Note di Matematica ISSN 1123-2536, e-ISSN 1590-0932
Note Mat. **33** (2013) no. 1, 139–170. doi:10.1285/i15900932v33n1p139

Commutator width in Chevalley groups

Roozbeh Hazrat
University of Western Sydney rhazrat@gmail.com

Alexei Stepanov [i]
Saint Petersburg State University and Abdus Salam School of Mathematical Sciences, Lahore
stepanov239@gmail.com

Nikolai Vavilov [ii]
Saint Petersburg State University nikolai-vavilov@yandex.ru

Zuhong Zhang [iii]
Beijing Institute of Technology zuhong@gmail.com

Abstract. The present paper is the [slightly expanded] text of our talk at the Conference "Advances in Group Theory and Applications" at Porto Cesareo in June 2011. Our main results assert that [elementary] Chevalley groups very rarely have finite commutator width. The reason is that they have very few commutators, in fact, commutators have finite width in elementary generators. We discuss also the background, bounded elementary generation, methods of proof, relative analogues of these results, some positive results, and possible generalisations.

Keywords: Chevalley groups, elementary subgroups, elementary generators, commutator width, relative groups, bounded generation, standard commutator formulas, unitriangular factorisations

MSC 2000 classification: 20G35, 20F12, 20F05

1 Introduction

In the present note we concentrate on the recent results on the commutator width of Chevalley groups, the width of commutators in elementary generators,

[i]This research was started within the framework of the RFFI/Indian Academy cooperation project 10-01-92651 and the RFFI/BRFFI cooperation project 10-01-90016. Currently the work is partially supported by the RFFI research project 11-01-00756 (RGPU) and by the State Financed research task 6.38.74.2011 at the Saint Petersburg State University. At the final stage the second author was supported also by the RFFI projects 13-01-00709 and 13-01-91150.

[ii]This research was started within the framework of the RFFI/Indian Academy cooperation project 10-01-92651 and the RFFI/BRFFI cooperation project 10-01-90016. Currently the work is partially supported by the RFFI research projects 11-01-00756 (RGPU) and 12-01-00947 (POMI) and by the State Financed research task 6.38.74.2011 at the Saint Petersburg State University. At the final stage the third author was supported also by the RFFI projects 13-01-00709 and 13-01-91150.

[iii]The work is partially supported by NSFC grant 10971011.

http://siba-ese.unisalento.it/ © 2013 Università del Salento

and the corresponding relative results. In fact, localisation methods used in the proof of these results have many further applications, both actual and potential: relative commutator formulas, multiple commutator formulas, nilpotency of K_1, description of subnormal subgroups, description of various classes of overgroups, connection with excision kernels, etc. We refer to our surveys [36, 31, 32] and to our papers [29, 35, 7, 40, 37, 38, 41, 76, 39, 33, 34] for these and further applications and many further related references.

2 Preliminaries

2.1 Length and width

Let G be a group and X be a set of its generators. Usually one considers symmetric sets, for which $X^{-1} = X$.

- The **length** $l_X(g)$ of an element $g \in G$ with respect to X is the minimal k such that g can be expressed as the product $g = x_1 \dots x_k$, $x_i \in X$.
- The **width** $w_X(G)$ of G with respect to X is the supremum of $l_X(g)$ over all $g \in G$. In the case when $w_X(G) = \infty$, one says that G does not have bounded word length with respect to X.

The problem of calculating or estimating $w_X(G)$ has attracted a lot of attention, especially when G is one of the classical-like groups over skew-fields. There are *hundreds* of papers which address this problem in the case when X is either

- the set of elementary transvections
- the set of all transvections or ESD-transvections,
- the set of all unipotents,
- the set of all reflections or pseudo-reflections,
- other sets of small-dimensional transformations,
- a class of matrices determined by their eigenvalues, such as the set of all involutions,
- a non-central conjugacy class,
- the set of all commutators,

etc., etc. Many further exotic generating sets have been considered, such as matrices distinct from the identity matrix in one column, symmetric matrices, etc., etc., etc. We do not make any attempt to list all such papers, there are simply far too many, and vast majority of them produce sharp bounds for classes of rings, which are trivial from our prospective, such as fields, or semi-local rings.

2.2 Chevalley groups

Let us fix basic notation. This notation is explained in [1, 4, 60, 74, 75, 2, 3, 92, 95, 93], where one can also find many further references.

- Φ is a reduced irreducible root system;
- Fix an order on Φ, let Φ^+, Φ^- and $\Pi = \{\alpha_1, \dots, \alpha_l\}$ are the sets of positive, negative and fundamental roots, respectively.
- Let $Q(\Phi)$ be the root lattice of Φ, $P(\Phi)$ be the weight lattice of Φ and P be any lattice such that $Q(\Phi) \leq P \leq P(\Phi)$;
- R is a commutative ring with 1;
- $G = G_P(\Phi, R)$ is the Chevalley group of type (Φ, P) over R;
- In most cases P does not play essential role and we simply write $G = G(\Phi, R)$ for any Chevalley group of type Φ over R;
- However, when the answer depends on P we usually write $G_{\mathrm{sc}}(\Phi, R)$ for the simply connected group, for which $P = P(\Phi)$ and $G_{\mathrm{ad}}(\Phi, R)$ for the adjoint group, for which $P = Q(\Phi)$;
- $T = T(\Phi, R)$ is a split maximal torus of G;
- $x_\alpha(\xi)$, where $\alpha \in \Phi$, $\xi \in R$, denote root unipotents G elementary with respect to T;
- $E(\Phi, R)$ is the [absolute] elementary subgroup of $G(\Phi, R)$, generated by all root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$;
- $E^L(\Phi, R)$ is the subset (not a subgroup!) of $E(\Phi, R)$, consisting of products of $\leq L$ root unipotents $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in R$;
- $H = H(\Phi, R) = T(\Phi, R) \cap E(\Phi, R)$ is the elementary part of the split maximal torus;
- $U^{\pm}(\Phi, R)$ is the unipotent radical of the standard Borel subgroup $B(\Phi, R)$ or its opposite $B^-(\Phi, R)$. By definition

$$U(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^+,\ \xi \in R \rangle.$$
$$U^-(\Phi, R) = \langle x_\alpha(\xi),\ \alpha \in \Phi^-,\ \xi \in R \rangle.$$

2.3 Chevalley groups versus elementary subgroups

Many authors not familiar with algebraic groups or algebraic K-theory do not distinguish Chevalley groups and their elementary subgroups. Actually, these groups are defined dually.

- Chevalley groups $G(\Phi, R)$ are [the groups of R-points of] algebraic groups. In other words, $G(\Phi, R)$ is defined as

$$G(\Phi, R) = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}[G], R),$$

where $\mathbb{Z}[G]$ is the affine algebra of G. By definition $G(\Phi, R)$ consists of solutions in R of certain algebraic equations.

- As opposed to that, *elementary* Chevalley groups $E(\Phi, R)$ are generated by elementary generators

$$E(\Phi, R) = \big\langle x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R \big\rangle.$$

When $R = K$ is a field, one knows relations among these elementary generators, so that $E(\Phi, R)$ can be defined by generators and relations. However, in general, the elementary generators are described by their action in certain representations.

By the very construction of these groups $E(\Phi, R) \leq G(\Phi, R)$ but, as we shall see, in general $E(\Phi, R)$ can be strictly smaller than $G(\Phi, R)$ even for fields. The following two facts might explain, why some authors confuse $E(\Phi, R)$ and $G(\Phi, R)$:

- Let $R = K$ be *any* field. Then $G_{\mathrm{sc}}(\Phi, K) = E_{\mathrm{sc}}(\Phi, K)$.
- Let $R = K$ be an *algebraically closed* field. Then $G_{\mathrm{ad}}(\Phi, K) = E_{\mathrm{ad}}(\Phi, K)$.

However, for a field K that is not algebraically closed one usually has strict inclusion $E_{\mathrm{ad}}(\Phi, K) < G_{\mathrm{ad}}(\Phi, K)$. Also, as we shall see, even for principal ideal domains $E_{\mathrm{sc}}(\Phi, R) < G_{\mathrm{sc}}(\Phi, R)$, in general.

2.4 Elementary generators

By the very construction Chevalley groups occur as subgroups of the general linear group $\mathrm{GL}(n, R)$. Let e be the identity matrix and e_{ij}, $1 \leq i, j \leq n$, be a matrix unit, which has 1 in position (i, j) and zeros elsewhere. Below we list what the elementary root unipotents, also known as elementary generators, look like for classical groups.

