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ON LICHNEROWICZ SMOOTH HOMOTOPY INVARIANTS FOR G-STRUCTURES

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Abstract. *In [10] A. Lichnerowicz introduced a remarkable smooth homotopy invariant $K(\Phi)$ for maps Φ between almost Kähler manifolds. The generalisations of $K(\Phi)$ for maps between Riemannian manifolds with given G-structures are described and some applications are given.*

1. INTRODUCTION

In 1969 A. Lichnerowicz found a smooth homotopy invariant, $K(\Phi)$, of maps $\Phi : M \to N$ between a compact special almost Hermitian manifold M and an almost Kähler manifold N. He defined this invariant in terms of Kähler forms of the manifold as

$$K(\Phi) = \int_M < \omega^M, \Phi^* \omega^N > dV_M.$$

He also showed that $K(\Phi)$ can be expressed by means of two partial energies $E'(\Phi)$ and $E''(\Phi)$ naturally associated to the map Φ, if one refers to the almost complex structures of M and N.

Interesting applications of $K(\Phi)$ were given in [10,11] and by others (see e.g. [5] for a report on this subject).

We were surprised very much in observing that the idea of the construction of $K(\Phi)$ can be applied to many different contexts. Under suitable general hypothesis a homotopy invariant $K_{\xi\eta}(\Phi)$ can be considered for smooth maps $\Phi : (M,g) \to (N,h)$ between Riemannian manifolds which admit «canonically» defined p-forms $\xi \in \Lambda^p M$ and $\eta \in \Lambda^p N$ playing a role of the Kähler 2-form in the complex case (Th. 2.1).

Indeed, we noticed that in the case of Riemannian manifolds with the holonomy group contained in the well known Berger list ([3], p. 301) such forms always exist and can be used in the definition of the homotopy invariant without any additional hypothesis (Th. 2.2).

In particular we show that even in the case of $U(n)$ the main Lichnerowicz invariant is only the first of a series of possible ones

It is also remarkable, although definitively obvious, that in the case of forms ξ and η of maximal degree (equals to the common dimension of the manifolds in question) the invariant $K_{\xi,\eta}(\Phi)$ is expressed by the degree of Φ up to a constant depending on $Vol(M)$.

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As an application of the homotopy invariants introduced in this paper we prove some results on the homotopicity for some classes of smooth maps $\Phi : M \to N$, where M and N are manifolds equipped with G-structures from the Berger list.

The other applications and relations between the homotopy invariants and appropriate partial energies will be a matter of further research.

LICHNEROWICZ-TYPE INVARIANTS FOR G-STRUCTURES

Let $G \subseteq SO(m)$ and $G' \subseteq SO(n)$ be two Lie groups. Suppose that $\eta_0 \in \Lambda^p \mathbf{R}^m$ and $\xi_0 \in \Lambda^p \mathbf{R}^n$ are given forms of degree p which are invariant by G and G' respectively.

Suppose that (M^m, g) and (N^n, h) are smooth, oriented, Riemannian manifolds with given G-structure and G'-structure subordinate to their respective $SO(m)-$ and $SO(n)$-structure.

In the following the manifold M^m will be always assumed to be compact.

Notice that on M^m and N^n there are canonically defined p-forms η and ξ corresponding to η_0 and ξ_0, respectively.

Definition 2.1. *Let $\Phi : (M^m, g) \to (N^n, h)$ be a smooth map. Define*

$$K_{\eta,\xi}(\Phi) = \int_M < \eta, \Phi^* \xi > dV_M .$$

where ξ and η are the p-forms defined above.

Theorem 2.1. *If ξ is closed and η co-closed then $K_{\eta,\xi}(\Phi)$ is a smooth homotopy invariant, i.e. it is constant on the connected components of $C(M, N)$ (space of smooth maps from M^m to N^n endowed with the usual topology of uniform convergence).*

A key point in proving the result of Lichnerowicz and its present generalisation is the following fundamental lemma ([11] p. 358; [5] p. 49).

Homotopy lemma. *Let $\Phi_t : M \to N$ be a smooth family of maps between the manifolds M and N, parametrized by a real number t. If ω is a closed p-form on N then*

$$(\partial/\partial t)(\Phi_t^* \omega) = d[\Phi_t^*(i_{\Psi(t)}\omega)],$$

where $i_{\Psi(t)}\omega$ denotes the interior product of the vector $\Psi(t) := (\partial/\partial t)(\Phi_t)$ with the form ω.

Proof of the theorem 2.1. If «*» denotes the Hodge operator on forms then we have (see, e.g. [12], p. 185)

$$K_{\eta,\xi}(\Phi) = \int_M \Phi^* \xi \wedge *\eta.$$

Let Φ_0 and Φ_1 br two maps from M to N, homotopic through the smooth family $\Phi_t, t \in [0,1]$. Since, by hypotesis $d(*\eta) = 0$, Homotopy Lemma yields

$$\Phi_1^*\xi - \Phi_0^*\xi = \int_{[0,1]} (\partial/\partial t)(\Phi_t^*\xi)dt = \int_{[0,1]} d[\Phi_t^*(i_{\Psi(t)}\omega)]dt = d\rho,$$

where $\rho := \int_{[0,1]} \Phi_t^*(i_{\Psi(t)}\omega)dt$.

