

Let $q\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} \frac{1}{n^n} p_2(P_n)$,for all $\sum_{n=0}^{\infty} P_n \in \mathcal{F}(U)$. Since $q\left(\sum_{n=0}^{\infty} P_n\right) \leq \sum_{n=0}^{\infty} ||P_n||_{\frac{1}{n}B}$ it followsthat q is τ_1 continuous. Since $(\mathcal{F}(U), \tau_1)$ is quasinormable there exists a τ_1-continuous seminorm $\tilde{q}$ which is associated with q. Again by proposition 4.11, if $\tau_1 = \tau_w$ or τ_δ, and by definition if $\tau_1 = \beta$, there exists a bounded subset B_1 of E such that for all n

$$\tilde{q}(P) \leq ||P||_{B_1} \quad \text{for all } P \in \mathcal{P}(^nE).$$

For each n, $q|_{\mathcal{P}(^nE)}$ is equivalent to $p_2|_{\mathcal{P}(^nE)}$ and hence $\tilde{q}|_{\mathcal{P}(^nE)}$ is associated to $p_2|_{\mathcal{P}(^nE)}$. The seminorm q_1 defined by

$$q_1\left(\sum_{n=0}^{\infty} P_n\right) = \sum_{n=0}^{\infty} \frac{1}{n^n} \tilde{q}(P_n) \text{ for } \sum_{n=0}^{\infty} P_n \in \mathcal{G}(U)$$

is τ_2 continuous since

$$q_1\left(\sum_{n=0}^{\infty} P_n\right) \leq \sum_{n=0}^{\infty} ||P_n||_{\frac{1}{n}B_1} \text{ for all } \sum_{n=0}^{\infty} P_n \in \mathcal{G}(U).$$

Hence (4.2) is satisfied by $(\mathcal{G}(U), \tau_2)$ and so $(\mathcal{G}(U), \tau_2)$ is quasinormable and this completes the proof.

Theorem 4.7 and propositions 4.11 and 4.12 show that the spaces $(\mathcal{H}_b(U), \beta)$, $(\mathcal{H}(U), \tau_w)$ and $(\mathcal{H}(U), \tau_\delta)$ are quasinormable for U a balanced open subset of a strict CB space E if E has the extension property. This will be the case if E has a representation as a strict inductive limit $\varinjlim_n E_n$ such that (E_n, E_{n+1}) has the polynomial extension property for all n. If E is a direct sum of Banach spaces, say $E = \sum_{j=1}^{\infty} F_j$, then $E = \varinjlim_n E_n$ where $E_n = \sum_{j=1}^{n} F_j$ for all n. Since E_n is complemented in E_{n+1}, $P \in \mathcal{P}(^m E_n)$ can be extended to E_{n+}, by the formula $\tilde{P}(x + y) = P(x)$ for x in E_n and y in F_{n+1}. Hence (E_n, E_{n+1}) has the polynomial extension property. We thus have the following example.

Proposition **4.13.** ***If** U **is a balanced open subset of a direct sum of Banach spaces then** $(\mathcal{H}(U), \tau_w)$, $(\mathcal{H}(U), \tau_\delta)$ **and** $(\mathcal{H}_b(U), \beta)$ **arequasinormable.***

Example 4.14. To obtain further examples-and in particular examples not covered by proposition 4.13-it is natural to consider a situation in which we can extend polynomials. This is the case by [6, 18,291 if $E = \varinjlim_n E_n$ and for each n

$$(*) \qquad E_n \hookrightarrow E_{n+1} \hookrightarrow E_n''.$$

The simplest case in which this may be realised is to take a non-reflexive Banach space E and to let $(E_n)_n$ denote an increasing sequence of Banach subspaces of E'' containing E. If $F = \varinjlim_n E_n$ then F has the extension property ((*) is satisfied since $E \hookrightarrow E_n$ implies $E'' \hookrightarrow E_n''$ and we have $E_n \hookrightarrow E_{n+} \hookrightarrow E'' \hookrightarrow E_n''$) and the spaces of holomorphic functions we considered on balanced domains of F are all quasinormable. It is not, however, clear if F is or is not isomorphic to a direct sum of Banach spaces. To give an example which is not a direct sum we use a space constructed by Moscatelli [48] (see also [46]).