- In the case $\Phi = \mathbb{A}_l$ one has $n = l + 1$. Root unipotents of $\mathrm{SL}(n, R)$ are [elementary] transvections

$$t_{ij}(\xi) = e + \xi e_{ij}, \qquad 1 \leq i \neq j \leq n, \quad \xi \in R.$$

- In the case $\Phi = \mathrm{D}_l$ one has $n = 2l$. We number rows and columns of matrices from $\mathrm{GL}(n, R)$ as follows: $1, \dots, l, -l, \dots, -1$. Then root unipotents of $\mathrm{SO}(2l, R)$ are [elementary] orthogonal transvections

$$T_{ij}(\xi) = e + \xi e_{ij} - \xi e_{-j,-i}, \qquad 1 \le i, j \le -1,\ i \ne \pm j, \quad \xi \in R.$$

- In the case $\Phi = \mathbf{C}_l$ also $n = 2l$ and we use the same numbering of rows and columns as in the even orthogonal case. Moreover, we denote ε_i the sign of i, which is equal to $+1$ for $i = 1, \dots, l$ and to -1 for $i = -1, \dots, -1$. In $\mathbf{C}_l$ there are two root lengths. Accordingly, root unipotents of $\mathrm{Sp}(2l, R)$ come in two stocks. Long root unipotents are the usual linear transvections $t_{i,-i}(\xi)$, $1 \le i \le -1$, $\xi \in R$, while short root unipotents are [elementary] symplectic transvections

$$T_{ij}(\xi) = e + \xi e_{ij} - \varepsilon_i \varepsilon_j \xi e_{-j,-i}, \qquad 1 \le i, j \le -1,\ i \ne \pm j, \quad \xi \in R.$$

- Finally, for $\Phi = \mathrm{B}_l$ one has $n = 2l+1$ and we number rows and columns of matrices from $\mathrm{GL}(n, R)$ as follows: $1, \dots, l, 0, -l, \dots, -1$. Here too there are two root lengths. The long root elements of the odd orthogonal group $\mathrm{SO}(2l+1, R)$ are precisely the root elements of the even orthogonal groups, $T_{ij}(\xi)$, $i \ne \pm j$, $i, j \ne 0$, $\xi \in R$. The short root elements have the form

$$T_{i0}(\xi) = e + \xi e_{i0} - 2\xi e_{-i,0} - \xi^2 e_{i,-1}, \qquad i \ne 0, \quad \xi \in R.$$

It would be only marginally more complicated to specify root elements of spin groups and exceptional groups, in their minimal faithful representations, see [93, 94].

2.5 Classical cases

Actually, most of our results are already new for classical groups. Recall identification of Chevalley groups and elementary Chevalley groups for the classical cases. The second column of the following table lists traditional notation of classical groups, according to types: $\mathbb{A}_l$ the special linear group, B_l the odd orthogonal group, $\mathbf{C}_l$ the symplectic group, and D_l the even orthogonal group. These groups are defined by algebraic equations. Orthogonal groups are not simply connected, the corresponding simply connected groups are the spin groups. The last column lists the names of their elementary subgroups, generated by the elementary generators listed in the preceding subsection.

Φ	$G(\Phi, R)$	$E(\Phi, R)$
$\mathbb{A}_l$	$\mathrm{SL}(l+1, R)$	$E(l+1, R)$
B_l	$\mathrm{Spin}(2l+1, R)$	$\mathrm{Epin}(2l+1, R)$
	$\mathrm{SO}(2l+1, R)$	$\mathrm{EO}(2l+1, R)$
$\mathbf{C}_l$	$\mathrm{Sp}(2l, R)$	$\mathrm{Ep}(2l, R)$
D_l	$\mathrm{Spin}(2l, R)$	$\mathrm{Epin}(2l, R)$
	$\mathrm{SO}(2l, R)$	$\mathrm{EO}(2l, R)$

Orthogonal groups [and spin groups] in this table are the *split* orthogonal groups. *Split* means that they preserve a bilinear/quadratic form of maximal Witt index. In the case of a field the group $\mathrm{EO}(n, K)$ was traditionally denoted by $\Omega(n, K)$ and called the kernel of spinor norm. Since the group $\mathrm{SO}(n, K)$ is not simply connected, in general $\Omega(n, K)$ is a proper subgroup of $\mathrm{SO}(n, K)$.

2.6 Dimension of a ring

Usually, dimension of a ring R is defined as the length d of the longest strictly ascending chain of ideals $I_0 < I_1 < \ldots < I_d$ of a certain class.

- The most widely known one is the Krull dimension $\dim(R)$ defined in terms of chains of prime ideals of R. Dually, it can be defined as the combinatorial dimension of $\mathrm{Spec}(R)$, considered as a topological space with Zariski topology.

Recall, that the combinatorial dimension $\dim(X)$ of a topological space X is the length of the longest *descending* chain of its *irreducible* subspaces $X_0 > X_1 > \ldots > X_d$. Thus, by definition,

$$\dim(R) = \dim(\mathrm{Spec}(R)).$$

However, we mostly use the following more accurate notions of dimension.

- The Jacobson dimension $\text{j-dim}(R)$ of R is defined in terms of j-ideals, in other words, those prime ideals, which are intersections of maximal ideals. Clearly, $\text{j-dim}(R)$ coincides with the combinatorial dimension of the *maximal* spectrum of the ring R, by definition, $\text{j-dim}(R) = \dim(\mathrm{Max}(R))$

Define dimension $\delta(X)$ of a topological space X as the smallest integer d such that X can be expressed as a *finite* union of Noetherian topological spaces of dimension $\leq d$. The trick is that these spaces do not have to be closed subsets of X.

- The Bass—Serre dimension of a ring R is defined as the dimension of its maximal spectrum, $\delta(R) = \delta(\mathrm{Max}(R))$.

Bass—Serre dimension has many nice properties, which make it better adapted to the study of problems we consider. For instance, a ring is semilocal iff $\delta(R) = 0$ (recall that a commutative ring R is called semilocal if it has finitely many maximal ideals).

2.7 Stability conditions

Mostly, stability conditions are defined in terms of stability of rows, or columns. In this note we only refer to Bass' stable rank, first defined in [9]. We will denote the [left] R-module of rows of length n by ${}^nR$, to distinguish it from the [right] R-module R^n of columns of height n.

A row $(a_1, \ldots, a_n) \in {}^nR$ is called *unimodular*, if its components $a_1, \ldots, a_n$ generate R as a right ideal,

$$a_1R + \ldots + a_nR = R.$$

or, what is the same, if there exist such $b_1, \ldots, b_n \in R$ that

$$a_1b_1 + \ldots + a_nb_n = 1.$$

The *stable rank* $\mathrm{sr}(R)$ of the ring R is the smallest such n that every unimodular row $(a_1, \ldots, a_{n+1})$ of length $n+1$ is *stable*. In other words, there exist elements $b_1, \ldots b_n \in R$ such that the row

$$(a_1 + a_{n+1}b_1, a_2 + a_{n+1}b_2, \ldots, a_n + a_{n+1}b_n)$$

of length n is unimodular. If no such n exists, one writes $\mathrm{sr}(R) = \infty$.

In fact, stable rank is a more precise notion of dimension of a ring, based on linear algebra, rather than chains of ideals. It is shifted by 1 with respect to the classical notions of dimension. The basic estimate of stable rank is Bass' theorem, asserting that $\mathrm{sr}(R) \leq \delta(R) + 1$.

Especially important in the sequel is the condition $\mathrm{sr}(R) = 1$. A ring R has stable rank 1 if for any $x, y \in R$ such that $xR + yR = R$ there exists a $z \in R$ such that $(x + yz)R = R$. In fact, rings of stable rank 1 are weakly finite (one-sided inverses are automatically two-sided), so that this last condition is equivalent to invertibility of $x + yz$. Rings of stable rank 1 should be considered as a class of 0-dimensional rings, in particular, all semilocal rings have stable rank 1. See [87] for many further examples and references.

2.8 Localisation

Let, as usual, R be a commutative ring with 1, S be a multiplicative system in R and $S^{-1}R$ be the corresponding localisation. We will mostly use localisation with respect to the two following types of multiplicative systems.

- Principal localisation: the multiplicative system S is generated by a non-nilpotent element $s \in R$, viz. $S = \langle s \rangle = \{1, s, s^2, \dots\}$. In this case we usually write $\langle s \rangle^{-1} R = R_s$.
- Maximal localisation: the multiplicative system S equals $S = R \setminus \mathfrak{m}$, where $\mathfrak{m} \in \mathrm{Max}(R)$ is a maximal ideal in R. In this case we usually write $(R \setminus \mathfrak{m})^{-1} R = R_{\mathfrak{m}}$.

We denote by $F_S : R \longrightarrow S^{-1}R$ the canonical ring homomorphism called the localisation homomorphism. For the two special cases mentioned above, we write $F_s : R \longrightarrow R_s$ and $F_{\mathfrak{m}} : R \longrightarrow R_{\mathfrak{m}}$, respectively.

Both $G(\Phi, \_)$ and $E(\Phi, \_)$ commute with direct limits. In other words, if $R = \varinjlim R_i$, where $\{R_i\}_{i \in I}$ is an inductive system of rings, then $G(\Phi, \varinjlim R_i) = \varinjlim G(\Phi, R_i)$ and the same holds for $E(\Phi, R)$. Our proofs crucially depend on this property, which is mostly used in the two following situations.

- First, let R_i be the inductive system of all finitely generated subrings of R with respect to inclusion. Then $X = \varinjlim X(\Phi, R_i)$, which reduces most of the proofs to the case of Noetherian rings.
- Second, let S be a multiplicative system in R and R_s, $s \in S$, the inductive system with respect to the localisation homomorphisms: $F_t : R_s \longrightarrow R_{st}$. Then $X(\Phi, S^{-1}R) = \varinjlim X(\Phi, R_s)$, which allows to reduce localisation with respect to any multiplicative system to principal localisations.