Therefore

$$\begin{aligned} K_{\eta,\xi}(\Phi_1) - K_{\eta,\xi}(\Phi_0) &= \int_M < \eta, \Phi_1^*\xi - \Phi_0^*\xi > dV_M = \int_M (\Phi_1^*\xi - \Phi_0^*\xi) \wedge *\eta \\ &= \int_M d\rho \wedge *\eta = \int_M d(\rho \wedge *\eta) = 0. \end{aligned}$$

Then, we obtain $K_{\eta,\xi}(\Phi_1) = K_{\eta,\xi}(\Phi_0)$, as required. ∎

Notice that in the case when $G \subseteq U(m'), G' \subseteq U(n'), m = 2m', n = 2n'$ and $\eta = \omega^M, \xi = \omega^N$, where ω^M and ω^N denote the Kähler forms on M and N, respectively, Theorem 2.1 is nothing but the result of Lichnerowicz [10] (see also [5], p. 48).

Let us return to the general case and assume that G, G', η_0, ξ_0 are as above. The following statement is a special case of Theorem 2.1:

Theorem 2.2. *Let (M^m, g) and (N^n, h) be Riemannian manifolds with holonomy groups G and G', respectively. Suppose that η and ξ are p-forms corresponding to η_0 and ξ_0 ([12], p. 113), which are invariant by the parallel transport. Then $K_{\eta,\xi}(\Phi)$ is a smooth homotopy invariant of the smooth maps $\Phi : (M^m, g) \to (N^n, h)$.*

Proof. It is sufficient to remark that since η and ξ are invariant by the parallel transport then they are closed and co-closed, respectively (cf. [12], p. 189). ∎

3. GENERALISATION OF LICHNEROWICZ INVARIANT IN THE COMPLEX CASE

Assume that $G = U(m)$ and $G' = U(n)$. Recall that the standard Kähler 2-form of $\mathbf{R}^{2m} \cong \mathbf{C}^m$ ([8])

$$\omega_0 = (i/2)(dz^1 \wedge dz^{\overline{1}} + \ldots + dz^m \wedge dz^{\overline{m}})$$

is $U(m)$-invariant. Also, the r-th exterior powers of $\omega_0, (\omega_0)^r := \omega_0 \wedge \ldots \wedge \omega_0$, are $U(m)$-invariant for $r = 1, \ldots, m$.

Let (M^{2m}, g) and (N^{2n}, h) be two almost Hermitian manifolds equipped with the Kähler forms ω^M and ω^N, respectively. For a smooth map $\Phi : (M^{2m}, g) \to (N^{2n}, h)$ we define

$$K_r(\Phi) = \int_M < (\omega^M)^r, (\Phi^*\omega^N)^r > dV_M, \qquad r = 1, \ldots, m.$$

Theorem 3.1. *Let* (M^{2m}, g) *and* (N^{2n}, h) *be almost Kähler manifolds. Then* $K_r(\Phi)$ *is a smooth homotopy invariant for* $r = 1, \ldots, m$.

Proof. By the hypothesis $d(\omega^M)^r = d(\omega^N)^r = 0$ for any positive integer r. By the below Lemma 3.1 we also have $d[*(\omega^M)^r] = 0$, then we apply Theorem 2.1. ∎

Remark 3.1. If we weaken the hypothesis and assume that (M^{2m}, g) is a special almost Hermitian manifold (i.e. $d(*\omega^M) = 0$) then for $r = 1$ we obtain nothing but just the Lichnerowicz invariant ([5], [10]).

Lemma 3.1. *Let* ω *be the fundamental 2-form on an almost Kähler manifold of real dimension 2m. Then*

$$*(\omega)^r = [r!/(m-r)!]\omega^{m-r}, \quad for\ r = 1, \ldots, m.$$

Proof. It is analogous to the proof of Lemma 4.1 in the quaternionic case. Let us only recall the well known identity (see, e.g. [5]):

$$\omega^m = m!vol(\mathbf{R}^{2m}),$$

where $vol(\mathbf{R}^{2m})$ is the volume form corresponding to the canonical Euclidean metric of $\mathbf{R}^{2m}$ with the canonical orientation. ∎

Now, let us give some applications of the invariants introduced above, in view also of the extension to other cases. Some of them are essentially known (see [5]) but this formulation seems to be important because it does not use the notions of the partial energies.

Definition 3.1. *Let* (N^{2n}, h) *be an almost Hermitian manifold with an almost complex structure* I *and* M^{2m} *any smooth, orientable, 2m-dimensional manifold* $(m \leq n)$. *Suppose that* $\Phi : M^{2m} \to N^{2n}$ *is a smooth immersion. We say that* (M^{2m}, Φ) *is an* immersed almost complex submanifold *of* (N^{2n}, h) *if the following condition is satisfied:*
for every point $p \in M^{2m}$ *the vector space* $\Phi_*(T_pM^{2m})$ *is a* I*-invariant subspace of* $T_{\Phi(p)}N^{2n}$.

Remark 3.2. Notice that the manifold M^{2m} (as above), endowed by the Riemannian metric $g := \Phi^*h$, admits a unique almost complex structure I' such that $I \circ \Phi_* = \Phi_* \circ I'$. With respect to I' the (M^{2m}, g) is an almost Hermitian manifold and Φ is holomorphic (Cf. [5], p. 48).

Proposition 3.1. *Let* (N^{2n}, h) *be an almost Kähler manifold with an almost complex structure* I *and* (M^{2m}, Φ) *be a compact immersed almost complex submanifold of* (N^{2n}, h). *Then, the immersion* $\Phi : M^{2m} \to N^{2n}$ *is not homotopic to a constant map. In particular, if*

M^{2m} is homeomorphic to the sphere $S^2(m = 1)$ the map Φ defines a non-trivial element in the homotopy group $\pi_2(N^{2n})$.

