Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **32** (2012) no. 2, 115–121. doi:10.1285/i15900932v32n2p115

Weakly normal subgroups and classes of finite groups

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Received: 1.3.2012; accepted: 13.7.2012.

Abstract. A subgroup K of a group G is said to be weakly normal in G if $K^g \leq N_G(K)$ implies $g \in N_G(K)$. In this paper we establish certain characterizations of solvable PST-groups using some weakly normal subgroups.

Keywords: Weakly normal subgroups, solvable PST-groups

MSC 2000 classification: 20D10, 20D20

1 Introduction and statements of results.

All groups are finite.

A subgroup H of a group G is said to be weakly normal in G provided that if $g \in G$ and $H^g \leq N_G(H)$, then $g \in N_G(H)$. It is not difficult to see that H is weakly normal in G if and only if whenever $H \lhd \langle H, H^g \rangle$, then $H \lhd \langle H, g \rangle$. For each group G let $WN(G)$ denote the set of all weakly normal subgroups of G. Notice that if H is normal in G or $N_G(H) = H$, then $H \in WN(G)$. Thus $WN(G)$ contains all maximal subgroups of G. Moreover, $WN(G)$ contains all the pronormal subgroups of G but there are weakly normal subgroups of G which are not pronormal (see [2, p. 28]). Also the join of two pronormal subgroups of a group need not be pronormal but belongs to $WN(G)$ (see [2, p. 28]).

In [5] the authors introduce the concept of a $\mathcal{H}$-subgroup of a group and prove a number of very interesting results about such subgroups. A subgroup X of a group G is called a $\mathcal{H}$-subgroup provided that $X^g \cap N_G(X) \leq X$ for all $g \in G$. The authors of [5] denote by $\mathcal{H}(G)$ the set of all $\mathcal{H}$-subgroups of G. Assume that $X \in \mathcal{H}(G)$ and $X^g \leq N_G(X)$. Then $g \in N_G(X)$ so that $\mathcal{H}(G) \subseteq WN(G)$. In particular, by Proposition 3 of [5], a Sylow subgroup of a normal subgroup of G belongs to $WN(G)$. Moreover, a Hall subgroup of G also is a WN-subgroup of G.

Example 1. ([3]). Let $G = S_4$ and $H = \langle (1\,2\,3\,4) \rangle$. Then $N_G(H) = \langle (1\,2\,3\,4), (1\,3) \rangle$, and if $g = (1\,2\,3)$, $H^g = (1\,4\,2\,3)$, whence $H^g \cap N_G(H) = \langle (1\,2)(3\,4) \rangle \not\leq H$.

Thus H is not an $\mathcal{H}$-group of G. Note that $N_G(H)$ has a unique cyclic group of order 4. Consequently, if $H^x \leq N_G(H)$, x an element of G, then $H^x = H$ and H is weakly normal in G.

However, Ballester-Bolinches and Esteban-Romero [3] proved the following very interesting results.

Theorem 1. *Let H be a weakly normal subgroup of the supersolvable group G. Then*

(a) If H is a p-group, p a prime, then H is an $\mathcal{H}$-subgroup of G.

(b) If every subgroup of H is weakly normal in G, then H is an $\mathcal{H}$-subgroup of G.

A group G is called a T-group if whenever H and K are subgroups of G such that $H \lhd K \lhd G$, then $H \lhd G$. W. Gaschütz [7] introduced the class of finite T-groups and characterized solvable T-groups. He proved that a solvable group G is a T-group if and only if the nilpotent residual L of G is a normal abelian Hall subgroup of G such that G acts by conjugation on L as power automorphisms and G/L is a Dedekind group.

Bianchi, Mauri, Herzog, and Verardi [5], Ballester-Bolinches and Esteban-Romero [3], and Csörgö and Herzog [6] provide a number of extensions of Gaschütz's [7] characterization of solvable T-groups.

Theorem 2. ([3, 5]). *Let G be a group. The following statements are equivalent:*

(a) G is a solvable T-group.

(b) Every subgroup of G belongs to $\mathcal{H}(G)$.

(c) Every subgroup of G belongs to $WN(G)$.

Theorem 3. ([6]) *Let G be a solvable group. Then G is a T-group if and only if there exists a subgroup L of G such that*

(a) L is a normal Hall subgroup of G.

(b) G/L is a Dedekind group.

(c) Every subgroup of L of prime power order belongs to $\mathcal{H}(G)$.

Throughout this paper $S_p(G)$ denotes the set of cyclic subgroups of a group G of prime order or of order 4. Also $\overline{S_p}(G)$ denotes the set of subgroups of G of prime power order. Note that $S_p(G) \subseteq \overline{S_p}(G)$.

Theorem A. Let G be a group. Then

(a) If $S_p(G) \subseteq WN(G)$, then G is supersolvable.

(b) If $\overline{S_p}(G') \subseteq WN(G)$, then G is supersolvable.

Theorem B. Let G be a group. Then $S_p(G) \leq WN(G)$ if and only if there exist subgroups L and D of G such that

(a) $G = L \rtimes D$, the semidirect product of L by D.

(b) L and D are nilpotent Hall subgroups of G.

(c) $S_p(L) \subseteq WN(G)$.

(d) $S_p(D) \subseteq WN(D)$.

Theorem C. A group G is a solvable T-group if and only if G has subgroups L and D such that

(a) $G = L \rtimes D$, the semidirect product of L by D.

(b) L and D are nilpotent Hall subgroups of G.

(c) $\overline{S_p}(L) \subseteq WN(G)$.

(d) $\overline{S_p}(D) \subseteq WN(D)$.

One of the purposes of this paper is to determine if a theorem like Theorem C might be proven for some classes of groups related to solvable T-groups. Such classes include solvable PST- and PT-groups. Let G be a group. A subgroup H of G is said to permute with a subgroup K of G if HK is a subgroup of G. H is said to be S-permutable if it permutes with all the Sylow subgroups of G. Kegel [8] showed that an S-permutable subgroup of G is subnormal in G. Kegel's result generalized Ore's result [11] that a permutable subgroup of G is subnormal in G. A group G is called a PST-(resp. PT-)group if S-permutability (resp. permutability) is a transitive relation in G, that is, if $H \subseteq K$ are subgroups of G such that H is S-permutative (resp. permutable) in K and K is S-permutable (resp. permutable) in G, then H is S-permutable (resp. permutable) in G. By Kegel's (resp. Ore's) result G is a PST-(resp. PT-)group if every subnormal subgroup of G is S-permutable (resp. permutable) in G. PST-groups have been studied in detaiil in [1, 2, 4].

Solvable PST- and PT-groups have been characterized by Agrawal [1]. He proved the following theorem.

Theorem 4. *Let G be a group. Then* *(a) G is a solvable PST-group if and only if it has a normal abelian Hall subgroup L such that G/L is nilpotent and G acts by conjugation on L as power automorphisms.*

(b) G is a solvable PT-group if and only if it is a solvable PST-group with Iwasawa Sylow subgroups.

G. Zacher [13] proved part (b) of Theorem 4 in 1964. An Iwasawa group is one in which every subgroup is permutable.

Theorem D. A group G is a solvable PST-group if and only if it has subgroups L and D such that

(a) $G = L \rtimes D$.

(b) L and D are nilpotent Hall subgroups of G.

(c) $\overline{S_p}(L) \subseteq WN(G)$.

Theorem E. Let L be a normal Hall subgroup of G such that

(a) G/L is a PST-group.

(b) Every subnormal subgroup of L is weakly normal in G.

Then G is a PST-group.

Theorem F. A group G is a solvable PST-group if and only if it has a normal nilpotent Hall subgroup L such that G/L is a solvable PST-group and $\overline{S_p}(L) \subseteq WN(G)$.

2 Preliminary results.

By Lemma 2.1 of [10] and Lemma 3 of [3] we have

Lemma 1. *Let H, K and N be subgroups of G. Then*

(a) If $H \leq K$ and $H \in WN(G)$, then $H \in WN(K)$.

(b) Let $N \leq H$. Then $H \in WN(G)$ if and only if $H/N \in WN(G/N)$.

(c) If $H \trianglelefteq \trianglelefteq K$ and $H \in WN(G)$, then $H \trianglelefteq K$.

Lemma 2. *Let G be a group.*

(a) Let M and L be subgroups of G such that $ML = LM$, $(|L|, |M|) = 1$ and $G = MN_G(L)$. Then $L \in WN(G)$.

(b) Let $N \lhd G$ and P a p-subgroup of G such that $(p, |N|) = 1$. If $P \in WN(G)$, then $PN \in WN(G)$ and $PN/N \in WN(G/N)$.

Proof. The exact same proofs of Lemmas 5 and 6 of [6] establish parts (a) and (b) of Lemma 2. QED

The next lemma contains results established by Li in [9].

Lemma 3. *Let G be a group, p a prime and P a Sylow p-subgroup of G.*

(a) If $p > 2$ and every minimal subgroup of P lies in the center of $N_G(P)$, then G is p-nilpotent.

(b) If $p = 2$ and every cyclic subgroup of P of order 2 or 4 is normal in $N_G(P)$, then G is 2-nilpotent.

(c) If G possesses a normal 2-complement N and if every minimal subgroup of any Sylow subgroup R of N is normal in $N_G(R)$, then G is supersolvable.

3 Proofs of the Theorems.

Proof of Theorem A. (a) Assume that $S_p(G) \subseteq WN(G)$. Let P be a Sylow p-subgroup of G, p a prime and let $U \leq P$ be a subgroup of order p or 4 if $p = 2$. Then $U \in S_p(G) \subseteq WN(G)$ and $U \lhd \lhd N_G(P)$. By part (c) of Lemma 1 $U \lhd N_G(P)$. Hence for every prime p the subgroups of $S_p(P)$ are normal in $N_G(P)$. Thus G is supersolvable by parts (b) and (c) of Lemma 3. Thus (a) is true. We also note (a) follows from Theorem 3.1 of [10].

Assume that $\overline{S_p}(G') \subseteq WN(G)$. We may assume that $G' \neq 1$. Note that $S_p(G') \subseteq \overline{S_p}(G') \subseteq WN(G')$ by part (a) of Lemma 1 and hence G' is supersolvable. Clearly G is solvable. Let M be a minimal normal subgroup of G contained in G'. M is an elementary abelian q-group for some prime q. Let $\langle x \rangle$ be a subgroup of M of order q. Then $\langle x \rangle \in WN(G)$ and by part (c) of Lemma 1 $\langle x \rangle \lhd G$. Consider $G/\langle x \rangle$ and note $G'/\langle x \rangle = (G/\langle x \rangle)'$. By part (b) of Lemma 1 and part (b) of Lemma 2 $\overline{S_p}(G'/\langle x \rangle) \leq WN(G/\langle x \rangle)$. Hence, by induction, $G/\langle x \rangle$ is supersolvable and so G is also supersolvable. Thus (b) is also true. QED

Proof of Theorem B. Assume that $S_p(G) \subseteq WN(G)$. By part (a) of Theorem A G is supersolvable. Thus $S_p(G) \subseteq WN(G) \subseteq \mathcal{H}(G)$ by part (a) of Theorem 1. By Theorem 10 of [6] there are subgroups L and D of G which satisfy L and D are Hall nilpotent subgroups of G such that $L \lhd G$ and $G = L \rtimes D$. Also $S_p(L)$ consists of normal subgroups of G and $S_p(D)$ consists of normal subgroups of D. Therefore, (a)–(d) holds.