2.9 K_1-functor

The starting point of the theory we consider is the following result, first obtained by Andrei Suslin [80] for $\mathrm{SL}(n, R)$, by Vyacheslav Kopeiko [48] for symplectic groups, by Suslin and Kopeiko [81] for even orthogonal groups and by Giovanni Taddei [83] in general.

Theorem 1. *Let Φ be a reduced irreducible root system such that $\mathrm{rk}(\Phi) \geq 2$. Then for any commutative ring R one has $E(\Phi, R) \trianglelefteq G(\Phi, R)$.*

In particular, the quotient

$$K_1(\Phi, R) = G_{\mathrm{sc}}(\Phi, R)/E_{\mathrm{sc}}(\Phi, R)$$

is not just a pointed set, it is a group. It is called K_1-functor.

The groups $G(\Phi, R)$ and $E(\Phi, R)$ behave functorially with respect to both R and Φ. In particular, to an embedding of root systems $\Delta \subseteq \Phi$ there corresponds the map $\varphi : G(\Delta, R) \longrightarrow G(\Phi, R)$ of the corresponding [simply connected] groups, such that $\varphi(E(\Delta, R)) \leq E(\Phi, R)$. By homomorphism theorem it defines the stability map $\varphi : K_1(\Delta, R) \longrightarrow K_1(\Phi, R)$.

In the case $\Phi = \mathbb{A}_l$ this K_1-functor specialises to the functor

$$\mathrm{SK}_1(n, R) = \mathrm{SL}(n, R)/E(n, R),$$

rather than the usual linear K_1-functor $K_1(n, R) = \mathrm{GL}(n, R)/E(n, R)$. In examples below we also mention the corresponding stable K_1-functors, which are defined as limits of $K_1(n, R)$ and $\mathrm{SK}_1(n, R)$ under stability embeddings, as n tends to infinity:

$$\mathrm{SK}_1(R) = \varinjlim \mathrm{SK}_1(n, R), \qquad K_1(R) = \varinjlim K_1(n, R).$$

Another basic tool are stability theorems, which assert that under some assumptions on Δ, Φ and R stability maps are surjective or/and injective. We do not try to precisely state stability theorems for Chevalley groups, since they depend on various analogues and higher versions of stable rank, see in particular [75, 64, 65, 66].

However, to give some feel, we state two classical results pertaining to the case of $\mathrm{SL}(n, R)$. These results, which are due to Bass and Bass—Vaserstein, respectively, are known as surjective stability of K_1 and injective stability of K_1. In many cases they allow to reduce problems about groups of higher ranks, to similar problems for groups of smaller rank.

Theorem 2. *For any $n \geq \mathrm{sr}(R)$ the stability map*

$$K_1(n, R) \longrightarrow K_1(n + 1, R)$$

is surjective. In other words,

$$\mathrm{SL}(n + 1, R) = \mathrm{SL}(n, R)E(n + 1, R).$$

Theorem 3. *For any $n \geq \mathrm{sr}(R) + 1$ the stability map*

$$K_1(n, R) \longrightarrow K_1(n + 1, R)$$

is injective. In other words,

$$\mathrm{SL}(n, R) \cap E(n + 1, R) = E(n, R).$$

2.10 K_1-functor: trivial or non-trivial

Usually, K_1-functor is non-trivial. But in some important cases it is trivial. Let us start with some obvious examples.

- $R = K$ is a field.
- More generally, R is semilocal
- R is Euclidean
- It is much less obvious that K_1 does not have to be trivial even for principal ideal rings. Let us cite two easy examples discovered by Ischebeck [43] and by Grayson and Lenstra [26], respectively.
- Let K be a field of algebraic functions of one variable with a perfect field of constants k. Then the ring $R = K \otimes_k k(x_1, \ldots, x_m)$ is a principal ideal ring. If, moreover, $m \geq 2$, and the genus of K is distinct from 0, then $\mathrm{SK}_1(R) \neq 1$.
- Let $R = \mathbb{Z}[x]$, and $S \subseteq R$ be the multiplicative subsystem of R generated by cyclotomic polynomials Φ_n, $n \in \mathbb{N}$. Then $S^{-1}R$ is a principal ideal ring such that $\mathrm{SK}_1(S^{-1}R) \neq 1$.

This is precisely why over a Euclidean ring it is somewhat easier to find Smith form of a matrix, than over a principal ideal ring.

However, there are some further examples, when K_1 is trivial. Usually, they are very deep. The first example below is part of the [almost] positive solution of the congruence subgroup problem by Bass—Milnor—Serre and Matsumoto [10, 60]. The second one is the solution of K_1-analogue of Serre's problem by Suslin [80].

- $R = \mathcal{O}_S$ is a Hasse domain.
- $R = K[x_1, \ldots, x_m]$ is a polynomial ring over a field.

2.11 K_1-functor, abelian or non-abelian

Actually, $K_1(\Phi, R)$ is not only non-trivial. Oftentimes, it is even non-abelian. The first such examples were constructed by Wilberd van der Kallen [45] and Anthony Bak [6]. In both cases proofs are of topological nature and use homotopy theory.

- Wilberd van der Kallen [45] constructs a number of examples, where $K_1(n, R)$ is non-abelian. For instance,

$$R = \mathbb{R}[x_1, x_2, y_1, y_2, y_3, y_4]/(x_1^2 + x_2^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 = 1)$$

is a 4-dimensional ring for which $[\mathrm{SL}(4,R),\mathrm{SL}(4,R)] \not\leq E(4,R)$. In fact, in this case even

$$[\mathrm{SL}(2,R),\mathrm{SL}(4,R)] \not\leq \mathrm{GL}(3,R)E(4,R).$$

- Anthony Bak [6] constructs examples of [finite dimensional] subrings R in the rings of continuous functions $\mathbb{R}^X$ and $\mathbb{C}^X$ on certain topological spaces X, for which not only $K_1(n,R)$, $n \geq 3$, is non-abelian, but even its nilpotency class can be arbitrarily large.

The question arises, as to how non-abelian $K_1(\Phi,R)$ may be. For finite dimensional rings this question was answered by Anthony Bak [6] for $\mathrm{SL}(n,R)$, for other even classical groups by the first author [29] and for all Chevalley groups by the first and the third authors [35].

Theorem 4. *Let Φ be a reduced irreducible root system such that* $\mathrm{rk}(\Phi) \geq 2$. *Further let R be a commutative ring of Bass—Serre dimension $\delta(R) = d < \infty$. Then $K_1(\Phi,R)$ is nilpotent of class $\leq d+1$.*

This theorem relies on a version of localisation method which Bak called localisation-completion [6]. This method turned out to be crucial for the proof of results we discuss in the present paper, see [36, 31] for more historical background and an introduction to this method in non-technical terms.

3 Main problems

3.1 Statement of the main problems

In this paper we discuss the following problem.

Problem 1. *Estimate the width of $E(\Phi,R)$ with respect to the set of elementary commutators*

$$X = \big\{[x,y] = xyx^{-1}y^{-1},\ x \in G(\Phi,R),\ y \in E(\Phi,R)\big\}.$$

Observe, that one could not have taken the set

$$X = \big\{[x,y] = xyx^{-1}y^{-1},\ x,y \in G(\Phi,R)\big\}$$

here, since $K_1(\Phi,R)$ maybe non-abelian.

It turns out that this problem is closely related to the following problem.

Problem 2. *Estimate the width of $E(\Phi,R)$ with respect to the set of elementary generators*

$$X = \big\{x_\alpha(\xi),\ \alpha \in \Phi,\ \xi \in R\big\}.$$

The answer in general will be highly unexpected, so we start with discussion of classical situations.

3.2 The group $\mathrm{SL}(2, R)$

Let us mention one assumption that is essential in what follows.

When R is Euclidean, expressions of matrices in $\mathrm{SL}(2, R)$ as products of elementary transvections correspond to continued fractions. Division chains in $\mathbb{Z}$ can be arbitrarily long, it is classically known that two consecutive Fibonacci numbers provide such an example. Thus, we get.

Fact 1. $\mathrm{SL}(2, \mathbb{Z})$ *does not have bounded length with respect to the elementary generators.*

Actually, behavious of the group $\mathrm{SL}(2, R)$ is exceptional in more than one respect. Thus, the groups $E(n, R)$, $n \geq 3$ are perfect. The group $E(2, R)$ is usually not.

Fact 2. $[\mathrm{SL}(2, \mathbb{Z}), \mathrm{SL}(2, \mathbb{Z})]$ *has index* 12 *in* $\mathrm{SL}(2, \mathbb{Z})$.

- Thus, in the sequel we always assume that $\mathrm{rk}(\Phi) \geq 2$.
- In fact, it is material for most of our results that the group $E(\Phi, R)$ is perfect. It usually is, the only counter-examples in rank ≥ 2 stemming from the fact that $\mathrm{Sp}(4, \mathrm{GF}\,2)$ and $G(\mathrm{G}_2, \mathrm{GF}\,2)$ are not perfect. Thus, in most cases one should add proviso that $E(\Phi, R)$ is actually perfect, which amounts to saying that R does not have residue field $\mathrm{GF}\,2$ for $\Phi = \mathrm{B}_2, \mathrm{G}_2$.