Proof. By Remark 3.2, consider M^{2m} as an almost Hermitian manifold. Then $\omega^M = \Phi^*\omega^N$ is the Kähler form on M^{2m}. Hence (M^{2m}, g) is almost Kähler and at every point $p \in M^{2m}$ we get $< \omega^M, \Phi^*\omega^N >_p = ||\omega^M||_p^2 = m$. In particular, $K_1(\Phi) \neq 0$ and the statement follows by the Theorem 3.1. ∎

Definition 3.3 ([7]). *Let (N^{2n}, h) be an almost Hermitian manifold with an almost complex structure I and M^{2m} any smooth 2m-dimensionam manifold. Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth immersion. We say that (M^{2m}, Φ) is an* immersed Lagrangian submanifold *of (N^{2n}, h) if at every point $p \in M^{2m}$ the vector subspaces $\Phi_*(T_pM^{2m})$ and $I\Phi_*(T_pM^{2m})$ are totally orthogonal in $T_{\Phi(p)}N^{2n}$.*

Proposition 3.2. *Let (N^{2n}, h) be an almost Kähler manifold. Suppose that M^{2m} is a compact, oriented, 2m-dimensional manifold $(m \leq n)$. Let $\Phi_1 : M^{2m} \to N^{2n}$ and $\Phi_2 : M^{2m} \to N^{2n}$ be two immersions such that (M^{2m}, Φ_1) and (M^{2m}, Φ_2) are almost complex and Lagrangian, respectively. Then Φ_1 and Φ_2 can not be homotopic.*

Proof. By Remark 3.2, we can consider M^{2m} as an almost Kähler manifold with respect to the Riemannian metric $g := \Phi^*h$ and the almost complex structure I' naturally induced by Φ_1^*. By the Definition 3.3 we get $\Phi_2^*(\omega^N) = 0$ (see [7], p. 87). Hence $K_1(\Phi_2) = 0$. On the other hand, by Proposition 3.1, we have $K_1(\Phi_1) \neq 0$. Then the statement follows immediately by the Theorem 3.1. ∎

The following propositions are simple generalisations of the known results (see e.g. [5]; Proposition 8.9, Corollary 8.12 and Proposition 8.21).

Proposition 3.3. *Let (M^{2m}, g) and (N^{2n}, h) be almost Kähler manifolds. Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth map such that the cohomology class $[\Phi^*(\omega^N)^r]$ equals $c_r[(\omega^M)^r]$ for some $r \in \{1, \ldots, \min(m, n)\}$ and some $c_r \in \mathbf{R}$. Then*

$$K_r(\Phi) = c_r[m!r!/(m - r)!]Vol(M).$$

In particular, $c_r \neq 0$ if and only if $K_r(\Phi) \neq 0$.

Proof. By hypothesis $\Phi^*(\omega^N)^r = c_r(\omega^M)^r + d\rho$ for some $\rho \in \Lambda^{r-1}M$. Then by Lemma 3.1 we get

$$\begin{aligned}K_r(\Phi) &= \int_M \Phi^*(\omega^N)^r \wedge *(\omega^M)^r = c_r \int_M (\omega^M)^r \wedge *(\omega^M)^r + \int_M d\rho \wedge *(\omega^M)^r \\ &= c_r[r!/(m-r)!] \int_M (\omega^M)^m + [r!/(m-r)!] \int_M d[\rho \wedge (\omega^M)^{m-r}] \\ &= c_r[m!r!/(m-r)!] Vol(M)\end{aligned}$$

as required. ∎

Proposition 3.4. *Let (M^{2m}, g) and (N^{2m}, h) be two connected, compact, almost Kähler manifolds of real dimension 2m. Assume that $\Phi : M^{2m} \to N^{2m}$ is a smooth map. Then*

$$K_m(\Phi) = (m!)^2 \deg(\Phi) \cdot Vol(N),$$

where $\deg(\Phi)$ *denote the degree of* Φ *(see, e.g. [1], p. 273).*

Proof. By the definition of $\deg(\Phi)$ we have

$$\int_M \Phi^*(dV_N) = \deg(\Phi) \cdot Vol(N).$$

Then, by Lemma 3.1 we get

$$\begin{aligned}\int_M \Phi^*(dV_N) &= (1/m!) \int_M \Phi^*(\omega^N)^m = [1/(m!)^2] \int_M \Phi^*(\omega^N)^m \wedge *(\omega^M)^m \\ &= [1/(m!)^2] K_m(\Phi)\end{aligned}$$

as required. ∎

Proposition 3.5. *Let (M^{2m}, g) and (N^{2n}, h) be almost Kähler manifolds. Suppose that $\Phi : M^{2m} \to N^{2n}$ is a smooth map such that the cohomology class $[\Phi^*\omega^N]$ equals $c[\omega^M]$ for some $c \in \mathbf{R}$. If rank $(d\Phi) < 2m$, then $K_r(\Phi) = 0$ for $r = 1, \ldots, m$.*

4. HOMOTOPY INVARIANTS IN THE QUATERNIONIC CASE AND APPLICATIONS

Let $Sp(m)$ be the group of the automorphisms of the right quaternionic vector space $\mathbf{H}^m$ which are unitary with respect to the canonical Hermitian product: $\zeta \cdot \zeta' = \sum \zeta^\alpha \overline{\zeta'^{\alpha}}, \zeta, \zeta' \in \mathbf{H}^m$.