Let $X_n = \ell_\infty$ and $Y_n = c_0$ for all n. Let $E = \left(\bigoplus_{n=1}^{\infty} Y_n\right)_{\ell_2}$. Then $E'' = \left(\bigoplus_{n=1}^{\infty} X_n\right)_{\ell_2}$.

For each n let $E_n = \left[\left(\bigoplus_{i\le n} X_i\right)_{\ell_2} \bigoplus_{\ell_2} \left(\bigoplus_{i>n} Y_i\right)\right]_{\ell_2}$. Then $E_n \hookrightarrow E_{n+1} \hookrightarrow E_n''$ for all n. By [48, lemma 3] the inductive limit $\varinjlim_n E_n$ is not isomorphic to a direct sum of Banach spaces. The spaces c_0 and ℓ_∞ may be replaced by Banach spaces X and X" provided X is not complemented in X".

The space considered in the previous example is called a standard strict *CB* space of Moscatelli type. The structure of these spaces has been investigated in recent years and we refer to [14, 45, 49] for details. We now characterize two collections of standard strict CB spaces of Moscatelli type by quasinormability of spaces of holomorphic functions.

A normal Banach sequence space is a Banach space $(\lambda, |.|)$ *satisfying*

(a) $\mathbf{C}^{(N)} \hookrightarrow \lambda \hookrightarrow \mathbf{C}^N$

(b) if $a = (a_n)_n \in \lambda$ and $b = (b_n)_n \in \mathbf{C}^N$ satisfy $|b_n| \leq |a_n|$ for all n then $b \in \lambda$ and $|b|_\lambda \leq |a|_\lambda$

Any Banach space with a 1-unconditional basis is a normal Banach sequence space.

If $(X_n)_n$ is a sequence of Banach spaces then

$$\lambda\left(\bigoplus_n X_n\right) := \{(x_n)_n; (\|x_n\|_n)_n \in \lambda\}$$

is a Banach space when normed by $\|(x_n)_n\| := |(\|x_n\|_n)|_\lambda$. Let Y denote a subspace of the Banach space X . For each n let $E_n = \lambda(\bigoplus_m X_m)$ where X_m = X for m $< n$ and X_m = Y for m $\geq$ n. Using coordinates we see that there is a natural inclusion from E_n into E_{n+1}. *The* strict inductive limit $E = \varinjlim_n E_n$ is called a ***standardstrict CB space of Moscatelli type*** and denoted by X(X, Y) . The space X(X, Y) may be identified algebraically with a subspace of X^N. If π_k and π^k denote, respectively, the projections on X^N onto the first k and all but the first k coordinates then the restriction of both π_k and π^k to X(X, Y) and each E_n gives rise to continuous projections. We have $\pi_k(\lambda(X, Y)) \cong X^k$, $\pi^k(\lambda(X, Y)) \cong$ X(X, Y) and $\pi_k(E_m) = \lambda(\ominus_m Y_m)$, Y_m = Y all m, for $k \geq$ m. Let B_X denote the unit ball of X and $\mathcal{B}_n$ the unit ball of E_n. By [14], $(\mathcal{B}_n)_{n=1}^{\infty}$ forms a fundamental sequence of bounded subsets of $\lambda(X, Y)$ and

$$\left\{\bigoplus_{k\in N} \epsilon_k B_k + \delta\mathcal{B}_1 ; \epsilon_k > 0 \text{ all } k \text{ and } \delta > 0\right\}$$

forms a fundamental system of neighbourhoods of the origin. A dual construction leads to a collection of Fréchet spaces called ***standard quojections of Moscatelli type.*** Specifically, if Y is a subspace of X we let $F_n = \lambda\left(\bigoplus_k X_k\right)$ where X_k = X for $k <$ n and $X_n = X/Y$ for n $\geq k$. This gives a sequence of quotient mappings $F_{n+1} \rightarrow F_n$ and our quojection is

$$X(X, X/Y) = \varprojlim F_n.$$

If λ'_β is also a normal Banach sequence space then X(X, $Y)'_\beta = \lambda'_\beta(X', X'/Y^\perp)$. If Z is an arbitrary Banach space then X(X, Y) x Z is a strict inductive limit of Banach spaces and in fact we have

$$\lambda(X, Y) \text{ x } Z = \varinjlim_n E_n \text{ x } Z$$

and we may think of Z as the fina1 coordinate space.