The reader may take these two points as standing assumptions for the rest of the note.

3.3 The answers for fields

The following result easily follows from Bruhat decomposition.

Theorem 5. *The width of* $G_{\mathrm{sc}}(\Phi, K)$ *with respect to the set of elementary generators is* $\leq 2|\Phi^+| + 4\,\mathrm{rk}(\Phi)$.

Rimhak Ree [67] observed that the commutator width of semisimple algebraic groups over an algebraically closed fields equals 1. For fields containing ≥ 8 elements the following theorems were established by Erich Ellers and Nikolai Gordeev [21] using Gauss decomposition with prescribed semi-simple part [16]. On the other hand, for very small fields these theorems were recently proven by Martin Liebeck, Eamonn O'Brien, Aner Shalev, and Pham Huu Tiep [51, 52], using explicit information about maximal subgroups and very delicate character estimates.

Actually, the first of these theorems in particular completes the answer to Ore conjecture, whether any element of a [non-abelian] finite simple group is a single commutator.

Theorem 6. *The width of* $E_{\rm ad}(\Phi,K)$ *with respect to commutators is* 1*.*

Theorem 7. *The width of* $G_{\rm sc}(\Phi,K)$ *with respect to commutators is* ≤ 2*.*

3.4 The answers for semilocal rings

The following results were recently published by Andrei Smolensky, Sury and the third author [73, 96]. Actually, their proofs are easy combinations of Bass' surjective stability [9] and Tavgen's rank reduction theorem [84]. The second of these decompositions, the celebrated Gauss decomposition, was known for semilocal rings, the first one was known for $\mathrm{SL}(n,R)$, see [20], but not in general.

Theorem 8. *Let* $\mathrm{sr}(R)=1$*. Then the*

$$E(\Phi,R)=U^+(\Phi,R)U^-(\Phi,R)U^+(\Phi,R)U^-(\Phi,R).$$

Corollary 1. *Let* $\mathrm{sr}(R)=1$*. Then the width of* $E(\Phi,R)$ *with respect to the set of elementary generators is at most* $M=4|\Phi^+|$*.*

Theorem 9. *Let* $\mathrm{sr}(R)=1$*. Then the*

$$E(\Phi,R)=U^+(\Phi,R)U^-(\Phi,R)H(\Phi,R)U(\Phi,R).$$

Corollary 2. *Let* $\mathrm{sr}(R)=1$*. Then the width of* $E(\Phi,R)$ *with respect to the set of elementary generators is at most* $M=3|\Phi^+|+4\,\mathrm{rk}(\Phi)$*.*

In particular, the width of $E(\Phi,R)$ over a ring with $\mathrm{sr}(R)=1$ with respect to commutators is always bounded, but its explicit calculation is a non-trivial task. Let us limit ourselves with the following result by Leonid Vaserstein and Ethel Wheland [90, 91].

Theorem 10. *Let* $\mathrm{sr}(R)=1$*. Then the width of* $E(n,R)$*,* $n\geq 3$*, with respect to commutators is* ≤ 2*.*

There are also similar results by You Hong, Frank Arlinghaus and Leonid Vaserstein [101, 5] for other classical groups, but they usually assert that the commutator width is ≤ 3 or ≤ 4, and sometimes impose stronger stability conditions such as $\mathrm{asr}(R)=1$, $\Lambda\,\mathrm{sr}(R)=1$, etc.

The works by Nikolai Gordeev and You Hong, where similar results are established for exceptional groups over local rings [subject to some mild restrictions on their residue fields] are still not published.

3.5 Bounded generation

Another nice class of rings, for which one may expect positive answers to the above problems, are Dedekind rings of arithmetic type.

Let K be an algebraic number field, i.e. either a finite algebraic extension of $\mathbb{Q}$, and further let S be a finite set of (non-equivalent) valuations of K, which contains all Archimedian valuations. For a non-Archimedian valuation $\mathfrak{p}$ of the field K we denote by $v_{\mathfrak{p}}$ the corresponding exponent. As usual, $R = \mathcal{O}_S$ denotes the ring, consisting of $x \in K$ such that $v_{\mathfrak{p}}(x) \geq 0$ for all valuations $\mathfrak{p}$ of K, which do not belong to S. Such a ring $\mathcal{O}_S$ is known as the Dedekind ring of arithmetic type, determined by the set of valuations S of the field K. Such rings are also called Hasse domains, see, for instance, [10]. Sometimes one has to require that $|S| \geq 2$, or, what is the same, that the multiplicative group $\mathcal{O}_S^*$ of the ring $\mathcal{O}_S$ is infinite.

Bounded generation of $\mathrm{SL}(n, \mathcal{O}_S)$, $n \geq 3$, was established by David Carter and Gordon Keller in [11, 12, 13, 14, 15], see also the survey by Dave Witte Morris [61] for a modern exposition. The general case was solved by Oleg Tavgen [84, 85]. The result by Oleg Tavgen can be stated in the following form due to the [almost] positive solution of the congruence subgroup problem [10, 60].

Theorem 11. *Let $\mathcal{O}_S$ be a Dedekind ring of arithmetic type,* $\mathrm{rk}(\Phi) \geq 2$. *Then the elementary Chevalley group $G(\Phi, \mathcal{O}_S)$ has bounded length with respect to the elementary generators.*

In Section 6 we discuss what this implies for the commutator width.

See also the recent works by Edward Hinson [42], Loukanidis and Murty [55, 62], Sury [79], Igor Erovenko and Andrei Rapinchuk [23, 24, 25], for different proofs, generalisations and many further references, concerning bounded generation.

3.6 van der Kallen's counter-example

However, all hopes for positive answers in general are completely abolished by the following remarkable result due to Wilberd van der Kallen [44].

Theorem 12. *The group* $\mathrm{SL}(3, \mathbb{C}[t])$ *does not have bounded word length with respect to the elementary generators.*

It is an amazing result, since $\mathbb{C}[t]$ is Euclidean. Since $\mathrm{sr}(\mathbb{C}[t]) = 2$ we get the following corollary

Corollary 3. *None of the groups* $\mathrm{SL}(n, \mathbb{C}[t])$, $n \geq 3$, *has bounded word length with respect to the elementary generators.*

See also [22] for a slightly easier proof of a slightly stronger result. Later Dennis and Vaserstein [20] improved van der Kallen's result to the following.

Theorem 13. *The group* $\mathrm{SL}(3, \mathbb{C}[t])$ *does not have bounded word length with respect to the commutators.*

Since for $n \geq 3$ every elementary matrix is a commutator, this is indeed stronger, than the previous theorem.

4 Absolute commutator width

Here we establish an amazing relation between Problems 1 and 2.

4.1 Commutator width in $\mathrm{SL}(n, R)$

The following result by Alexander Sivatsky and the second author [72] was a major breakthrough.

Theorem 14. *Suppose that $n \geq 3$ and let R be a Noetherian ring such that $\dim \mathrm{Max}(R) = d < \infty$. Then there exists a natural number $N = N(n, d)$ depending only on n and d such that each commutator $[x, y]$ of elements $x \in E(n, R)$ and $y \in \mathrm{SL}(n, R)$ is a product of at most N elementary transvections.*

Actually, from the proof in [72] one can derive an efficient upper bound on N, which is a *polynomial* with the leading term $48n^6d$.

It is interesting to observe that it is already non-trivial to replace here an element of $\mathrm{SL}(n, R)$ by an element of $\mathrm{GL}(n, R)$. Recall, that a *ring of geometric origin* is a localisation of an affine algebra over a field.

Theorem 15. *Let $n \geq 3$ and let R be a ring of geometric origin. Then there exists a natural number N depending only on n and R such that each commutator $[x, y]$ of elements $x \in E(n, R)$ and $y \in \mathrm{GL}(n, R)$ is a product of at most N elementary transvections.*

Let us state another interesting variant of the Theorem 14, which may be considered as its stable version. Its proof crucially depends on the Suslin—Tulenbaev proof of the Bass—Vaserstein theorem, see [82].

Theorem 16. *Let $n \geq \mathrm{sr}(R) + 1$. Then there exists a natural number N depending only on n such that each commutator $[x, y]$ of elements $x, y \in \mathrm{GL}(n, R)$ is a product of at most N elementary transvections.*

Actually, [72] contains many further interesting results, such as, for example, analogues for the Steinberg groups $\mathrm{St}(n, R)$, $n \geq 5$. However, since this result depends on the centrality of $K_2(n, R)$ at present there is no hope to generalise it to other groups.

4.2 Decomposition of unipotents

The proof of Theorem 14 in [72] was based on a combination of localisation and decomposition of unipotents [77]. Essentially, in the simplest form decomposition of unipotents gives finite polynomial expressions of the conjugates

$$gx_\alpha(\xi)g^{-1}, \qquad \alpha \in \Phi,\ \xi \in R,\ g \in G(\Phi, R),$$

as products of factors sitting in proper parabolic subgroups, and, in the final count, as products of elementary generators.