Recall that the enhancement $Sp(m) \cdot Sp(1)$ of $Sp(m)$ is the group of $\mathbf{R}$-linear automorphisms $T_{A,q}$ of $\mathbf{H}^m$ defined by

$$T_{A,q}(\zeta) := A\zeta \cdot q, \quad \zeta \in \mathbf{H}^m,$$

where "." means the quaternionic multiplication by a unitary quaternion $q \in Sp(1)$ and A is a transformation from $Sp(m)$.

The group $Sp(m) \cdot Sp(1)$ preserves the Euclidean product $g_o(\zeta,\zeta') = Re(\zeta \cdot \overline{\zeta'})$ in $\mathbf{H}^m \cong \mathbf{R}^{4m}$, hence it is a subgroup of $SO(4m)$. Moreover, the multiplication by a quaternion $\lambda \in \mathbf{H}$ is preserved up to a conjugation in $\mathbf{H}$, i.e. if $\omega_\lambda(\zeta) := \zeta \cdot \lambda$ then $T_{A,q}[\omega_\lambda(\zeta)] = \omega_{q^{-1}\lambda q}[T_{A,q}(\zeta)]$.

So, if $(1, i_1, i_2, i_3)$ is a normal base of $\mathbf{H}$ (i.e. $i_1^2 = i_2^2 = i_3^2 = -1$ and $i_1 i_2 i_3 = -1$), defined up to the conjugation, then the right multiplication by $-i_1, -i_2$ and $-i_3$ determines a triple (I_1, I_2, I_3) of complex structures of $\mathbf{R}^{4m} = \mathbf{H}^m$ satisfying the following conditions:

$$I_1^2 = I_2^2 = I_3^2 = -Id, \qquad I_1 I_2 I_3 = -Id,$$

where Id stands for the identity mapping in $\mathbf{R}^{4m}$.

Any two such triples (I_1, I_2, I_3) and (I_1', I_2', I_3') are related by a transformation

$$I_h' = \sum_{1,3} c_{hk} I_k, \qquad h = 1,2,3,$$

with $(c_{hk}) \in SO(3)$.

Definition 4.1. *The* standard enhanced quaternionic structure *of* $\mathbf{H}^m$ *is the 3-dimensional subspace* Q_o *of* $End_R\mathbf{H}^m$ *generated by (any one of) such triple* (I_1, I_2, I_3), *called an* admissible hypercomplex base *of* Q_o *(we also write* $(I_1, I_2, I_3) \in Q_o$*)*

Let $(I_1, I_2, I_3) \in Q_o$. Consider the 2-forms ω_1, ω_2 and ω_3 defined by

$$\omega_k(\zeta,\zeta') := g_o(I_k\zeta,\zeta'), \qquad k = 1,2,3.$$

Definition 4.2 (cf. *[3], p. 419). Define*

$$\Omega_o := \omega_1 \wedge \omega_1 + \omega_2 \wedge \omega_2 + \omega_3 \wedge \omega_3.$$

Remark 4.1. Ω_o *is a well defined 4-form, indipendent of* $(I_1, I_2, I_3) \in Q_o$ *and invariant by the group* $Sp(m) \cdot Sp(1)$. *Moreover, it is non-degenerate, being*

$$(\Omega_o)^m = (2m+1)!vol(\mathbf{R}^{4m})$$

(see, e.g. [2], p. 85).

Definition 4.3. *1)A Riemannian manifold* (M^{4m}, g) *with a given* $Sp(m) \cdot Sp(1)$*-structure is called an* almost Hermitian quaternionic manifold. *The 4-form* Ω *corresponding to* Ω_o *is called the* fundamental 4-form *(or* Kähler form*) of* (M^{4m}, g). *Yet, we will denote by* Q *the 3-dimensional subbundle of* $EndTM^{4m}$ *corresponding to* Q_o.
2) An almost quaternionic Hermitian manifold (M^{4m}, g) *is called an* almost quaternionic Kähler manifold *if the fundamental 4-form* Ω *is closed.*
3) An almost quaternionic Hermitian manifold (M^{4m}, g) *is called a* quaternionic Kähler manifold *if its holonomy group is contained in* $Sp(m) \cdot Sp(1)$.

Remark 4.2. The most important example of an almost quaternionic Kähler manifold is the quaternionic projective space $\mathbf{HP}^m$ with standard metric (cf. [14] or [4]).

Remark 4.3. 1) Since $Sp(1) \cdot Sp(1) \cong SO(4)$, every 4-dimensional oriented Riemannian manifold is naturally an almost quaternionic Kähler manifold. 2) Recently, see [15], p. 136, it was shown that if $4m > 8$ then the holonomy group of an almost quaternionic Kähler manifold is contained in $Sp(m) \cdot Sp(1)$.

Proposition 4.1. *Let* (M^{4m}, g) *and* (N^{4n}, h) *be almost quaternionic Kähler manifolds* (M^{4m} *being compact). Suppose that* $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ *is a smooth map. Then*

$$K_\Omega^r(\Phi) := \int_M < (\Omega^M)^r, (\Phi^*\Omega^N)^r > dV_M,$$

are smooth homotopy invariants for $r = 1, \dots, m$.