The following lemma is a variation of a fundamental lemma which first appeared in [55] and which was subsequently modified in [19] and [16]. Our version is modeled on that given in [16, lemma 1] and the proof there can be modified to prove our lemma. Proposition 4.16 is also motivated by a result in [16].

Lemma 4.15. *Let Y denote a subspace of a Banach space X and suppose Y has the approximation property. If n is a positive integer and for every* $\epsilon > 0$ *there exists* $\alpha > 0$ *such that*

$$\overline{\Gamma}\left(\bigotimes_{n,s} B_Y\right)\bigcap \alpha\Gamma\left(\bigotimes_{n,s} B_X\right) \subset \epsilon\overline{\Gamma}\left(\bigotimes_{n,s} B_Y\right),$$

where the closures are taken in $\bigotimes_{n,\pi,s} X$ *, then the canonica1 mapping*

$$J_n : \bigotimes_{n,\pi,s} Y \to \bigotimes_{n,\pi,s} X$$

is a monorphism.

Proposition 4.16. *Let X(X, Y) denote a standard strict* $\mathcal{LB}$ *space of Moscatelli type and let Z denote a Banach space. Suppose Y and Z have the approximation property and* $(\mathcal{P}({}^n(X(X,Y)\times Z)), \tau_w)$ *is quasinormable. Then the canonica1 inclusions*

$$I_n : \bigotimes_{n,\pi,s}(Y \times Y \times Z) \to \bigotimes_{n,\pi,s}(X \times Y \times Z) \tag{4.6}$$

and

$$J_n : \bigoplus_{n,\pi,s}(Y \times Z) \to \bigoplus_{n,\pi,s}(X \times Z) \tag{4.7}$$

are monomorphism.

Proof. If $(\mathcal{P}({}^n(\lambda(X, Y) \times Z)), \tau_w)$ is quasinormable, then since it is infrabarrelled, its strong dual satisfies the strict Mackey condition. By [41, p. 186], $\bigotimes_{n,\pi,s}(\lambda(X, Y) \times Z)$ is infrabarrelled and hence is isomorphic to a subspace of its second dual $(\mathcal{P}({}^n(\lambda(X, Y) \times Z)), \tau_w)'_\beta$. Hence $\bigotimes_{n,\pi,s}(\lambda(X,Y) \times Z)$ also satisfies the strict Mackey condition. Hence there exists a positive integer m, a sequence of positive real numbers $(\epsilon_k)_k$ and $\delta > 0$ such that

$$\overline{\Gamma}\left(\bigotimes_{n,s}\mathcal{B}_1 \times B_Z\right)\bigcap\Gamma\left(\bigotimes_{n,s}((\bigoplus_k \epsilon_k B_X + \delta\mathcal{B}_1) \times B_Z)\right)$$
$$\subset \epsilon\overline{\Gamma}\left(\bigotimes_{n,s}(\mathcal{B}_m \times B_Z)\right),$$

where the closures are taken in $\bigotimes_{n,\pi,s}(\lambda(X,Y) \times Z)$.

Projecting onto the m^{th}, $(m+1)^{st}$ and fina1 coordinates gives

$$\overline{\Gamma}\left(\bigotimes_{n,s}(B_Y \times B_Y \times B_Z)\right)\bigcap\Gamma\left(\bigotimes_{n,s}((\epsilon_m B_X + \delta B_Y) \times (\epsilon_{m+1} B_X + \delta B_Y) \times B_Z)\right)$$
$$\subset \epsilon\overline{\Gamma}\left(\bigotimes_{n,s} B_Y \times B_Y \times B_Z\right),$$

where the closures are taken in $\bigotimes_{n,\pi,s}(X \times Y \times Z)$. Let $\alpha = \inf\{1, \epsilon_m\epsilon_{m+1}, \delta^2\}$. Hence

$$\overline{\Gamma}\left(\bigotimes_{n,s}(B_Y \times B_Y \times B_Z)\right)\bigcap\alpha\Gamma\left(\bigotimes_{n,s}(B_X \times B_Y \times B_Z)\right)$$
$$\subset \epsilon\overline{\Gamma}\left(\bigotimes_{n,s} B_Y \times B_Y \times B_Z\right),$$

where the closures are taken in $\bigotimes_{n,s}(X \times Y \times Z)$. An application of lemma 4.15 shows that (4.6) is true and (4.7) follows by projecting onto the m^{th} and final coordinates.