Roughly speaking, decomposition of unipotents allows to plug in explicit polynomial formulas as the induction base — which is the most difficult part of all localisation proofs! — instead of messing around with the length estimates in the conjugation calculus.

To give some feel of what it is all about, let us state an immediate corollary of the Theme of [77]. Actually, [77] provides explicit polynomial expressions of the elementary factors, rather than just the length estimate.

Fact 3. *Let R be a commutative ring and $n \geq 3$. Then any transvection of the form $gt_{ij}(\xi)g^{-1}$, $1 \leq i \neq j \leq n$, $\xi \in R$, $g \in \mathrm{GL}(n, R)$ is a product of at most $4n(n-1)$ elementary transvections.*

It is instructive to compare this bound with the bound resulting from Suslin's proof of Suslin's normality theorem [80]. Actually, Suslin's direct factorisation method is more general, in that it yields elementary factorisations of a broader class of transvections. On the other hand, it is less precise, both factorisations coincide for $n = 3$, but asymptotically factorisation in Fact 3 is better.

Fact 4. *Let R be a commutative ring and $n \geq 3$. Assume that $u \in R^n$ is a unimodular column and $v \in {}^nR$ be any row such that $vu = 0$. Then the transvection $e + uv$ is a product of at most $n(n-1)(n+2)$ elementary transvections.*

Let us state a counterpart of the Theorem 14 that results from the Fact 3 alone, *without* the use of localisation. This estimate works for *arbitrary* commutative rings, but depends on the length of the elementary factor. Just wait until subsection 4.5!

Theorem 17. *Let $n \geq 3$ and let R be a commutative ring. Then there exists a natural number $N = N(n, M)$ depending only on n and M such that each commutator $[x, y]$ of elements $x \in E^M(n, R)$ and $y \in \mathrm{SL}(n, R)$ is a product of at most N elementary transvections.*

It suffices to expand a commutator $[x_1 \dots x_M, y]$, where x_i are elementary transvections, with the help of the commutator identity $[xz, y] = {}^x[z, y] \cdot [x, y]$, and take the upper bound $4n(n-1) + 1$ for each of the resulting commutators $[x_i, y]$. One thus gets $N \leq M^2 + 4n(n-1)M$.

However, such explicit formulas are only available for linear and orthogonal groups, and for exceptional groups of types E_6 and E_7. Let us state the estimate resulting from the proof of [93, Theorems 4 and 5].

Fact 5. *Let R be a commutative ring and $\Phi = \mathrm{E}_6, \mathrm{E}_7$. Then any root element of the form $gx_\alpha(\xi)g^{-1}$, $\alpha \in \Phi$, $\xi \in R$, $g \in G(\Phi, R)$ is a product of at most $4 \cdot 16 \cdot 27 = 1728$ elementary root unipotents in the case of $\Phi = \mathrm{E}_6$ and of at most $4 \cdot 27 \cdot 56 = 6048$ elementary root unipotents in the case of $\Phi = \mathrm{E}_7$.*

Even for symplectic groups — not to say for exceptional groups of types $\mathrm{E}_8, \mathrm{F}_4$ and G_2! — it is only known that the elementary groups are generated by root unipotents of certain classes, which afford reduction to smaller ranks, but no explicit polynomial factorisations are known, and even no polynomial length estimates.

This is why generalisation of Theorem 14 to Chevalley groups requires a new idea.

4.3 Commutator width of Chevalley groups

Let us state the main result of [78]. While the main idea of proof comes from the work by Alexander Sivatsky and the second author [72], most of the actual calculations are refinements of conjugation calculus and commutator calculus in Chevalley groups, developed by the first and the third authors in [35].

Theorem 18. *Let $G = G(\Phi, R)$ be a Chevalley group of rank $l \geq 2$ and let R be a ring such that $\dim \mathrm{Max}(R) = d < \infty$. Then there exists a natural number N depending only on Φ and d such that each commutator $[x, y]$ of elements $x \in G(\Phi, R)$ and $y \in E(\Phi, R)$ is a product of at most N elementary root unipotents.*

Here we cannot use decomposition of unipotents. The idea of the second author was to use the *second localisation* instead. As in [72] the proof starts with the following lemma, where M has the same meaning as in Subsection 3.4.

Lemma 1. *Let $d = \dim(\mathrm{Max}(R))$ and $x \in \mathrm{G}(\Phi, R)$. Then there exist $t_0, \ldots, t_k \in R$, where $k \leq d$, generating R as an ideal and such that $F_{t_i}(x) \in \mathrm{E}^M(\Phi, R_{t_i})$ for all $i = 0, \ldots, k$.*

Since $t_0, \ldots, t_k$ are unimodular, their powers also are, so that we can rewrite y as a product of y_i, where each y_i is congruent to e modulo a high power of t_i. In the notation of the next section this means that $y_i \in E(\Phi, R, t_i^m R)$.

When the ring R is Noetherian, $G(\Phi, R, t_i^m R)$ injects into $\mathrm{G}(\Phi, R_{t_i})$ for some high power t_i^m. Thus, it suffices to show that $F_{t_i}([x, y_i])$ is a product of bounded number of elementary factors without denominators in $E(\Phi, R_{t_i})$. This is the first localisation.

The second localisation consists in applying the same argument again, this time in R_{t_i}. Applying Lemma 1 once more we can find $s_0, \ldots, s_d$ forming a unimodular row, such that the images of y_i in $E(\Phi, R_{t_i s_j})$ are products of at most M elementary root unipotents with denominators s_j. Decomposing

$F_{s_j}(x) \in E(\Phi, R_{s_j})$ into a product of root unipotents, and repeatedly applying commutator identities, we eventually reduce the proof to proving that the length of each commutator of the form

$$\Big[x_\alpha\Big(\frac{t_i^l}{s_j}a\Big), x_\beta\Big(\frac{s_j^n}{t_i}b\Big)\Big]$$

is bounded.

4.4 Commutator calculus

Conjugation calculus and commutator calculus consists in rewriting conjugates (resp. commutators) with denominators as products of elementary generators *without* denominators.

Let us state a typical technical result, the base of induction of the commutator calculus.

Lemma 2. *Given $s, t \in R$ and $p, q, k, m \in \mathbb{N}$, there exist $l, m \in \mathbb{N}$ and $L = L(\Phi)$ such that*

$$\Big[x_\alpha\Big(\frac{t^l}{s^k}a\Big), x_\beta\Big(\frac{s^n}{t^m}b\Big)\Big] \in E^L(\Phi, s^p t^q R).$$

A naive use of the Chevalley commutator formula gives $L \leq 585$ for simply laced systems, $L \leq 61882$ for doubly laced systems and $L \leq 797647204$ for $\Phi = \mathrm{G}_2$. And this is just the first step of the commutator calculus!

Reading the proof sketched in the previous subsection upwards, and repeatedly using commutator identities, we can eventually produce bounds for the length of commutators, ridiculous as they can be.

Recently in [34] the authors succeeded in producing a similar proof for Bak's unitary groups, see [28, 47, 8, 36] and references there. The situation here is in many aspects more complicated than for Chevalley groups. In fact, Bak's unitary groups are not always algebraic, and all calculations should be inherently carried through in terms of *form ideals*, rather then ideals of the ground ring. Thus, the results of [34] heavily depend on the *unitary* conjugation calculus and commutator calculus, as developed in [29, 37].

4.5 Universal localisation

Now something truly amazing will happen. Some two years ago the second author noticed that the width of commutators is bounded by a universal constant that depends on the type of the group alone, see [76]. Quite remarkably, one can obtain a length bound that does not depend either on the dimension of the ring, or on the length of the elementary factor.

Theorem 19. *Let $G = G(\Phi, R)$ be a Chevalley group of rank $l \geq 2$. Then there exists a natural number $N = N(\Phi)$ depending on Φ alone, such that each commutator $[x, y]$ of elements $x \in G(\Phi, R)$ and $y \in E(\Phi, R)$ is a product of at most N elementary root unipotents.*

What is remarkable here, is that there is no dependence on R whatsoever. In fact, this bound applies even to infinite dimensional rings! Morally, it says that in the groups of points of algebraic groups there are very few commutators.

Here is a very brief explanation of how it works. First of all, Chevalley groups are representable functors, $G(\Phi, R) = \mathrm{Hom}(\mathbb{Z}[G], R)$, so that there is a *universal element* $g \in \mathrm{G}(\Phi, \mathbb{Z}[G])$, corresponding to $\mathrm{id} : \mathbb{Z}[G] \longrightarrow \mathbb{Z}[G]$, which specialises to *any* element of the Chevalley group $G(\Phi, R)$ of the same type over *any* ring.

But the elementary subgroup $E(\Phi, R)$ is not an algebraic group, so where can one find universal elements?

The real know-how proposed by the second author consists in construction of the universal coefficient rings for the principal congruence subgroups $G(\Phi, R, sR)$ (see the next section, for the definition), corresponding to the principal ideals. It turns out that this is enough to carry through the same scheme of the proof, with bounds that do not depend on the ring R.