Proof. The statement follows directly by the Theorem 2.1. It is enough to notice that the forms $(\Omega^M)^r$ and $(\Omega^N)^r$ are evidently closed and co-closed by the following lemma. ∎

Lemma 4.1. *Let* Ω *be the fundamental 4-form on an almost quaternionic Kähler manifold of real dimension 4m. Then*

$$*(\Omega^r) = a_r \Omega^{m-r}, \quad r = 1, \dots, m$$

where $a_r = \|\Omega^r\|^2/(2m+1)!$. *In particular,* $a_1 = 12m/(2m)!$.

Proof. We follow the idea of Bonan ([4], p. 59). Observe that $*(\Omega_o^r)$ and Ω_o^{m-r} are both of the same degree and invariant by the group $Sp(m) \cdot Sp(1)$ for $r = 1, \dots, m$. Suppose that $*(\Omega_o^r) \neq k\Omega_o^{m-r}$ for any real number k. Then, it would imply that on $\mathbf{HP}^m$ (whose holonomy group is $Sp(m) \cdot Sp(1)$) the harmonic forms $*(\Omega^{\mathbf{HP}})^r$ and $(\Omega^{\mathbf{HP}})^{m-r}$ are not proportional. But it contradicts the well known fact that the Betti number $b_{4(m-r)}(\mathbf{HP}^m)$ is equal to 1. Hence

$$*(\Omega_o^r) = a_r \Omega_o^{m-r}, \quad r = 1, \dots, m$$

for some coefficients a_r which can be computed as follows. By Remark 4.1 we get

$$(\Omega_o^r) \wedge *(\Omega_o^r) = a_r (\Omega_o)^m = a_r (2m+1)!\, vol(\mathbf{R}^{4m}).$$

On the other hand by the definition of Hodge operator $''*''$ we have

$$(\Omega_o^r) \wedge *(\Omega_o^r) = \|\Omega_o^r\|^2 vol(\mathbf{R}^{4m}).$$

Hence

$$a_r = \|\Omega_o^r\|^2 / (2m+1)!.$$

Finally, the indicated value of a_1 follows from the formula $\|\Omega_o\|^2 = 12m(2m+1)$, that we will establish in the proof of Proposition 4.2. ∎

Definition 4.4. *Let* (M^{4m}, g) *and* (N^{4n}, h) *be two almost quaternionic Hermitian manifolds and* $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ *a smooth map.* Φ *is called* Q-holomorphic *if the following condition is satisfied:*

for every point $p \in M^{4m}$ and each hypercompex base (I_1', I_2', I_3') belonging to Q_p^M there exists a hypercomplex base $(I_1, I_2, I_3) \in Q_{\Phi(p)}^N$ such that

$$I_a(\Phi_*)_p = (\Phi_*)_p I_a' \qquad for\ a = 1, 2, 3 \tag{4.1}$$

Example 4.1. Any 4-dimensional, oriented, Riemannian manifold (M^4, g) can be considered as an almost quaternionic Kähler manifold (see Remark 4.3). A diffeomorphism $\Phi : (M^4, g) \to (M^4, g)$ is Q-holomorphic iff it is conformal and preserves the fixed orientation.

Remark 4.3. Let (N^{4n}, h) be an almost quaternionic Hermitian (resp. Kähler) manifold and M^{4m} any smooth, orientable, $4m(\leq 4n)$-dimensional manifold. Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion an the following condition is satisfied:

for every point $p \in M^{4m}$ the space $\Phi_*(T_p M^{4m})$ is a quaternionic subspace of $T_{\Phi(p)} N^{4n}$.

Consider on M^{4m} the Riemannian metric $g := \Phi^* h$. Then, there is a unique (natural) quaternionic structure Q^M on M^{4m} such that (M^{4m}, g) is an almost quaternionic Hermitian (resp., Kähler) manifold endowed with the fundamental 4-form $\Omega^M := \Phi^* \Omega^N$ and $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic map. (M^{4m}, Φ) is called an *immersed almost quaternionic submanifold* of (N^{4n}, h).

The following proposition is a partial extension to the quaternionic case of the results of Lichnerowicz ([11], p. 379; see also [5], Cor. 8.20).

Proposition 4.2. *Let (M^{4m}, g) and (N^{4n}, h) be two almost quaternionic Kähler manifolds. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a Q-holomorphic isometric immersion. Then*

$$K^1_\Omega(\Phi) = 12m(2m+1)Vol(M)$$

In particular Φ can not be homotopic to a constant map.

Proof. Let $p \in M^{4m}$. Chose orthonormal bases of the form

$$(e_1, I'_1 e_1, I'_2 e_1, I'_3 e_1, \dots, e_m, I'_1 e_m, I'_2 e_m, I'_3 e_m),$$

and

$$(f_1, I_1 f_1, I_2 f_1, I_3 f_1, \dots, f_n, I_1 f_n, I_2 f_n, I_3 f_n),$$

in $T_p M^{4m}$ and $T_{\Phi(p)} N^{4n}$ respectively, where (I'_1, I'_2, I'_3) is a hypercomplex base of Q^M_p, (I_1, I_2, I_3) is a hypercomplex base of $Q^N_{\Phi(p)}$ and the suitable condition (4.1) holds.