If E is a locally convex space we write E'' in place of $(E'_\beta)'_\beta$. If E is infrabarrelled then E'' is the natural bidual in the sense of Grothendieck [34].

Proposition 4.17. *(a) If E is an infraharrelled locally convex space then (E, E'') has the polynomial extension property.*
(b) If E is an infrabarrelled $\mathcal{DF}$ space then

$$(*) \qquad (\mathcal{P}(^nE''), \beta)/_{\{P\in\mathcal{P}(^nE''); P|_E=0\}} \cong (\mathcal{P}(^nE), \beta)$$

and

$$(**)\qquad (\mathcal{H}_b(E''),\beta)/_{\{f\in\mathcal{H}_b(E'');f|_E=0\}} \cong (\mathcal{H}_b(E),\beta).$$

Proof. (a) We have included the infrabarrelled the infrabarrelled hypothesis as our definition of the polynomial extension property only applies to subspaces and E is a subspace of E'' if and only if E is infrabarrelled.

If n is a positive integer and $\mathcal{F}$ is a locally bounded subset of $\mathcal{P}({}^nE)$ then there exists a $p \in cs(E)$ and $M > 0$ such that

$$\sup_{P\in\mathcal{F}}\{|P(x)|; p(x) \leq 1\} \leq M < \infty.$$

Let $\hat{E}_p$ denote the completion of the space $(E/p^{-1}(0), p)$ and let i_p denote the canonica1 quotient mapping of E into $\hat{E}_p$. For each $P \in \mathcal{F}$ there exists a unique $\tilde{P} \in \mathcal{P}({}^n(\hat{E}_p))$ such that $P = \tilde{P} \circ i_p$. Hence

$$\sup_{P\in\mathcal{F}} \|\tilde{P}\|_{B(\hat{E}_p)} \leq M < \infty.$$

By [18, theorem 3] each $P \in \mathcal{P}({}^n(\hat{E}_p))$ has an isometric extension $\hat{P}$ to $(\hat{E}_p)''$ i.e.

$$\|\hat{P}\|_{B(\hat{E}_p)''} = \|P\|_{B_{\hat{E}_p}}.$$

Let J_E denote the canonical mapping from a locally convex space into its second dual E''. We let i_p^{tt} denote the second transpose of the mapping i_p. If $P \in \mathcal{F}$ the above gives the following commutative diagram.

$$\begin{array}{ccccc}
E & \xrightarrow{i_p} & \hat{E}_p & & \\
 & \searrow & \swarrow & & \\
J_E \downarrow & P \quad \mathbb{C} & \tilde{P} & \downarrow J_{E''} & \\
 & \hat{\tilde{P}} & & & \\
 & \nwarrow & & & \\
E'' & \xrightarrow[i_p^{tt}]{} & (\hat{E}_p)'' & &
\end{array}$$

Let $P'' = \hat{\tilde{P}} \circ i_p^{tt}$. Then $P'' \in \mathcal{P}({}^nE'')$ and $P''|_{J_E(E)} = P$.

Since i_p^{tt} is continuous $V := (i_p^{tt})^{-1}(B_{(\hat{E}_p)''})$ is a neighbourhood of 0 in E". Hence

$$\sup_{p\in\mathcal{F}} ||P''||_V = \sup_{PE3} ||\hat{\tilde{P}}||_{B_{(\hat{E}_p)''}} = M < \infty.$$

(b). By (a) the restriction mapping

$$R_E : \mathcal{P}(^nE'') \to \mathcal{P}(^nE)$$
$$P \to P|_E$$

is surjective and $\ker(R_E) = \{P \in \mathcal{P}(E''); P|_E = 0\}$.

