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Products of Locally Supersoluble Groups

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Abstract. A group G is said to be FC-*hypercentral* if every non-trivial homomorphic image of G contains some non-trivial element having only finitely many conjugates. It is proved that if the FC-hypercentral group $G = AB = AC = BC$ is factorized by three locally supersoluble subgroups A,B and C, and the commutator subgroup of G is nilpotent, then G is locally supersoluble.

Keywords: Locally supersoluble group; FC-hypercentral group

MSC 2000 classification: 20F16

1 Introduction

Almost fifty years ago, O.H. Kegel [5] proved that if a finite group G has a triple factorization of the form

$$G = AB = AC = BC,$$

where A and B are nilpotent subgroups and C is a supersoluble subgroup, then G is supersoluble. This result was later generalized by F.G. Peterson replacing supersolubility by the assumption that the subgroup C belongs to a saturated formation $\mathfrak{F}$ containing all finite nilpotent groups and proving that in this case G itself is an $\mathfrak{F}$-group (see [1], Theorem 2.5.10). Peterson also produced an example to show that a finite group $G = AB = AC = BC$, factorized by a nilpotent subgroup A and two supersoluble subgroups B and C, need not be supersoluble. As in many problems concerning supersoluble groups, the main obstacle here is the behaviour of the commutator subgroup. In fact, although there exist finite non-supersoluble groups which are the product of two supersoluble normal subgroups, R. Baer [3] proved that if G is a finite group with nilpotent commutator subgroup, then every collection of supersoluble normal subgroups of G generates a supersoluble subgroup.

It follows from Baer's theorem that any finite group with nilpotent commutator subgroup contains a largest supersoluble normal subgroup. This can be applied to show that if a finite group G with nilpotent commutator subgroup has a triple factorization $G = AB = AC = BC$, where A,B and C are supersoluble subgroups, then G itself is supersoluble. The situation is much more complicated for infinite groups, as Y.P. Sysak constructed groups which are not locally supersoluble but have a triple factorization by abelian subgroups (see [1], Section 6.1); observe that such groups are metabelian by the famous theorem of Itô. On the other hand, the above result on finite factorized groups has been extended in [4] to the case of groups with finite abelian section rank. The aim of this short paper is to prove that a similar result also holds for another class of groups satisfying a suitable finiteness condition.

A group G is said to be *FC-hypercentral* if every non-trivial homomorphic image of G contains some non-trivial element having only finitely many conjugates. It should be observed that in the example of Sysak quoted above every non-trivial element has infinitely many conjugates. FC-hypercentral groups (which can be defined equivalently by means of the *upper FC-central series* of a group) are known to be a large class of infinite groups with many relevant properties (see for instance [7] Part 1, Chapter 4); in particular, it is easy to show that any locally supersoluble FC-hypercentral group is hypercyclic (i.e. it has an ascending normal series with cyclic factors). Our main result here is the following.

Theorem *Let the FC-hypercentral group $G = AB = AC = BC$ be the product of three locally supersoluble subgroups A, B, C. If the commutator subgroup G' of G is nilpotent, then G is locally supersoluble.*

Observe finally that a similar problem for groups with a triple factorization by locally nilpotent subgroups has been studied in [2].

Most of our notation is standard and can be found in [7]; for properties concerning factorized groups, we refer to the monograph [1].

2 Proof of the theorem

A famous theorem by N. Itô states that any group $G = AB$ which is the product of two abelian subgroups is metabelian (see [1], Theorem 2.1.1). This result has been for a long time the only significant information on arbitrary products of groups. More recently, D.J.S. Robinson and S.E. Stonehewer [8] have obtained information on the chief factors of products of abelian groups, that will be crucial in the proof of our main theorem. As a consequence of their result, it turns out that any group $G = AB = AC = BC$ with a triple factorization by abelian subgroups satisfies a generalized nilpotency condition.

Lemma 1. *Let the group $G = AB$ be the product of two abelian subgroups A and B. Then every chief factor of G is centralized either by A or by B.*

Corollary 1. *Let the group $G = AB = AC = BC$ be the product of three abelian subgroups A, B, C. Then all chief factors of G are central.*

The following lemma is a result of D.H. McLain [6], showing that finitely generated FC-hypercentral groups are close to nilpotent groups.

Lemma 2. *Let G be a finitely generated FC-hypercentral group. Then G contains a nilpotent subgroup of finite index.*

Lemma 3. *Let the FC-hypercentral group $G = AB = AK = BK$ be the product of two abelian subgroups A and B and a locally supersoluble normal subgroup K. Then G is locally supersoluble.*

Proof. Assume for a contradiction that the statement is false, and choose a counterexample G which has no cyclic non-trivial normal subgroups. As the group G is FC-hypercentral, its (non-trivial) normal subgroup K contains an element $u \neq 1$ with only finitely many conjugates. Then the normal subgroup $M = \langle u \rangle^G$ is finitely generated, so that it is nilpotent-by-finite by Lemma 2 and its Fitting subgroup U is a nilpotent subgroup of finite index.

Suppose first that M is finite, and let N be a minimal normal subgroup of G such that $N \leq M$. It follows from Lemma 1 that N either is centralized by A or by B. Assume without loss of generality that $[N, A] = \{1\}$, and let $z \neq 1$ be an element of N such that the cyclic subgroup $\langle z \rangle$ is normal in K. Then

$$N = \langle z \rangle^G = \langle z \rangle^{AK} = \langle z \rangle^K = \langle z \rangle$$

is cyclic, a contradiction.

Suppose now that M is infinite, so that also U is an infinite normal subgroup of G. As U is finitely generated, it contains a torsion-free characteristic subgroup of finite index, which is of course a non-trivial normal subgroup of G. Therefore among all counterexamples it is possible to choose the group G in such a way that it contains a finitely generated torsion-free nilpotent (non-trivial) normal subgroup V with minimal torsion-free rank. If p is any prime number, the normal section V/V^p is a finite non-trivial p-group and hence it contains a maximal proper G-invariant subgroup $V(p)/V^p$. Application of Lemma 1 yields that $V/V(p)$ is centralized either by A or by B. Thus without loss of generality it can be assumed that there exists an infinite set π of prime numbers such that

$[V, A]$ is contained in $V(p)$ for each $p \in \pi$. It follows that

$$[V, A] \leq W = \bigcap_{p \in \pi} V(p).$$

On the other hand, W is a normal subgroup of G and the index $|V : W|$ is infinite, so that $W = \{1\}$ by the minimal choice of G. Therefore $[V, A] = \{1\}$, and hence any cyclic normal subgroup of K contained in V is also normal in $G = AK$. This last contradiction completes the proof of the lemma. $\boxed{QED}$

Lemma 4. *Let G be a group whose chief factors are central, and let A be a finitely generated torsion-free abelian normal subgroup of G. Then A is contained in $Z_r(G)$ for some non-negative integer r.*

Proof. Let r be the Prüfer rank of the abelian subgroup A. If p is any prime number, the normal section A/A^p of G has order p^r and so it is contained in $Z_r(G/A^p)$. Thus

$$[A, \underbrace{G, \ldots, G}_{r}] \leq A^p$$

for each p, and hence

$$[A, \underbrace{G, \ldots, G}_{r}] \leq \bigcap_{p} A^p = \{1\}.$$

Therefore A is contained in $Z_r(G)$. $\boxed{QED}$

In the following lemma, a group G is said to be *hypercentral* if it coincides with its hypercentre, or equivalently if any non-trivial homomorphic image of G has non-trivial centre. Hypercentral groups are known to be locally nilpotent; moreover, an FC-hypercentral group is hypercentral if and only if it is locally nilpotent. Recall also that a group G is an *FC-group* if every element of G has only finitely many conjugates. The study of FC-groups is one of the most relevant topics of the theory of infinite groups; among the elementary results on this subject, we mention that the commutator subgroup of any FC-group is periodic.

Lemma 5. *Let G be an FC-hypercentral group whose chief factors are central. Then G is hypercentral.*

Proof. As the hypotheses are inherited by homomorphic images, it is clearly enough to show that G has non-trivial centre (provided that G is not trivial). If G contains a finite non-trivial normal subgroup, then it also has a (finite) minimal normal subgroup N, so that $N \leq Z(G)$ and $Z(G) \neq \{1\}$. Assume now that G has no finite non-trivial normal subgroups, so that it contains an element

a of infinite order with only finitely many conjugates. Then the normal closure $\langle a\rangle^G$ is an FC-group with no periodic non-trivial normal subgroups, and hence it is a finitely generated torsion-free abelian group. It follows from Lemma 4 that $\langle a\rangle^G$ is contained in $Z_r(G)$ for some non-negative integer r, and so $Z(G) \neq \{1\}$ also in this case. The lemma is proved. $\boxed{QED}$

As we mentioned in the introduction, Baer proved that in any finite group with nilpotent commutator subgroup, the subgroup generated by supersoluble normal subgroups is likewise supersoluble. It is easy to prove that Baer's result can be extended to infinite groups in the following way.