Suppose that Φ is Q-holomorphic. It is clear that $\Omega^M = \Phi^*\Omega^N$ and $< \Omega^M, \Phi^*\Omega^N >_p = \|\Omega^M\|^2_p$. Notice that the only components of Ω^M_p which are different from 0 are those that correspond (up to the permutations) to the 4-ples of vectors

$$(4.2) \qquad (e_t, I_a e_t, e_s, I_a e_s) \quad ,(I_b e_t, I_c e_t, I_b e_s, I_c e_s) \quad for\ t,s = 1,\dots,m\,, t \neq s$$

and

$$(4.3) \qquad (e_t, I_a e_t, I_b e_s, I_c e_s) \quad for\ t,s = 1,\dots,m,$$

for any circular permutation (a,b,c) of $(1,2,3)$. It is easy to see that, up to the permutations, there are $3m(m-1)$ different components of the type (4.2), $3m(m-1)$ different components of the type (4.3) with $t \neq s$ and m different components of the type (4.3) with $t = s$. By a simple calculation we get

$$\Omega(e_t, I_a e_t, e_s, I_a e_s) = \Omega(I_b e_t, I_c e_t, I_b e_s, I_c e_s) = \Omega(e_t, I_a e_t, I_b e_s, I_c e_s) = 2$$
$$for\ t \neq s$$

and

$$\Omega(e_t, I_a e_t, I_b e_t, I_c e_t) = 6\,.$$

Since $\|\Omega^M\|^2_P = 3m(m-1)2^2 + 3m(m-1)2^2 + m6^2 = 12m(2m+1)$, then by integrating we get the required formula. ∎

Remark 4.4. The proposition 4.2, with a slight modification, holds when Φ is a conformal immersion.

Let us recall that 4-dimensional immersed quaternionic submanifolds M^4 of a quaternionic Kähler manifold $(N^{4n}, h), n > 1$, are totally geodesic (see e.g. [16]) and semi-conformally flat. In the case when the scalar curvature of (N^{4n}, h) is positive the only possible types for compact M^4 are $\mathbf{HP}^1 \cong S^4$ and $\mathbf{CP}^2$ ([13]).

Corollary 4.5. *Every immersed quaternionic submanifold of (N^{4n}, h) which is isometric to $\mathbf{HP}^1$ defines a non-trivial element in the group $\pi_4(N^{4n})$.*

Remark 4.5. The above fact was well known in the case when (N^{4n}, h) is the quaternionic projective space $\mathbf{HP}^m$ and $\Phi : \mathbf{HP}^1 \to \mathbf{HP}^m$ is a canonical immersion of a quaternionic projective line (see, e.g. [17], p. 30).

The following three propositions are simple extensions of those considered in the complex case, Section 3.

Proposition 4.3. *Let (M^{4m}, g) and (N^{4n}, h) be almost quaternionic Kähler manifolds (M^{4m} being compact). Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map such that the cohomology class $[\Phi^*(\Omega^N)^r]$ equals $c_r[(\Omega^M)^r]$ for some $c_r \in \mathbf{R}, r \in \{1, \ldots, \min(n, m)\}$. Then*

$$K^r_\Omega(\Phi) = c_r a_r (2m+1)! Vol(M).$$

Proposition 4.4. *Let (M^{4m}, g) and (N^{4m}, h) be two connected, compact, almost quaternionic Kähler manifolds of the same dimension $4m$. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map. Then*

$$K^m_\Omega(\Phi) = [(2m+1)!]^2 \deg(\Phi) Vol(N).$$

Proposition 4.5. *Let (M^{4m}, g) and (N^{4m}, h) be two almost quaternionic Kähler manifolds with M^{4m} compact. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ is a smooth map such that $[\Phi^*\Omega^N] = c[\Omega^M]$ for some $c \in \mathbf{R}$ and $rank(d\Phi) < 4m$. Then $K^r_\Omega(\Phi) = 0$ for $r = 1, \ldots, m$.*

Corollary 4.6. *Let (M^{4m}, g) and (N^{4n}, h) be two almost quaternionic Kähler manifolds. Suppose that $\Phi : (M^{4m}, g) \to (N^{4n}, h) (m \geq n)$ is a Q-holomorphic submersion (not necessarily Riemannian). If $\dim H^4(M^{4m}, \mathbf{R}) = 1 (e.g. M^{4m} = \mathbf{HP}^m)$ then $m = n$.*

To prove the Corollary 4.6 we need the following:

Remark 4.6. Let $\Phi : (M^{4m}, g) \to (N^{4n}, h)$ be a Q-holomorphic submersion. Then on M^{4m} there exists a metric g' such that (M^{4m}, g') is an almost quaternionic Kähler manifold and $\Phi' := \Phi : (M^{4m}, g') \to (N^{4n}, h)$ is a (Q-holomorphic) Riemannian submersion. The construction of g' is the following.

Let V and H be the vertical and horizontal distributions defined by Φ on M^{4m} of dimension $4(m-n)$ and $4n$, respectively: $V := Ker\Phi_*, H := V^\perp$. Note that for each point $p \in M^{4m}$ the spaces V_p and H_p are quaternionic subspaces of T_pM^{4m}. We define the new metric g' by requiring that the restrictions of g' and g to V_p coincide and that the restriction of Φ_* to H_p is an isometry, being V_p and H_p still orthogonal.

Proof of Corollary 4.6. By Remark 4.6 we can suppose that Φ is a Riemannian submersion, say Q- holomorphic. Then, analogously like in the proof of the Proposition 4.2, we get

$$K^1_\Omega(\Phi) = 12n((2n+1)Vol(M).$$

If $m > n$, the proposition 4.5 would give $K^1_\Omega(\Phi) = 0$, that is a contradiction. ∎

The following definition and proposition are analogous to the Definition 3.3 and Proposition 3.2.