Since E and E'' are $\mathcal{DF}$-spaces it follows that ($\mathcal{P}(^nE), \beta$) and $(\mathcal{P}(^nE''), \beta)$ are Fréchet spaces. Since bounded subsets of E are also bounded subsets of E'' it follows that R_E is continuous. An application of the open mapping theorem shows that (*) is true.

If $f = \sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(E)$ and there exists a neighbourhood V of 0 in E such that $\limsup_{n\to\infty} \left\| \frac{\hat{d}^n f(0)}{n!} \right\|_V^{\frac{1}{n}} = 0$ then an examination ofthe estimates in (a) shows that each $\frac{\hat{d}^n f(0)}{n!}$ has an extension to E'' as $\frac{\widetilde{\hat{d}^n f(0)}}{n!}$ and there exists a neighbourhood $\tilde{V}$ of 0 in E'' such that

$$\limsup_{n\to\infty} \left\| \frac{\widetilde{\hat{d}^n f(0)}}{n!} \right\|_{\tilde{V}}^{\frac{1}{n}} = 0.$$

Such holomorphic functions are said to be of uniformly bounded type (see the proof of proposition 4.8). By [33, proposition 7] each element of $\mathcal{H}_b(E)$ is of this type and so the mapping

$$R_E : \mathcal{H}_b(E'') \to \mathcal{H}_b(E)$$
$$f \to f|_E$$

is surjective and a further application of the open mapping theorem completes the proof.

Our proof of the above proposition was motivated by results in [18] and [33]. We remark that the proof in (a) shows that the restriction mapping on germs $\mathcal{H}(0_{E''}) \to \mathcal{H}(0_E)$ is also surjective. Further results regarding completeness and extensions and their relationship with one another can be found in [6, 25, 43].

By [18, theorem 1] any Banach space X has the following property; each $P \in \mathcal{P}(^nX)$, n arbitrary, has an extension P_d to X'' such that for every convex bounded subset B of X and every $x^{**} \in \overline{B}^{\sigma(X'',X')}$ there exists a net $(x_\alpha)_\alpha$ in B, which converges in the weak* topology to x^{**}, and $P(x_\alpha) \to P_d(x^{**})$ as $\alpha \to \infty$.

Proposition 4.18. *If Y is a subspace of the Banach space X and (Y x Y', X x Y') has the polynomial extension property then Y'' is complemented in X''.*

Proof. Let $P(y, y') = y'(y)$ for $(y,y') \in Y \times Y'$. Since $P \in \mathcal{P}(^n(Y \times Y'))$ there exists $\tilde{P} \in \mathcal{P}(^n(X \times Y'))$ which extend P. Let $\tilde{P}_d$ denote the extension of $\tilde{P}$ to X" x Y"', described above, and let Q be its restriction to X" x Y' If $x^{**} \in X''$ and $y' \in Y'$ we let

$$[\pi(x^{**})](y') = Q(x^{**}, y').$$

Since Q is continuous, $\pi(x^{**}) \in Y''$ and $\pi : X'' \to Y''$ is continuous and linear. If $y^{**} \in Y'', ||y^{**}|| \leq 1$, and $y' \in Y'$ then the weak* closure of $B_Y \times \{y'\}$, contains $y^{**} \times \{y'\}$. Hence there exists a net in B_Y, $(y_\alpha)_\alpha$, which converges weak* to y^{**}, such that

$$Q(y^{**}, y') = \lim_{\alpha \to \infty} Q(y_\alpha, y') = \lim_{\alpha \to \infty} y'(y_\alpha) = y^{**}(y') = [\pi(y^{**})](y')$$

Hence $\pi(y^{**}) = y^{**}$ for all $y^{**} \in Y''$. The mapping π is the required projection of X" onto Y".

We are now in a position to give our characterizations of two collections of Moscatelli type spaces.

Theorem 4.19. *Let $\lambda(X, Y)$ denote a standard strict CB-space of Moscatelli type and suppose Y has the approximation property The following are equivalent.*
(a) For all positive integers n the canonical mappings

$$J_n : \bigotimes_{n,\pi,s}(Y \times Y) \to \bigotimes_{n,\pi,s}(X \times Y)$$

are monomorphism,
(b) $\lambda(X, Y)$ has the extension property,
(c) $(\mathcal{H}_b(U), \beta), (\mathcal{H}(U), \tau_w)$ and $(\mathcal{H}(U), \tau_\delta)$ are quasinormable for any balanced open subset U of $\lambda(X, Y)$,
(d) $\mathcal{P}((^n(\lambda(X, Y)) . \beta)$ is quasinormable for every positive integer n.

Proof. (a)$\Rightarrow$(b). If (a) holds then the pair (E_n, E_{n-}) has the polynomial estension property and hence $\lambda(X, Y)$ has the extension property and (b) holds.

(b)$\Rightarrow$(c). If holds then theorem 4.7 and proposition 4. 11 imply that (c) holds.

(c)$\Rightarrow$ (d). This is trivial since $(\mathcal{P}(^n(\lambda(X, Y))) \beta)$ is a complemented subspace of any of the spaces in (c).

(d)$\Rightarrow$ (a). This follows from (4.6) in proposition (4.16).

Theorem 4.20. *Let $\lambda(X, Y)$ denote a standard strict CL3 space of Moscatelli type. Suppose Y' has the approximation property and λ is a reflexive normal Banach sequence space. The following are equivalent.*

(a) Y'' is complemented in X'',

(b) $\lambda'(X', X'/Y^{\perp})$ is a product of Banach spaces,

(c) $\lambda(X'',Y'') = (\lambda(X,Y)'_{\beta})'_{\beta}$ is a direct sum of Banach spaces,

(d) $(\mathcal{H}_b(U),\beta),(\mathcal{H}(U),\tau_w)$ and $(\mathcal{H}(U),\tau_\delta)$ are quasinormable for any balanced open subset of $\lambda(X, Y) \times Z$ where Z is any Banach space with the approximation property,

(e) $(\mathcal{P}(^n\lambda(X, Y) \times Z)), \beta)$ is quasinormable for all n and for any Banach space Z with the approximation property.

Proof. (a)$\Rightarrow$(b). If Y'' is complemented in X'' then $X'/Y^{\perp}$ is complemented in X' and

$$(\lambda(X, Y))'_{\beta} = \lambda'(X', X'/Y^{\perp}) = E, \times Z^N$$

where Z is a topological complement of $X'/Y^{\perp}$ in X'. Hence $\lambda(X, Y)'_{\beta}$ is a product of Banach spaces.

(b)$\Rightarrow$(c). If (b) holds then $((\lambda(X, Y))'_{\beta})'_{\beta} = \lambda(X'', Y'')$ is a direct sum of Banach spaces. By proposition 4.13 and 4.17, (c) is true when $U = E$.

If (d) is true for $U = E$ then trivially (e) is true.

If (e) holds then lemma 4.2 and (4.7) imply that the pair (Y x Y', X x Y') has the polynomial extension property. By lemma 4.18, Y'' is complemented in X'' and hence (e)$\Rightarrow$ (a).

If (e) holds then lemma 4.2 and (4.6) imply that (Y x Y x Z, X x Y x Z) has the polynomial extension property. By theorem 4.6 and proposition 4.11 this implies that (d) $\Rightarrow$ (e) for arbitrary balanced U. This completes the proof.

Remark 4.21. Theorem 4.20 can be strengthened by noticing that we did not use all the hypothesis at all stages. For instance in condition (d) and (e) we can replace Z by the single Banach space Y'. Clearly it suffices in condition (e) to have this condition for one value of $n \geq 2$. The result also shows that if we have quasinormability for one balanced domain then we have it for all balanced domains. The general method of proof can be adapted to more general *CB* spaces of Moscatelli type-spaces of the form $([(X_n)_n, (Y_n)_n])$ and also to the nonsymmetric case-in fact condition (a) shows that we get the same results for both cases.

Example 4.22. It is now simple to give both positive and negative examples. If Y is a closed non-complemented subspace of a reflexive Banach space X (any Banach space not isomorphic to a Hilbert space contains such a subspace) then $(\mathcal{P}(^n\lambda(X, Y) \times Y')), \beta)$ is not quasinormable for any $n \geq 2$.

On the other hand theorems 4.19 and 4.20 show that positive results not covered by proposition 4.13 are rather similar to that given in example 4.14. Further examples are obtained by taking a subspace Y of X such that Y is an M-ideal in X but not isomorphic to an M-summand in X . This occurs for instance if Y is a closed but non-weak* closed ideal $\mathcal{I}$ in a W*-algebra d since in this case $\mathcal{I}^{**}$ is a M-summand in $\mathcal{A}^{**}$ and hence is complemented (for further information on M-ideals and M-summands we referto [36, chapter 3]). Further examples may be found in [16] and [19].

We now turn to the case of a direct sum of Fréchet-Schwartz spaces. In this case we trivially have the extension property but condition (4.3) is not always true (see example 4.26).

Proposition 4.23. *Let* $E = \sum_{n=1}^{\infty} F_n$ *denote a direct sum of Fréchet-Schwartz space and suppose each* F_n *has a Schauder basis and admits a continuous norm. Let* U *denote a balanced open subset of E and suppose p is a* τ_δ *continuous seminorm on* $\mathcal{H}(U)$ *. There exists a positive integer m such that for each positive integer n there exists a compact subset* K_n *of* E_m *satisfying*

$$p(P) \leq ||P||_{K_n} \quad \textit{for all } P \in \mathcal{P}(^nE).$$

In particular, if $f \in \mathcal{H}(U)$ *and* $f|_{U \cap \sum_{j=1}^{m} F_j} = 0$ *then* $p(f) = 0$.

Proof. Let π_n denote the projection from E onto $E_n := \sum_{j=1}^{n} F_j$. We first suppose that for all n there exists n_j and $P_n \in \mathcal{P}(^{n_j}E)$ such that $P_n|_{E_n} = 0$ and $p(P_n) \neq 0$. Since $P_n \circ \pi_m \to P_n$ locally as $m \to \infty$ we may suppose that for each n there exists j_n such that $P_n = P \circ \pi_{j_n}$. Now fix n. The space E_{j_n} is a Fréchet-Schwartz space with continuous norm and Schauder basis. Let T_ℓ denote the projection in E_{j_n} onto the first ℓ coordinates and let

$$Q_\ell(x) = P_n(T_\ell(\pi_{j_n}(x))) \quad \text{for all } x \in E.$$

Since $\tau_0 = \tau_\omega$ on $\mathcal{P}(^{n_j}E_{j_n})$ (see proposition 3.5) it follows that $Q_\ell \to P_n$ as $\ell \to \infty$ in $(\mathcal{P}(^{n_j}E), \tau_\omega)$. Hence we can choose ℓ sufficiently large so that $p(Q_{\ell_n}) \neq 0$. Let $R_n = nQ_{\ell_n}/p(Q_{\ell_n})$. We have $p(R_n) = n$ and $R_n|_{E_n} = 0$ and if W_m is the unit ball of a continuous norm on E_m then $||R_n||_{W_m} < \infty$ for all m.

We consider two possibilities;

(a) there exists a positive integer k such that $R_n \in \mathcal{P}(^kE)$ for an infinite number of n. The method of example 4.1 O(c) can be adapted to show that this gives a contradiction,

(b) there exists an infinite subset M of N such that $n \neq m$ in M imply $n_j \neq m_j$. By proposition 4.10(b), $\sum_{n \in M} P_n \in H(E)$. Since $\sum_{n \in M} p(P_n) = \infty$ this is impossible.

Hence both possibilities lead to the contradiction and we have shown that there exists a positive integer m such that if $f \in \mathcal{H}(U)$ and $f|_{U \cap E_m} = 0$ then p(f) = 0 .

Since $\tau_0 = \tau_w$ on $\mathcal{P}(^nE_m)$ for any positive integers m and n the seminonn $p|_{E_m}$ on $P(^nE_m)$ defined by

$$p|_{E_m}(P) = p(P \circ \pi_m) \quad \text{for all } P \in \mathcal{P}(^nE_m)$$

is well defined and τ_0 continuous. Hence there exists for each n a compact subset K_n of E_m suchthat $p|_{E_m}(P) \leq ||P||_{K_n}$ for all $P \in \mathcal{P}(^nE_m)$. If $P \in \mathcal{P}(^nE)$ then $(P - P \circ \pi_m)|_{E_m} = 0$ and hence

$$p(P) = p(P \circ \pi_m) \leq ||P||_{K_n}$$

for $P \in P(^nE)$ and this completes the proof.