Lemma 6. *Let G be a group with locally nilpotent commutator subgroup, and let H and K be locally supersoluble normal subgroups of G. Then the subgroup HK is locally supersoluble.*

A relevant theorem by P. Hall states that if G is any group containing a nilpotent normal subgroup N such that the factor group G/N' is nilpotent, then G itself is nilpotent (see [7] Part 1, Theorem 2.27). Actually, P. Hall's methods apply to many group theoretical properties; in particular the following well known result will allow to restrict ourselves to the case of metabelian groups.

Lemma 7. *Let G be a group, and let N be a nilpotent normal subgroup of G. If the factor group G/N' is locally supersoluble, then G itself is locally supersoluble.*

Recall finally that a normal subgroup N of a group G is said to be *hypercyclically embedded* in G if it has an ascending series with cyclic factors consisting of normal subgroups of G. Any group G has a largest hypercyclically embedded normal subgroup H, which is of course characteristic, and G is locally supersoluble if and only G/H has the same property.

Proof of the Theorem − As the subgroup G' is nilpotent, by Lemma 7 it is enough to prove that the factor group G/G'' is locally supersoluble. Moreover, the hypotheses are obviously inherited by homomorphic images, and hence replacing G by G/G'', it can be assumed without loss of generality that the group G is metabelian.

Let K be the largest hypercyclically embedded normal subgroup of AG'; clearly, K is normal in G, and so we may consider the factor group $\bar{G} = G/K$. Since G' is abelian, the intersection $A \cap G'$ is a hypercyclically embedded normal subgroup of AG'. Therefore $A \cap G'$ is contained in K, and hence $\bar{A} \cap \bar{G}' = \{1\}$. In particular, the subgroup $\bar{A}$ is abelian. Consider now the largest hypercyclically embedded normal subgroup $\bar{L}$ of $\bar{B}\bar{G}'$, and put $\hat{G} = \bar{G}/\bar{L}$. As before, we have that $\bar{B} \cap \bar{G}'$ is a subgroup of $\bar{L}$, so that $\hat{B} \cap \hat{G}' = \{1\}$ and $\hat{B}$ is abelian. Finally, let $\hat{N}$ be the largest hypercyclically embedded normal subgroup of $\hat{C}\hat{G}'$, and write $\tilde{G} = \hat{G}/\hat{N}$. Clearly, $\hat{C} \cap \hat{G}' \leq \hat{N}$, so that $\tilde{C} \cap \tilde{G}' = \{1\}$ and $\tilde{C}$ is abelian. Therefore

$$\tilde{G} = \tilde{A}\tilde{B} = \tilde{A}\tilde{C} = \tilde{B}\tilde{C}$$

is the product of its abelian subgroups $\tilde{A}$,$\tilde{B}$ and $\tilde{C}$, so that all chief factors of $\tilde{G}$ are central by Corollary 1, and it follows from Lemma 5 that the group $\tilde{G}$ is hypercentral. Since the normal subgroup $\hat{N}$ is hypercyclically embedded in $\hat{C}\hat{G}'$, we obtain that $\hat{C}\hat{G}'$ is locally supersoluble. Therefore the group

$$\hat{G} = \hat{A}\hat{B} = \hat{A}(\hat{C}\hat{G}') = \hat{B}(\hat{C}\hat{G}')$$

has a triple factorization by the abelian subgroups $\hat{A}$ and $\hat{B}$ and the locally supersoluble normal subgroup $\hat{C}\hat{G}'$, and hence $\hat{G}$ is locally supersoluble by Lemma 3. As the normal subgroup $\bar{L}$ is hypercyclically embedded in $\bar{B}\bar{G}'$, it follows that $\bar{B}\bar{G}'$ is locally supersoluble.

Repeat now the above argument following a different order, i.e. considering before the largest hypercyclically embedded normal subgroup $\bar{V}$ of $\bar{C}\bar{G}'$ and the factor group $G^* = \bar{G}/\bar{V}$, and then the largest hypercyclically embedded normal subgroup W^* of $B^*(G^*)'$. In this way, it turns out that also the normal subgroup $\bar{C}\bar{G}'$ of $\bar{G}$ is locally supersoluble. Therefore the metabelian group

$$\bar{G} = (\bar{B}\bar{G}')(\bar{C}\bar{G}')$$

is the product of two locally supersoluble normal subgroups, and so it is likewise locally supersoluble by Lemma 6. As the normal subgroup K is hypercyclically embedded in AG', it follows that AG' is locally supersoluble. Repeating once again the argument with a different order (i.e. working before with B and then with A), we obtain that also BG' is locally supersoluble. Thus

$$G = (AG')(BG')$$

is the product of two locally supersoluble normal subgroups, and so G is locally supersoluble by Lemma 6. The theorem is proved. $\boxed{QED}$

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