Definition 4.5 ([7]). *Let (N^{4n}, h) be an almost quaternionic Hermitian manifold and M^{4m} any smooth $4m$-dimensional manifold. Suppose that $\Phi : M^{4m} \to N^{4n}$ is a smooth immersion. We say that (M^{4m}, ϕ) is an* immersed Lagrangian submanifold *of (N^{4n}, h) if at every point $p \in M^{4m}$ and any hypercomplex base $(I_1, I_2, I_3) \in Q^N_{\Phi(p)}$ the four subspaces $\Phi_*(T_pM^{4m}), I_a\Phi_*(T_pM^{2m}) (a = 1, 2, 3)$ are totally orthogonal in $T_{\Phi(p)}N^{4n}$.*

Proposition 4.6. *Let (N^{4n}, h) be an almost quaternionic Kähler manifold. Suppose that M^{4m} is a compact, oriented, $4m$-dimensional manifold $(m \leq n)$. Let $\Phi_1 : M^{4m} \to N^{4n}$ and $\Phi_2 : M^{4m} \to N^{4n}$ be two immersions such that (M^{4m}, Φ_1) and (M^{4m}, Φ_2) are almost quaternionic and Lagrangian, respectively. Then Φ_1 and Φ_2 can not be homotopic.*

We omit the proof, which is similar to that one of Proposition 3.2.

5. SPECIAL CASES: HOLONOMY GROUPS G_2 AND $SPIN(7)$

In this section we consider G_2-and $Spin(7)$-structures. We must point out that the existence of *compact* manifolds with holonomy group G_2 or $Spin(7)$ is still an open problem. Nevertheless, if the holonomy group is contained in G_2 or $Spin(7)$ one can easily construct some examples.

Let $(\mathbf{R}^7, g_0)$ be the standard Euclidean 7-dimensional space with the standard orientation and metric g_0. It is well known that by an appropriate identification of $\mathbf{R}^7$ with the imaginary octonions (endowed with the octonion multiplication) one can define an alternating vector-cross-product $P : \mathbf{R}^7 \times \mathbf{R}^7 \to \mathbf{R}^7$ satisfying the following properties:

1) $P(X,Y)$ is orthogonal to both vectors X and Y,

2) $||P(X,Y)||^2 = ||X||^2||Y||^2 - [g_0(X,Y)]^2$.

(Such a vectorial product is unique up to the isometries).

The Lie group G_2 is characterised as the subgroup of $SO(7)$ which preserves P (see, e.g. [6], [7]).

The 3-form $\alpha_0 \in \Lambda^3\mathbf{R}^7$ defined by

$$\alpha_0(X,Y,Z) := g_0(X,P(Y,Z)), \qquad X,Y,Z \in \mathbf{R}^7$$

is invariant by G_2.

Every 7-dimensional, oriented, Riemannian manifold (M^7,g) with a given G_2-structure becomes naturally equipped with a fundamental 3-form α, corresponding to α_0 (we can also consider a fundamental 4-form $\beta := *\alpha$).

The following proposition is an immediate application of the Theorem 2.2.

Proposition 5.1. *Let (M^7,g) and (N^7,h) be Riemannian manifolds with holonomy groups contained in G_2. Suppose that M^7 is compact and $\Phi : M^7 \to N^7$ is a smooth map. Then*

$$K_\alpha(\Phi) := \int_M < \alpha^M, \Phi^*\alpha^N > dV_M \quad \text{and} \quad K_\beta(\Phi) := \int_M < \beta^M, \Phi^*\beta^N > dV_M$$

are smooth homotopy invariants.

Recall that a 3-dimensional vector subspace V^3 of $\mathbf{R}^7$ is called *special* if it is closed under P. Then it admits an orthonormal base of the form $(X,Y,P(X,Y))$ (see, e.g. [6]). Similarly, we say that a 4-dimensional vector subspace V^4 of $\mathbf{R}^7$ is *special* if it is orthogonal to a 3-dimensional special subspace. Correspondingly, if (M^7,g) is a Riemannian manifold with a given G_2-structure one can introduce a notion of a *special submanifold* of dimension 3 or 4.

Proposition 5.2. *Let (N^7,h) be a Riemannian manifold with holonomy group contained in G_2. Then an immersion $\Phi : M^3 \to N^7$ (resp. $\Phi : M^4 \to N^7$) of a compact, oriented, special 3-dimensional (resp. 4-dimensional) submanifold of (N^7,h) can not be homotopic to a constant map. In particular, if M^3 (resp. M^4) is homeomorphic to the sphere S^3 (resp. S^4) then the map Φ defines a non-trivial element in the homotopy group $\pi_3(N^7)$ (resp. $\pi_4(N^7)$).*

Proof. For simplicity consider only the 3-dimensional case. Observe that for any 3-dimensional oriented Riemannian manifold (M^3,g) and for any smooth map $\Phi : M^3 \to N^7$ we can consider the quantity

$$K_{dV_M,\alpha}(\Phi) := \int_M < dV_M, \Phi^*\alpha^N > dV_M,$$

which is homotopy invariant by Theorem 2.1.

Now, let M^3 be a special submanifold immersed by Φ. Consider on M^3 the Riemannian metric $g := \Phi^*\, h$. Then $dV_M = \Phi^*\alpha^N$ (eventually by exchanging the orientation of M^3) and we have

$$< dV_M, \Phi^*\alpha^N >_p = 1.$$

Hence, $K_{dV_M,\alpha}(\Phi)$ is different from zero, then Φ can not be homotopic to a constant map.∎

We can obtain similar results for the group $Spin(7)$. Let us recall its definition.

By the appropriate identification of $\mathbf{R}^8$ with the Cayley algebra of octonions one can define an alternating three-linear product:

$$P^\wedge : \mathbf{R}^8 \times \mathbf{R}^8 \times \mathbf{R}^8 \to \mathbf{R}^8$$

satisfying the following properties:

1) $P^\wedge(X_1, X_2, X_3)$ is orthogonal to each vector X_1, X_2 and X_3,

2) $\|P^\wedge(X_1, X_2, X_3)\|^2 = \det[g_0(X_i, X_j)]$, $i, j = 1, 2, 3$.

(There are two of such products up to isometries of $\mathbf{R}^8$).

The Lie group $Spin(7)$ is the authomorphism group of $\mathbf{R}^8$ which preserves such fixed product $P^\wedge$.

The 4-form $\gamma_0 \in \Lambda^4 R^8$ defined by

$$\gamma_0(X, Y, Z, T) = g_0(X, P^\wedge(Y, Z, T))$$

is obviously invariant by the group $Spin(7)$ as well the form $\gamma_0' := *\gamma_0$.

Every 8-dimensional oriented Riemannian manifold (M^8, g) with a given $Spin(7)$-structure is naturally equipped with a fundamental 4-form γ. In analogy to the case of G_2 we have the following:

Proposition 5.3. *Let (M^8, g) and (N^8, h) be Riemannian manifolds with the holonomy group $Spin(7)$. Suppose that M^8 is compact and $\Phi : M^8 \to . N^8$ is a smooth map. Then*

$$K_\gamma(\Phi) := \int_M < \gamma^M, \Phi^*\gamma^N > dV_M \text{ and } K_{\gamma'}(\Phi) := \int_M < \gamma'^M, \Phi^*\gamma'^N > dV_M$$

are smooth homotopy invariant.

Remark 5.1. One can also introduce a notion of *special subspace* of $\mathbf{R}^8$ with respect to $P^\wedge$, i.e. a 4-dimensional subspace closed under $P^\wedge$, to find an analogue of Proposition 5.2.

Remark 5.2. The hypothesis on (M^7, g) (resp. (M^8, g)) in the Proposition 5.1 (resp. 5.3) can be weakened by supposing that a G_2-structure (resp. $Spin(7)$-structure) is given and $d(*\alpha^M) = 0$ (resp. $d(*\gamma^M) = 0$).

REFERENCES

[1] M. Berger, B. Gostiaux, *Geometrie différentielle*, Ed. Armand Colin, Paris, 1972.

[2] A.L. Besse, *Manifolds all of whose geodesics are closed*, Ergeb. der Math., **13**, Springer, 1978.

[3] A.L. Besse, *Einstein manifolds* , Ergeg. der Math., **3** Folge Band **10**, Springer-Verlag, Berlin and New York, 1987 .

[4] E. Bonan, *Sur les G-structures de type quaternionen*, Cahiers de Top. et Geom. Diff. **9**, pp. 389-461, 1967.

[5] J. Eelles, L. Lémaire, *Selected topics in harmonic maps*, Conference Board of the Mathematical Sciences, American Mathematical Society, Providence, Rhode Island, 50, 1983.

[6] A. Gray, *Vector cross product on manifolds*, Trans. Amer. Math. Soc., **14**, pp. 465-504, 1968.

[7] R. Harvey, H.B. Lawson Jr., *Calibrated Geometries*, Acta Math., **148**, pp. 47-157, 1982.

[8] S. Kobayashi, K. Nomizu, *Foundations of differential geometry*, Vol. I-II, Intersciences, 1963.

[9] V.Y. Kraines, *Topology of quaternionic manifolds*, Trans. Amer. Math. Soc., **122**, pp. 357-367, 1966.

[10] A. Lichnerowicz, *Applications harmoniques et variétés Kähleriennes*, Rend. Sem. Mat. Fis. di Milano, **39**, pp. 1-12, 1969.

[11] A. Lichnerowicz, *Applications harmoniques et variétés Kähleriennes*, Symp. Math. III, pp. 341-402, Bologna, 1970.

[12] A. Lichnerowicz, *Théorie globale des connexions et del groupes d'holonomie*, Roma, Ed. Cremonese, 1955.

[13] S. Marchiafava, *Su alcune sottovarietà che ha interesse considerare in una varietà Kähleriana quaternionale*, Rend. di Mat., Serie VII, Vol. 10, pp. 493-529, Roma, 1990.

[14] E. Martinelli, *Variétés à structure quaternionienne généralisées*, Rev. Roumaine Math. Pures Appl., **10**, pp. 915-922, 1965.

[15] S.M. Salamon, *Riemannian geometry and holonomy groups*, Pitman Res. Notes Math. Ser., **201**, Oxford, 1989.

[16] H. Tasaki, *Quaternionic submanifolds in quaternionic symmetric spaces*, Tōhoku Math. J., **38**, pp. 513-538, 1986.

[17] W.T. Wu, *Sur les classes caractéristiques des structures fibrés sphériques*, Publications de l'Institut de Mathematiques de l'Université de Strassbourg, Ed. Hermann, Vol. XI, Paris, 1952.

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