5 Relative commutator width

In the absolute case the above results on commutator width are mostly published. In this section we state relative analogues of these results which are announced here for the first time.

5.1 Congruence subgroups

Usually, one defines congruence subgroups as follows. An ideal $A \trianglelefteq R$ determines the reduction homomorphism $\rho_A : R \longrightarrow R/A$. Since $G(\Phi, \_)$ is a functor from rings to groups, this homomorphism induces reduction homomorphism $\rho_A : G(\Phi, R) \longrightarrow G(\Phi, R/A)$.

- The kernel of the reduction homomorphism ρ_A modulo A is called the principal congruence subgroup of level A and is denoted by $G(\Phi, R, A)$.
- The full pre-image of the centre of $G(\Phi, R/A)$ with respect to the reduction homomorphism ρ_A modulo A is called the full congruence subgroup of level A, and is denoted by $C(\Phi, R, A)$.

But in fact, without assumption that $2 \in R^*$ for doubly laced systems, and without assumption that $6 \in R^*$ for $\Phi = \mathrm{G}_2$, the genuine congruence subgroups should be defined in terms of admissible pairs of ideals (A, B), introduced by Abe, [1, 4, 2, 3], and in terms of form ideals for symplectic groups. One of these ideals corresponds to short roots and another one corresponds to long roots.

In [30] we introduced a more general notion of congruence subgroups, corresponding to admissible pairs: $G(\Phi, R, A, B)$ and $C(\Phi, R, A, B)$, Not to overburden the note with technical details, we mostly tacitly assume that $2 \in R^*$ for $\Phi = \mathrm{B}_l, \mathbf{C}_l, \mathrm{F}_4$ and $6 \in R^*$ for $\Phi = \mathrm{G}_2$. Under these simplifying assumption one has $A = B$ and $G(\Phi, R, A, B) = G(\Phi, R, A)$ and $C(\Phi, R, A, B) = C(\Phi, R, A)$. Of course, using admissible pairs/form ideals one can obtained similar results without any such assumptions.

5.2 Relative elementary groups

Let A be an additive subgroup of R. Then $E(\Phi, A)$ denotes the subgroup of E generated by all elementary root unipotents $x_\alpha(\xi)$ where $\alpha \in \Phi$ and $\xi \in A$. Further, let L denote a nonnegative integer and let $E^L(\Phi, A)$ denote the *subset* of $E(\Phi, A)$ consisting of all products of L or fewer elementary root unipotents $x_\alpha(\xi)$, where $\alpha \in \Phi$ and $\xi \in A$. In particular, $E^1(\Phi, A)$ is the set of all $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in A$.

In the sequel we are interested in the case where $A = I$ is an ideal of R. In this case we denote by

$$E(\Phi, R, I) = E(\Phi, I)^{E(\Phi, R)}$$

the *relative* elementary subgroup of level I. As a *normal* subgroup of $E(\Phi, R)$ it is generated by $x_\alpha(\xi)$, $\alpha \in \Phi$, $\xi \in A$. The following theorem [74, 86, 88] lists its generators as a subgroup.

Theorem 20. *As a subgroup $E(\Phi, R, I)$ is generated by the elements*

$$z_\alpha(\xi, \zeta) = x_{-\alpha}(\zeta) x_\alpha(\xi) x_{-\alpha}(-\zeta),$$

where $\xi \in I$ for $\alpha \in \Phi$, while $\zeta \in R$.

It is natural to regard these generators as the *elementary generators* of $E(\Phi, R, I)$. For the special linear group $\mathrm{SL}(n, \mathcal{O}_S)$, $n \geq 3$, over a Dedekind ring of arithmetic type Bernhard Liehl [54] has proven bounded generation of the elementary relative subgroups $E(n, \mathcal{O}_S, I)$ in the generators $z_{ij}(\xi, \zeta)$. What is remarkable in his result, is that the bound does not depend on the ideal I. Also, he established similar results for $\mathrm{SL}(2, \mathcal{O}_S)$, provided that $\mathcal{O}_S^*$ is infinite.

5.3 Standard commutator formula

The following result was first proven by Giovanni Taddei [83], Leonid Vaserstein [88] and Eiichi Abe [2, 3].

Theorem 21. *Let Φ be a reduced irreducible root system of rank ≥ 2, R be a commutative ring, $I \trianglelefteq R$ be an ideal of R. In the case, where $\Phi = \mathrm{B}_2$ or $\Phi = \mathrm{G}_2$ assume moreover that R has no residue fields $\mathbb{F}_2$ of 2 elements. Then the following standard commutator formula holds*

$$\big[G(\Phi, R), E(\Phi, R, I)\big] = \big[E(\Phi, R), C(\Phi, R, I)\big] = E(\Phi, R, I).$$

In fact, in [30] we established similar result for relative groups defined in terms of admissible pairs, rather then single ideals. Of course, in all cases, except Chevalley groups of type F_4, it was known before, [8, 63, 18].

With the use of level calculations the following result was established by You Hong [100], by analogy with the Alec Mason and Wilson Stothers [59, 56, 57, 58]. Recently the first, third and fourth authors gave another proof, of this result, in the framework of relative localisation [38], see also [97, 40, 98, 31, 37, 41, 32, 39, 33, 76] for many further analogues and generalisations of such formulas.

Theorem 22. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geq 2$. *Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two ideals of R. Then*

$$[E(\Phi, R, A), G(\Phi, R, B)] = [E(\Phi, R, A), E(\Phi, R, B)].$$

5.4 Generation of mixed commutator subgroups

It is easy to see that the mixed commutator $[E(\Phi, R, A), E(\Phi, R, B)]$ is a subgroup of level AB, in other words, it sits between the relative elementary subgroup $E(\Phi, R, AB)$ and the corresponding congruence subgroup $G(\Phi, R, AB)$.

Theorem 23. *Let Φ be a reduced irreducible root system,* $\mathrm{rk}(\Phi) \geq 2$. *Further, let R be a commutative ring, and $A, B \trianglelefteq R$ be two ideals of R. When $\Phi = \mathrm{B}_2, \mathrm{G}_2$, assume that R does not have residue field of 2 elements, and when $\Phi = \mathbf{C}_l$, $l \geq 2$, assume additionally that any $a \in R$ is contained in the ideal $a^2R + 2aR$. Then*

$$E(\Phi, R, AB) \leq [E(\Phi, R, A), E(\Phi, R, B)] \leq \\ [G(\Phi, R, A), G(\Phi, R, B)] \leq G(\Phi, R, AB).$$

It is not too difficult to construct examples showing that in general the mixed commutator subgroup $[E(\Phi, R, A), E(\Phi, R, B)]$ can be strictly larger than $E(\Phi, R, AB)$. The first such examples were constructed by Alec Mason and Wilson Stothers [59, 57] in the ring $R = \mathbb{Z}[i]$ of Gaussian integers.

In this connection, it is very interesting to explicitly list generators of the mixed commutator subgroups $[E(\Phi, R, A), E(\Phi, R, B)]$ *as subgroups.* From Theorem 20 we already know most of these generators. These are $z_\alpha(\xi\zeta, \eta)$, where $\xi \in A$, $\zeta \in B$, $\eta, \vartheta \in R$. But what are the remaining ones?

In fact, using the Chevalley commutator formula it is relatively easy to show that $[E(\Phi,R,A),E(\Phi,R,B)]$ is generated by its intersections with the fundamental SL_2's. Using somewhat more detailed analysis the first and the fourth author established the following result, initially for the case of $\mathrm{GL}(n,R)$, $n\geq 3$, see [41] and then, jointly with the third author, for all other cases, see [32, 39].

Theorem 24. *Let R be a commutative ring with 1 and A, B be two ideals of R. Then the mixed commutator subgroup $\big[E(\Phi,R,A),E(\Phi,R,B)\big]$ is generated as a normal subgroup of $E(n,R)$ by the elements of the form*

- $\big[x_\alpha(\xi), {}^{x_{-\alpha}(\eta)}x_\alpha(\zeta)\big]$,
- $\big[x_\alpha(\xi), x_{-\alpha}(\zeta)\big]$,
- $x_\alpha(\xi\zeta)$,

where $\alpha\in\Phi$, $\xi\in A$, $\zeta\in B$, $\eta\in R$.

Another moderate technical effort allows to make it into a natural candidate for the set of elementary generators of $[E(\Phi,R,A),E(\Phi,R,B)]$.

Theorem 25. *Let R be a commutative ring with 1 and I, J be two ideals of R. Then the mixed commutator subgroup $\big[E(\Phi,R,A),E(\Phi,R,B)\big]$ is generated as a group by the elements of the form*

- $\big[z_\alpha(\xi,\eta), z_\alpha(\zeta,\vartheta)\big]$,
- $\big[z_\alpha(\xi,\eta), z_{-\alpha}(\zeta,\vartheta)\big]$,
- $z_\alpha(\xi\zeta,\eta)$,

where $\alpha\in\Phi$, $\xi\in A$, $\zeta\in B$, $\eta,\vartheta\in R$.

5.5 Relative commutator width

Now we are all set to address relative versions of the main problem. The two following results were recently obtained by the second author, with his method of universal localisation [76], but they depend on the construction of generators in Theorems 20 and 25. Mostly, the preceding results were either published or prepublished in some form, and announced at various conferences. These two theorems are stated here for the first time.

Theorem 26. *Let R be a commutative ring with 1 and let $I\trianglelefteq R$, be an ideal of R. Then there exists a natural number $N=N(\Phi)$ depending on Φ alone, such that any commutator $[x,y]$, where*

$$x\in G(\Phi,R,I),\quad y\in E(\Phi,R)\qquad or\qquad x\in G(\Phi,R),\quad y\in E(\Phi,R,I)$$

is a product of not more that N elementary generators $z_\alpha(\xi,\zeta)$, $\alpha\in\Phi$, $\xi\in I$, $\zeta\in R$.

Theorem 27. *Let R be a commutative ring with 1 and let $A, B \trianglelefteq R$, be ideals of R. there exists a natural number $N = N(\Phi)$ depending on Φ alone, such that any commutator*

$$[x,y], \qquad x\in G(\Phi,R,A), \quad y\in E(\Phi,R,B)$$

is a product of not more that N elementary generators listed in Theorem 25.

Quite remarkably, the bound N in these theorems does not depend either on the ring R, or on the choice of the ideals I, A, B. The proof of these theorems is not particularly long, but it relies on a whole bunch of universal constructions and will be published in **??**. From the proof, it becomes apparent that similar results hold also in other such situations: for any other functorial generating set, for multiple relative commutators [41, 39], etc.

6 Loose ends

Let us mention some positive results on commutator width and possible further generalisations.

6.1 Some positive results

There are some obvious bounds for the commutator width that follow from unitriangular factorisations. For the $\mathrm{SL}(n,R)$ the following result was observed by van der Kallen, Dennis and Vaserstein. The proof in general was proposed by Nikolai Gordeev and You Hong in 2005, but is still not published, as far as we know.

Theorem 28. *Let $\mathrm{rk}(\Phi)\geq 2$. Then for any commutative ring R an element of $U(\Phi,R)$ is a product of not more than two commutators in $E(\Phi,R)$.*

Combining the previous theorem with Theorem 8 we get the following corollary.

Corollary 4. *Let $\mathrm{rk}(\Phi)\geq 2$ and let R be a ring such that $\mathrm{sr}(R)=1$. Then the any element of $E(\Phi,R)$ is a product of ≤ 6 commutators.*

This focuses attention on the following problem.

Problem 3. *Find the shortest factorisation of $E(\Phi,R)$ of the form*

$$E = UU^-UU^-\dots U^\pm.$$

Let us reproduce another result from the paper by Andrei Smolensky, Sury and the third author [96]. It is proven similarly to Theorem 8, but uses Cooke—Weinberger [17] as induction base. Observe that it depends on the Generalised Riemann's Hypothesis, which is used to prove results in the style of Artin's conjecture on primitive roots in arithmetic progressions. Lately, Maxim Vsemirnov succeeded in improving bounds and in some cases eliminating dependence on GRH. In particular, Cooke—Weinberger construct a division chain of length 7 in the non totally imaginary case, the observation that it can be improved to a division chain of length 5 is due to Vsemirnov [99]. Again, in the form below, with $G(\Phi, \mathcal{O}_S)$ rather than $E(\Phi, \mathcal{O}_S)$, it relies on the almost positive solution of the congruence subgroup problem [10, 60].

Theorem 29. *Let $R = \mathcal{O}_S$ be a Dedekind ring of arithmetic type with infinite multiplicative group. Then under the Generalised Riemann Hypothesis the simply connected Chevalley group $G_{\mathrm{sc}}(\Phi, \mathcal{O}_S)$ admits unitriangular factorisation of length* 9,

$$G_{\mathrm{sc}}(\Phi, \mathcal{O}_S) = UU^-UU^-UU^-UU^-U.$$

In the case, where $\mathcal{O}_S$ has a real embedding, it admits unitriangular factorisation of length 5,

$$G_{\mathrm{sc}}(\Phi, \mathcal{O}_S) = UU^-UU^-U.$$

Corollary 5. *Let $\mathrm{rk}(\Phi) \geq 2$ and let $\mathcal{O}_S$ be a Dedekind ring of arithmetic type with infinite multiplicative group. Then the any element of $G_{\mathrm{sc}}(\Phi, \mathcal{O}_S)$ is a product of ≤ 10 commutators. In the case, where $\mathcal{O}_S$ has a real embedding, this estimate can be improved to ≤ 6 commutators.*

6.2 Conjectures concerning commutator width

We believe that solution of the following two problems is now at hand. In Section 2 we have already cited the works of Frank Arlinghaus, Leonid Vaserstein, Ethel Wheland and You Hong [90, 91, 101, 5], where this is essentially done for *classical* groups, over rings subject to $\mathrm{sr}(R) = 1$ or some stronger stability conditions.

Problem 4. *Under assumption $\mathrm{sr}(R) = 1$ prove that any element of elementary group $E_{\mathrm{ad}}(\Phi, R)$ is a product of ≤ 2 commutators in $G_{\mathrm{ad}}(\Phi, R)$.*

Problem 5. *Under assumption $\mathrm{sr}(R) = 1$ prove that any element of elementary group $E(\Phi, R)$ is a product of ≤ 3 commutators in $E(\Phi, R)$.*

It may well be that under this assumption the commutator width of $E(\Phi, R)$ is always ≤ 2, but so far we were unable to control details concerning semisimple factors.

It seems, that one can apply the same argument to higher stable ranks. Solution of the following problem would be a generalisation of [19, Theorem 4].

Problem 6. *If the stable rank* $\mathrm{sr}(R)$ *of* R *is finite, and for some* $m \geq 2$ *the elementary linear group* $E(m, R)$ *has bounded word length with respect to elementary generators, then for all* Φ *of sufficiently large rank any element of* $E(\Phi, R)$ *is a product of* ≤ 4 *commutators in* $E(\Phi, R)$.

Problem 7. *Let* R *be a Dedekind ring of arithmetic type with infinite multiplicative group. Prove that any element of* $E_{\mathrm{ad}}(\Phi, R)$ *is a product of* ≤ 3 *commutators in* $G_{\mathrm{ad}}(\Phi, R)$.

Some of our colleagues expressed belief that any element of $\mathrm{SL}(n, \mathbb{Z})$, $n \geq 3$, is a product of ≤ 2 commutators. However, for Dedekind rings with *finite* multiplicative groups, such as $\mathbb{Z}$, at present we do not envisage any *obvious* possibility to improve the generic bound ≤ 4 even for large values of n. Expressing elements of $\mathrm{SL}(n, \mathbb{Z})$ as products of 2 commutators, if it can be done at all, should require a lot of specific case by case analysis.

6.3 The group $\mathrm{SL}(2, R)$: improved generators

One could also mention the recent paper by Leonid Vaserstein [89] which shows that for the group $\mathrm{SL}(2, R)$ it is natural to consider bounded generation not in terms of the elementary generators, but rather in terms of the generators of the pre-stability kernel $\tilde{E}(2, R)$. In other words, one should also consider matrices of the form $(e + xy)(e + yx)^{-1}$.

Theorem 30. *The group* $\mathrm{SL}(2, \mathbb{Z})$ *admits polynomial parametrisation of total degree* ≤ 78 *with* 46 *parameters.*

The idea is remarkably simple. Namely, Vaserstein observes that $\mathrm{SL}(2, \mathbb{Z})$ coincides with the pre-stability kernel $\tilde{E}(2, \mathbb{Z})$. All generators of the group $\tilde{E}(2, \mathbb{Z})$, not just the elementary ones, admit polynomial parametrisation. The additional generators require 5 parameters each.

It only remains to verify that each element of $\mathrm{SL}(2, \mathbb{Z})$ has a small length, with respect to this new set of generators. A specific formula in [89] expresses an element of $\mathrm{SL}(2, \mathbb{Z})$ as a product of 26 elementary generators and 4 additional generators, which gives $26 + 4 \cdot 5 = 46$ parameters mentioned in the above theorem.

6.4 Bounded generation and Kazhdan property

The following result is due to Yehuda Shalom [70], Theorem 8, see also [71, 46].

Theorem 31. *Let R be an m-generated commutative ring, $n \geq 3$. Assume that $E(n,R)$ has bounded width C in elementary generators. Then $E(n,R)$ has property T. In an appropriate generating system S the Kazhdan constant is bounded from below*

$$\mathcal{K}(G,S) \geq \frac{1}{C \cdot 22^{n+1}}.$$

Problem 8. *Does the group* $\mathrm{SL}(n,\mathbb{Z}[x])$*, $n \geq 3$, has bounded width with respect to the set of elementary generators?*

If this problem has positive solution, then by Suslin's theorem and Shalom's theorem the groups $\mathrm{SL}(n,\mathbb{Z}[x])$ have Kazhdan property T. Thus,

Problem 9. *Does the group* $\mathrm{SL}(n,\mathbb{Z}[x])$*, $n \geq 3$, have Kazhdan property T?*

If this is the case, one can give a uniform bound of the Kazhdan constant of the groups $\mathrm{SL}(n,\mathcal{O})$, for the rings if algebraic integers. It is known that these group have Kazhdan property, but the known estimates depend on the discriminant of the ring $\mathcal{O}$.

Problem 10. *Prove that the group* $\mathrm{SL}(n,\mathbb{Q}[x])$ *does not have bounded width with respect to the elementary generators.*

It is natural to try to generalise results of Bernhard Liehl [54] to other Chevalley groups. The first of the following problems was stated by Oleg Tavgen in [84]. As always, we assume that $\mathrm{rk}(\Phi) \geq 2$. Otherwise, Problem 12 is open for the group $\mathrm{SL}(2,\mathcal{O}_S)$, provided that the multiplicative group $\mathcal{O}_S^*$ is infinite.

Problem 11. *Prove that over a Dedekind ring of arithmetic type the relative elementary groups $E(\Phi,\mathcal{O}_S,I)$ have bounded width with respect to the elementary generators $z_\alpha(\xi,\zeta)$, with a bound that does not depend on I.*

Problem 12. *Prove that over a Dedekind ring of arithmetic type the mixed commutator subgroups $[E(\Phi,\mathcal{O}_S,A),E(\Phi,\mathcal{O}_S,B)]$ have bounded width with respect to the elementary generators constructed in Theorem* 25*, with a bound that does not depend on A and B.*

6.5 Not just commutators

It is very challenging to understand, to which extent such behaviour is typical for more general classes of group words. There are a lot of recent results showing that the verbal length of the finite simple groups is strikingly small [68, 69, 49, 50, 53, 27]. In fact, under some natural assumptions for large finite simple groups this verbal length is 2. We do not expect similar results to hold for rings other than the zero-dimensional ones, and some arithmetic rings of dimension 1.

Powers are a class of words in a certain sense opposite to commutators. Alireza Abdollahi suggested that before passing to more general words, we should first look at powers.

Problem 13. *Establish finite width of powers in elementary generators, or lack thereof.*

An answer – in fact, *any* answer! – to this problem would be amazing. However, we would be less surprised if for rings of dimension ≥ 2 the verbal maps in $G(R)$ would have very small images.

Acknowledgements. The authors thank Francesco Catino, Francesco de Giovanni and Carlo Scoppola for an invitation to give a talk on commutator width at the Conference in Porto Cesareo, which helped us to focus thoughts in this direction. Also, we would like to thank Nikolai Gordeev and You Hong for inspiring discussions of positive results on commutator width and related problems, Sury and Maxim Vsemirnov for discussion of arithmetic aspects, Alireza Abdollahi for suggestion to look at powers, Anastasia Stavrova amd Matthias Wendt for some very pertinent remarks concerning localisation, and correcting some misprints in the original version of our lemmas of conjugation calculus and commutator calculus.

References

[1] E. Abe: *Chevalley groups over local rings*, Tôhoku Math. J., **21** (1969), n. 3, 474–494.

[2] E. Abe: *Chevalley groups over commutative rings*, Proc. Conf. Radical Theory (Sendai, 1988), Uchida Rokakuho, Tokyo (1989), 1–23.

[3] E. Abe: *Normal subgroups of Chevalley groups over commutative rings*, Algebraic K-Theory and Algebraic Number Theory (Honolulu, HI, 1987), Contemp. Math., **83**, Amer. Math. Soc., Providence, RI (1989), 1–17.

[4] E. Abe, K. Suzuki: *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J., **28** (1976), n. 1, 185–198.

[5] F. A. Arlinghaus, L. N. Vaserstein, You Hong: *Commutators in pseudo-orthogonal groups*, J. Austral. Math. Soc., Ser. A, **59** (1995), 353–365.

[6] A. Bak: *Nonabelian K-theory: the nilpotent class of K_1 and general stability*, K-Theory, **4** (1991), 363–397.

[7] A. Bak, R. Hazrat and N. Vavilov: *Localization completion strikes again: relative K_1 is nilpotent by abelian*, J. Pure Appl. Algebra, **213** (2009), 1075–1085.

[8] A. Bak, N. Vavilov: *Structure of hyperbolic unitary groups I: elementary subgroups*, Algebra Colloquium, **7** (2000), n. 2, 159–196.

[9] H. Bass: *K-theory and stable algebra*, Publ. Math. Inst Hautes Etudes Sci., **22** (1964), 5–60.

[10] H. Bass, J. Milnor, J.-P. Serre: *Solution of the congruence subgroup problem for* SL_n $(n \geq 3)$ *and* Sp_{2n} $(n \geq 2)$, Publ. Math. Inst. Hautes Etudes Sci., **33** (1967), 59–137.

[11] D. Carter, G. E. Keller: *Bounded elementary generation of* $\mathrm{SL}_n(\mathcal{O})$, Amer. J. Math., **105** (1983), 673–687.

[12] D. Carter, G. E. Keller: *Elementary expressions for unimodular matrices*, Commun. Algebra, **12** (1984), 379–389.

[13] D. Carter, G. E. Keller: *Bounded elementary expressions in* $\mathrm{SL}(2,\mathcal{O})$, Preprint Univ. Virginia, (1985), 1–11.

[14] D. Carter, G. E. Keller: *The congruence subgroup problem for non standard models*, Preprint Univ. Virginia, (1985), 1–44.

[15] D. Carter, G. E. Keller, E. Paige: *Bounded expressions in* $\mathrm{SL}(2,\mathcal{O})$, Preprint Univ. Virginia, (1985), 1–21.

[16] V. Chernousov, E. Ellers, N. Gordeev: *Gauss decomposition with prescribed semisimple part: short proof*, J. Algebra, **229** (2000), 314–332.

[17] G. Cooke, P. J. Weinberger: *On the construction of division chains in algebraic number rings, with applications to* SL_2, Commun. Algebra, **3** (1975), 481–524.

[18] D. L. Costa, G. E. Keller: *On the normal subgroups of* $\mathrm{G}_2(A)$, Trans. Amer. Math. Soc. **351**, 12 (1999), 5051–5088.

[19] R. K. Dennis, L. N. Vaserstein: *On a question of M. Newman on the number of commutators*, J. Algebra, **118** (1988), 150–161.

[20] R. K. Dennis, L. N. Vaserstein: *Commutators in linear groups*, K-theory, **2** (1989), 761–767.

[21] E. Ellers, N. Gordeev: *On the conjectures of J. Thompson and O. Ore*, Trans. Amer. Math. Soc., **350** (1998), 3657–3671.

[22] I. V. Erovenko: $\mathrm{SL}_n(F[x])$ *is not boundedly generated by elementary matrices: explicit proof*, Electronic J. Linear Algebra, **11** (2004), 162–167.

[23] I. V. Erovenko: Bounded generation of S-arithmetic orthogonal groups, Ph. D. Thesis, Univ. Virginia, (2002), 1–101.

[24] I. V. Erovenko, A. S. Rapinchuk: *Bounded generation of some S-arithmetic orthogonal groups*, C. R. Acad. Sci., **333** (2001), n. 5, 395–398.

[25] I. V. Erovenko, A. S. Rapinchuk: *Bounded generation of S-arithmetic subgroups of isotropic orthogonal groups over number fields*, J. Number Theory, **119** (2008), n. 1, 28–48.

[26] D. R. Grayson: SK_1 *of an interesting principal ideal domain*, J. Pure Appl. Algebra, **20** (1981), 157–163.

[27] R. M. Guralnick, G. Malle: *Products of conjugacy classes and fixed point spaces*, J. Amer. Math. Soc., **25** (2012), n. 1, 77–121.

[28] A. J. Hahn, O. T. O'Meara: The classical groups and K-theory, Springer, Berlin et al. 1989.

[29] R. Hazrat: *Dimension theory and nonstable K_1 of quadratic modules*, K-Theory, **27** (2002), 293–328.

[30] R. Hazrat, V. Petrov, N. Vavilov: *Relative subgroups in Chevalley groups*, J. K-theory, **5** (2010), 603–618.

[31] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang: *The yoga of commutators*, J. Math. Sci. (N. Y.) **179** (2011), n. 6, 662–678.

[32] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang: *The yoga of commutators, further applications*, J. Math. Sci. (N. Y.), (2012), to appear.

[33] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang: *Multiple commutator formula.* II, (2012), to appear.

[34] R. Hazrat, A. Stepanov, N. Vavilov, Z. Zhang: *On the length of commutators in unitary groups*, (2012), to appear.

[35] R. Hazrat, N. Vavilov: K_1 *of Chevalley groups are nilpotent*, J. Pure Appl. Algebra, **179** (2003), 99–116.

[36] R. Hazrat, N. Vavilov: *Bak's work on the K-theory of rings (with an appendix by Max Karoubi)*, J. K-Theory, **4** (2009), 1–65.

[37] R. Hazrat, N. Vavilov, Z. Zhang: *Relative unitary commutator calculus, and applications*, J. Algebra, **343** (2011), 107–137.

[38] R. Hazrat, N. Vavilov, Z. Zhang: *Relative commutator calculus in Chevalley groups*, J. Algebra, (2012), 1–35, to appear. Preprint `arXiv:1107.3009v1 [math.RA]`.

[39] R. Hazrat, N. Vavilov, