C-totally real pseudo-parallel submanifolds of S-space forms

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Abstract. Let $\mathcal{M}(c)$ be a $(2n + s)$-dimensional S-space form of constant $f$-sectional curvature $c$ and $M$ be an $n$-dimensional C-totally real, minimal submanifold of $\mathcal{M}(c)$. We prove that if $M$ is pseudo parallel and $Ln - \frac{1}{4}(n(c + 3s) + c - s) \geq 0$, then $M$ is totally geodesic.

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1 Introduction

Given an isometric immersion $F : M \rightarrow \mathcal{M}$, let $h$ be the second fundamental form and $\nabla$ the Van der Weather- Bortolloti connection of $M$. Then Deprez defined the immersion to be semi-parallel if

$$R(X,Y).h = (\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} - \nabla_{[X,Y]}) h = 0,$$  \hspace{1cm} (1)

holds for any vectors $X, Y$ tangent to $M$. Deprez mainly paid attention to the case of semi-parallel immersion in a real space form [see [9], [10]]. Later, Lumiste showed that a semi-parallel submanifold is the second order envelop of the family of parallel submanifolds [14].

In [11], authors obtained some results on hypersurfaces in 4-dimension space form $N^4(c)$ satisfying the curvature condition

$$\mathcal{R}.h = LQ(g,h).$$  \hspace{1cm} (2)

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The submanifolds satisfying (1.2) are called pseudo-parallel [1, 2].

In [1], authors have shown that if $F$ is a pseudo-parallel immersion with $H(p) = 0$ and $L(p) - c \geq 0$, then the point $p$ is a geodesic point of $M$.

In the present paper, we generalize their results for the case of $M$, that is a submanifold of $S$-space form $\mathbb{M}(c)$ of constant $f$--sectional curvature $c$.

We prove the following result:

**Theorem 1.** Let $\mathbb{M}(c)$ be a $(2n+s)$--dimensional $S$-space form of constant $f$--sectional curvature $c$ and $M$ be an $n$-dimensional $C$-totally real, minimal submanifold of $\mathbb{M}(c)$. If $M$ is pseudo-parallel and $Ln - \frac{1}{4}(n(c+3s)+c-s) \geq 0$, then $M$ is totally geodesic.

2 Preliminaries

Let $(M, g)$ be an $n$-dimensional ($n \geq 3$) connected semi-Riemannian manifold of class $C^\infty$. We denote by $\nabla$, $R$ and $S$ the Levi-Civita connection, Riemannian curvature tensor, and Ricci tensor of $(M, g)$, respectively. The Ricci operator $Q$ is defined by $g(QX,Y) = S(X,Y)$, where $X, Y \in \chi(M)$, $\chi(M)$ being the Lie algebra of vector fields on $M$. Now we define endomorphisms $R(X,Y)$ and $X \Lambda_A Y$ of $\chi(M)$ by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, \quad (3)$$

$$(X \Lambda_A Y)Z = A(Y,Z)X - A(X,Z)Y, \quad (4)$$

where $X, Y, Z \in \chi(M)$ and $A$ is a symmetric $(0,2)$-tensor.

A concircular curvature tensor $Z$ is defined by

$$Z(X,Y) = R(X,Y) - \frac{\kappa}{n(n-1)}(X \Lambda_A Y),$$

where $\kappa$ is scalar curvature of $M$.

Let $F : M \to \mathbb{M}(c)$ be an isometric immersion of an $n$-dimensional Riemannian manifold $M$ into a $(2n+1)$--dimensional real space form $\mathbb{M}(c)$. We denote by $\nabla$ and $\nabla$ the Levi-Civita connection of $M$ and $\mathbb{M}(c)$, respectively. Also, we denote by $N(M)$ its normal bundle. Then for vector fields $X, Y$ which are tangent to $M$, the second fundamental form $h$ is given by the formula $h(X,Y) = \nabla_X Y - \nabla_X Y$. Furthermore, for $\xi \in N(M)$, $A_\xi : TM \to TM$ denotes the Weingarten operator in the $\xi$--direction, $A_\xi X = \nabla^\perp_X \xi - \nabla_X \xi$, where $\nabla^\perp$ denotes normal connection on $M$. The second fundamental form $h$ and $A_\xi$ are related by $\bar{g}(h(X,Y), \xi) = g(A_\xi X, Y)$, where $g$ is the induced metric of $\bar{g}$ for any vector fields $X, Y$ tangent to $M$. The mean curvature vector $H$ of $M$ is defined as

$$H = \frac{1}{n} \text{tr}(h).$$
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The covariant derivative $\nabla h$ of $h$ is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^\perp(h(Y, Z)) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

where $\nabla h$ is a normal bundle valued tensor of type $(0, 3)$ and is called the third fundamental form of $M$. The equation of Codazzi implies that $\nabla h$ is symmetric and hence

$$(\nabla_X h)(Y, Z) = (\nabla_Y h)(X, Z) = (\nabla_Z h)(X, Y).$$

Here, $\nabla$ is called the Van der Weeder - Bortolotti connection of $M$. If $\nabla h = 0$, then $F$ is called parallel [13].

The basic equations of Gauss and Ricci are

$$g(R(\nabla_X Y) Z, W) = cg(\nabla_X Y Z) + g(h(\nabla_X Y, Z), h(Y, W)) - g(h(\nabla_X Y, Z), h(Y, W)),$$

where $R$ is curvature tensor belonging to the connection $\nabla$.

3 S-space forms

Let $\mathcal{M}$ be a $(2m + s)$-dimensional framed metric manifold [18] (or almost r-contact metric manifolds [17]) with a framed metric structure $(f, \xi_\alpha, \eta^\alpha, g)$, $\alpha \in \{1, 2, \ldots, s\}$, where $f$ is a $(1, 1)$ tensor field defining an $f$-structure of
rank 2m, \( \xi_1, \xi_2, \ldots, \xi_s \) are vector fields, \( \eta^1, \eta^2, \ldots, \eta^s \) are 1-forms and \( \bar{g} \) is a Riemannian metric on \( \overline{M} \) such that for all \( X, Y \in T\overline{M} \) and \( \alpha, \beta \in \{1, 2, \ldots, s\} \),

\[
f^2 = -I + \eta^\alpha \otimes \xi_\alpha, \quad \eta^\alpha(\xi_\alpha) = \delta^\alpha_\beta, \quad f(\xi_\alpha) = 0, \quad \eta^\alpha \alpha f = 0, \tag{11}
\]

\[
\bar{g}(fX, fY) = \bar{g}(X, Y) - \sum_\alpha \eta^\alpha(X)\eta^\alpha(Y), \tag{12}
\]

\[
\Omega(X, Y) \equiv \bar{g}(X, fY) = -\Omega(Y, X), \quad \bar{g}(X, \xi_\alpha) = \eta^\alpha(X). \tag{13}
\]

A framed metric structure is an S-structure [3] if \([f, f] + 2d\eta^\alpha \otimes \xi_\alpha = 0 \) and \( \Omega = \eta^\alpha \otimes \xi_\alpha \) for all \( \alpha \in \{1, 2, \ldots, s\} \). When \( s = 1 \), a framed metric structure is an almost contact metric structure, while an S-structure is a Sasakian-structure. When \( s = 0 \), a framed metric structure is an almost Hermitian structure, while an S-structure is Käehler structure. If a framed metric structure on \( \overline{M} \) is an S-structure, then it is known [3] that

\[
(\nabla_X f)Y = \sum_\alpha (\bar{g}(fX, fY)\xi_\alpha + \eta^\alpha(Y) f^2 X), \tag{14}
\]

\[
\nabla \xi_\alpha = -f, \quad \alpha \in \{1, 2, \ldots, s\}. \tag{15}
\]

The converse may also be proved. In case of Sasakian structure (i.e. \( s = 1 \)) (3.4) implies (3.5). In Kähler case (i.e. \( s = 0 \)), we get \( \nabla f = 0 \). For \( s > 1 \), examples of S-structure are given in [3] [4] [5].

A plane section in \( T_p \overline{M} \) is called a \( f \)-section if there exists a vector \( X \in T_p \overline{M} \) orthogonal to \( \xi_1, \xi_2, \ldots, \xi_s \) such that \( \{X, fX\} \) span the section. The sectional curvature of a \( f \)-section is called a \( f \)-sectional curvature. It is known that [14] in an S-manifold of constant \( f \)-sectional curvature \( c \)

\[
\bar{R}(X, Y)Z = \sum_{\alpha, \beta} \{\eta^\alpha(X)\eta^\beta(Z)f^2 Y - \eta^\alpha(Y)\eta^\beta(Z)f^2 X
\]

\[
- \bar{g}(fX, fY)\eta^\alpha(Y)\xi_\beta + \bar{g}(fY, fZ)\eta^\alpha(X)\xi_\beta
\]

\[
+ \frac{(c + 3s)}{4} \{ -\bar{g}(fY, fZ)f^2 X + \bar{g}(fX, fZ)f^2 Y \}
\]

\[
+ \frac{(c - s)}{4} \{ \bar{g}(X, fZ)fY - \bar{g}(Y, fZ)fX + 2\bar{g}(X, fY)fZ \}, \tag{16}
\]

for all \( X, Y, Z \in T\overline{M} \), where \( \bar{R} \) is curvature tensor of \( \overline{M} \). An S-manifold of constant \( f \)-sectional curvature \( c \) is called an S-space form \( \overline{M}(c) \).

A submanifold \( M \) of an S-space form \( \overline{M}(c) \) is called a C-totally real submanifold if and only if \( f(T_x M) \subset T_x^+ M \), for all \( x \in M(T_x M \) and \( T_x^+ M \) are respectively the tangent space and normal space of \( M \) at \( x \)). When \( \xi_\alpha \) is tangent to \( M \), \( M \) is a C-totally real submanifold if and only if \( \nabla_X \xi_\alpha = 0 \), for all
X ∈ M, α ∈ {1, 2, . . . , s}, where ∇ is the connection on M induced from Levi-Civita connection ∇ on M. It is to see that the C-totally real submanifolds M of M are submanifolds with ξα ∈ T⊥M.

We already know that [1] if M is an n-dimensional C-totally real submanifold of a (2m+s)−dimensional S-space form M(c), then following statements are equivalent:

(i) M is totally geodesic.
(ii) M is of constant curvature K = \frac{1}{4}(c+3s).
(iii) The Ricci tensor S = \frac{1}{4}(n-1)(c+3s)g.
(iv) The scalar curvature κ = \frac{1}{4}n(n-1)(c+3s).

Following the argument as in [11], we can prove

**Theorem 2.** Let M be a minimal, C-totally real submanifold of an S-space form M(c), then

\[ \kappa > \frac{n^2(n-2)}{2(2n-1)}(c+3s), \]

implies that M is totally geodesic.

Following the argument as in [10], we can prove:

**Proposition 1.** If M is an n-dimensional C-totally real submanifold of an S-space form M(c). Then the following conditions are equivalent:

(i) M is minimal.
(ii) The mean curvature vector H of M is parallel.

4 Main Results

**Theorem 3.** Let M(c) be a (2n+s)−dimensional S-space form of constant f-sectional curvature c and M be an n-dimensional C-totally real, minimal submanifold of M(c). If M is pseudo parallel and Ln − \frac{1}{4}(n(c+3s) + c - s) ≥ 0, then M is totally geodesic.

**Proof.** Let M be an n-dimensional C-totally real submanifold of a (2n+s)−dimensional S-space form M(c) of constant f-sectional curvature c. We choose an orthonormal basis \{e_1, e_2, . . . , e_n, fe_1 = e_1^*, . . . , fe_n = e_n^*, e_{n+1}^* = ξ_1, . . . , e_{n+s}^* = ξ_s\}. Then for 1 ≤ i, j ≤ n, n + 1 ≤ α ≤ 2n + s, the components of second fundamental form h are given by

\[ h_{ij}^α = g(h(e_i, e_j), e_α). \] (17)
Similarly, the components of first and second covariant derivative of $h$ are given by

$$h^\alpha_{ijk} = \nabla_{e_k} h^\alpha_{ij},$$

(18)

and

$$h^\alpha_{ijkl} = \nabla_{e_l \nabla_{e_k} h^\alpha_{ij}} = \nabla_{e_l} h^\alpha_{ij},$$

(19)

respectively. It is well known that

$$h^*_{ij} = h^*_{kj} = h^*_{ji}, \quad h^{(n+1)*} = 0.$$}

If $F$ is pseudo-parallel, then by definition, the condition

$$\mathcal{R}(e_l, e_k).h = L[(e_l \Lambda g_e)k]h$$

(20)

is fulfilled where

$$[(e_l \Lambda g_e)k](e_i, e_j) = -h((e_l \Lambda g_e)k)e_i - h(e_i, (e_l \Lambda g_e)e_k)e_j,$$

(21)

for $1 \leq i, j, k, l \leq n$.

Now using (2.2) in (4.5), we get

$$(e_i, e_j) = -g(e_k, e_i)h(e_l, e_j) + g(e_l, e_i)h(e_k, e_j)$$

$$- g(e_k, e_j)h(e_l, e_i) + g(e_j, e_l)h(e_k, e_i).$$

(22)

By (2.9) we have

$$(\mathcal{R}(e_l, e_k).h)(e_i, e_j) = (\nabla_{e_l} \nabla_{e_k} h)(e_i, e_j) - (\nabla_{e_k} \nabla_{e_l} h)(e_i, e_j).$$

(23)

Making use of (4.1), (4.3), (4.6) and (4.7), the pseudo-parallelity condition (4.4) gives us

$$h^\alpha_{ijkl} = h^\alpha_{ijkl} - L\{\delta_{ki} h^\alpha_{lj} - \delta_{li} h^\alpha_{kj} + \delta_{kj} h^\alpha_{il} - \delta_{lj} h^\alpha_{ki}\},$$

(24)

where $g(e_i, e_j) = \delta_{ij}$ and $1 \leq i, j, k, l \leq n$, $n + 1 \leq \alpha \leq 2n + s$.

Recall that the Laplacian $\triangle h^\alpha_{ij}$ of $h^\alpha_{ij}$ is defined by

$$\triangle h^\alpha_{ij} = \sum_{i,j,k=1}^{n} h^\alpha_{ijkl},$$

(25)

Then we obtain

$$\frac{1}{2} \triangle(||h||^2) = \sum_{i,j,k,l=1}^{n} \sum_{\alpha=n+1}^{2n+s} h^\alpha_{ij} h^\alpha_{ijkl} + ||\nabla h||^2,$$

(26)
From equations (4.14) and (4.15), we have respectively. In view of (4.1) and (4.3), we obtain are the square of the length of second and third fundamental forms of $M$, respectively. In view of (4.1) and (4.3), we obtain

$$h_{ij}^a h_{ijkk}^a = g(h(e_i, e_j), e_\alpha)g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), e_\alpha) = g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), g(h(e_i, e_j), e_\alpha), e_\alpha) = g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), h(e_i, e_j)).$$

Therefore, due to (4.13), equation (4.10) becomes

$$\frac{1}{2} \Delta(||h||^2) = \sum_{i,j,k=1}^n g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), h(e_i, e_j)) + ||\nabla h||^2. \tag{30}$$

Further, by the use of (4.4), (4.6) and (4.7), we get

$$g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), h(e_i, e_j)) = g((\nabla_{e_k} \nabla_{e_k} h)(e_i, e_j), h(e_i, e_j)) - g(e_k, e_j)g(h(e_k, e_i), h(e_i, e_j)) + g(e_k, e_i)g(h(e_j, e_k), h(e_i, e_j)) - g(e_k, e_k)g(h(e_i, e_j), h(e_i, e_j)). \tag{31}$$

From equations (4.14) and (4.15), we have

$$\frac{1}{2} \Delta(||h||^2) = \sum_{i,j,k=1}^n \left[ g((\nabla_{e_k} \nabla_{e_k} h)(e_k, e_i), h(e_i, e_j)) - L \left\{ g(e_i, e_j)g(h(e_k, e_i), h(e_i, e_j)) - g(e_k, e_j)g(h(e_k, e_i), h(e_i, e_j)) + g(e_k, e_i)g(h(e_j, e_k), h(e_i, e_j)) - g(e_k, e_k)g(h(e_i, e_j), h(e_i, e_j)) \right\} \right] + ||\nabla h||^2. \tag{32}$$

Further by definitions

$$||h||^2 = \sum_{i,j=1}^n g(h(e_i, e_j), h(e_i, e_j)),$$

$$H^\alpha = \sum_{k=1}^n h_{kk}^\alpha,$$

$$||H||^2 = \frac{1}{n^2} \sum_{\alpha=n+1}^{2n+s} (H^\alpha)^2,$$
and after some calculations, we get
\[ \frac{1}{2} \Delta (||h||^2) = \sum_{i,j=1}^{n} \sum_{\alpha=n+1}^{2n+s} h_{ij}^{\alpha} (\nabla_{e_i} \nabla_{e_j} H^\alpha) - L\{n^2||H||^2 - n||h||^2\} + ||\nabla h||^2. \] (33)

Using minimality condition, equation (4.17) reduces to
\[ \frac{1}{2} \Delta (||h||^2) = Ln||h||^2 + ||\nabla h||^2. \] (34)

Now, using the arguments as Blair has shown in [9], we have
\[ \frac{1}{2} \Delta (||h||^2) = ||\nabla h||^2 - \sum_{\alpha,\beta=n+1}^{2n+s} \left\{ [T_r(A_\alpha o A_\beta)]^2 + ||[A_\alpha, A_\beta]||^2 \right\} + \frac{1}{4}(n(c + 3s) + c - s)||h||^2. \] (35)

From (4.18) and (4.19), we have
\[ 0 = (Ln - \frac{1}{4}(n(c + 3s) + c - s)||h||^2 + \sum_{\alpha,\beta=n+1}^{2n+s} \left\{ [T_r(A_\alpha o A_\beta)]^2 + ||[A_\alpha, A_\beta]||^2 \right\}, \]
if \( Ln - \frac{1}{4}(n(c + 3s) + c - s) \geq 0 \), then \( T_r(A_\alpha o A_\beta) = 0 \).

In particular, \( ||A_\alpha||^2 = T_r(A_\alpha o A_\beta) = 0 \), then \( h = 0 \) and hence \( M \) is totally geodesic.

**Corollary 1.** Let \( \overline{M}(c) \) be a \((2n+s)\)-dimensional S-space form of constant \( f \)-sectional curvature \( c \) and \( M \) be an \( n \)-dimensional C-totally real, minimal submanifold of \( \overline{M}(c) \). If \( M \) is semi-parallel (i.e. \( R.h = 0 \)) and \( n(c+3s)+c-s \leq 0 \), then it is totally geodesic.

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References


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