Conversely, assume that G has Hall subgroups L and D which satisfy (a)–(d). By part (c) of Lemma 1 the subgroups in $S_p(L)$ are all normal in G and the subgroups in $S_p(D)$ are normal in D. Thus, by Theorem 10 of [6], $S_p(G) \subseteq \mathcal{H}(G)$. But $\mathcal{H}(G) \subseteq WN(G)$.

This completes the proof of Theorem B. QED

Proof of Theorem C. Assume G satisfies (a)–(d) and let X be a q-subgroup of L, q a prime. By (b) L is nilpotent and so $X \lhd \lhd G$. By (c) $X \in WN(G)$ and hence, by part (c) of Lemma 1, $X \lhd G$. Thus L is a Dedekind group. Now let us assume X has order q. Then L/X is a Hall nilpotent subgroup of G/X and $\overline{S_p}(L/X) \subseteq WN(G/X)$ by part (b) of Lemma 1. Also DX/X is a nilpotent Hall subgroup of G/X and it is isomorphic to D. Thus $\overline{S_p}(DX/X) \subseteq WN(DX/X)$. This means that G/X satisfies (a)–(d) with respect to L/X and DX/X. By induction on the order of G, G/X is supersolvable and so G is supersolvable. This means L is abelian and G acts on L as power automorphisms. By Gaschütz's Theorem [7] it is enough to show D is a Dedekind group. By part (a) of Theorem 1 $\overline{S_p}(L) \subseteq \mathcal{H}(G)$ and $\overline{S_p}(D) \subseteq \mathcal{H}(D)$. Let r be a prime divisor of the order of D and let R be the Sylow r-subgroup of D. Note that $\overline{S_p}(R) = \mathcal{H}(R)$. Let Y be a subgroup of R. Then $Y \lhd R$ by part (c) of Lemma 1. Thus D is a Dedekind group.

Conversely, assume that G is a solvable T-group. Then, by Gaschütz's Theorem [7], the nilpotent residual L of G is a normal abelian Hall subgroup of G on which G acts as power automorphisms and G/L is a Dedekind group. Let D be a system normalizer of G. By the Gaschütz, Shenkman and Carter Theorem, Theorem 9.2.7 of [12, p. 264], $G = L \rtimes D$. Thus (a) and (b) hold. Since all the subgroups of L are normal in G it follows that $\overline{S_p}(L) \subseteq WN(G)$. Likewise, since D is a Dedekind group, $\overline{S_p}(D) \subseteq WN(D)$. This completes the proof. QED

Proof of Theorem D. Assume that G is a solvable PST-group and let L be the nilpotent residual of G. By part (a) of Theorem 4 L is an abelian normal Hall subgroup of G on which G acts by conjugation as power automorphisms. Let D be a system normalizer of G. By Theorem 9.2.7 of [12, p. 264] $G = L \rtimes D$. Note that D is a nilpotent Hall subgroup of G. Let X be a subgroup of L. Then $X \lhd G$ and so $\overline{S_p}(L) \subseteq WN(G)$. Therefore, (a), (b) and (c) are true.

Conversely, assume (a), (b) and (c) hold for the group G. Then G has nilpotent Hall subgroups L and D such that $L \lhd G$ and $G = L \rtimes D$. Clearly G is solvable. Note that G/L is nilpotent and hence is a PST-group. Let X be a subgroup of L of prime power order. Then $X \in \overline{S_p}(L) \subseteq WN(G)$ and $X \lhd \lhd G$. By part (c) of Lemma 1 $X \lhd G$. Therefore, by Theorem 2.4 of [1] G is a solvable PST-group. QED

Proof of Theorem E. Let L be a normal Hall subgroup of G such that G/L is a PST-group and every subnormal subgroup of L belongs to $WN(G)$. Let X be a subnormal subgroup of L. Then X is subnormal in G and hence X is normal in G by part (c) of Lemma 1. Thus G is a PST-group by Theorem 2.4 of [1]. QED

Proof of Theorem F. This follows from Theorem 2.3 of [1] and Theorem E. QED

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