The basis requirement in proposition 4.23 could be avoided if we know the answer to the following question.

If E is a Fréchet space with continuous norm does E contain a neighbourhood basis at the origin, $\mathcal{V}$, such that for all n

$$\bigcap_{V \in \mathcal{V}} \{P \in \mathcal{P}(^nE); ||P||_V < \infty\}$$

is τ_w dense in $P(^nE)$?

It is easily seen that the proof of the above proposition can be adapted to obtain the following.

Proposition 4.24. *If U is a balanced open subset of a direct sum of Fréchet-Schwartz spaces $E = \sum_{n=1}^{\infty} E_n$ each having a Schauder basis and admitting a continuous norm then $\tau_0 = \tau_w$ on $\mathcal{H}(U)$. If $\tau_0 = \tau_\delta$ on $\mathcal{H}(U \cap E_m)$ for all m then $\tau_0 = \tau_\delta$ on $\mathcal{H}(U)$.*

Ifwe let $F_n \approx s$ (the space of rapidly decreasing sequences) then $E \approx \mathcal{D}$ (the space of test functions on R^n) and we recover, in proposition 4.24, a result proved in [12]. With the same hypotheses we have the following.

Corollary 4.25. *$(\mathcal{H}(U), \tau_0)$ is a Schwartz space and $(\mathcal{H}(U), \tau_\delta)$ is an ultrabornological Schwartz space.*

Proof. The result for τ_0 follows either from theorem 1.7 or from theorem 4.7 and propositions 4.23 and 4.24. Since $\tau_0 = \tau_w = \tau_\delta$ on $\mathcal{P}({}^nE)$ the result for τ_0 implies that the bounded subsets of $(\mathcal{P}({}^nE), \tau_\delta)$ are precompact. Lemma 3.4 then implies that the bounded subsets of $(\mathcal{H}(U), \tau_\delta)$ are precompact. By theorem 4.7 and proposition 4.23, $(\mathcal{H}(U), \tau_\delta)$ is quasinormable and this implies that $(\mathcal{H}(U), \tau_\delta)$ is a Schwartz space.

The Fréchet-Schwartz spaces in propositions 4.23, 4.24 and Corollary 4.25 may be replaced by Fréchet-Montel spaces for which $\tau_0 = \tau_w$ on $\mathcal{P}({}^nE_m)$ for all n and m (see for instance corollary 3.5).

Finally we give an example which shows that the continuous norm hypothesis in propositions 4.23 and 4.24 is necessary and the example also shows that condition (4.3) of theorem 4.7 is not always satisfied by strict $\mathcal{LF}$ spaces.

Example 4.26. Let $F_1 = \mathbf{C}^N$ and $F_i = \mathbf{C}$ for $i \geq 2$. Then $E = \sum_{i=1}^{\infty} F_i = \mathbf{C}^N \times \mathbf{C}^{(N)}$ is a direct sum of Fréchet-Schwartz spaces each of which has a Schauder basis. Let

$u_n = (0, \ldots, 1, 0, \ldots)$ in F_1 (1 in the n^{th} position) and let v_n denote a unit vector in F_n for $n \geq 2$

The seminorm

$$p\left(\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!}\right) = \sum_{n=0}^{\infty} \left| \frac{\hat{d}^2 f(0)}{2!}(u_n + v_n) - \frac{\hat{d}^2 f(0)}{2!}(v_n) \right|$$

for $\sum_{n=0}^{\infty} \frac{\hat{d}^n f(0)}{n!} \in \mathcal{H}(U)$, U any balanced open subset of E, is τ_w but not τ_0 continuous ([21]). Hence the conclusion of proposition 4.24 is not valid in this case. The seminorm p does not have the property given in proposition 4.23. On the other hand E is a reflexive A-nuclear space with basis and hence $(\mathcal{H}(E), \tau_0)$, $(\mathcal{H}(E), \tau_w)$ and $(\mathcal{H}(E), \tau_w)$ are all Schwartz (and even nuclear) spaces and so the conclusion of corollary 4.25 is valid for this space (when $U = E$). We refer to [21] and [23, chapters 1 and 5] for further details.

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Seán Dineen
University College Dublin
Belfield
Dublin 4
Ireland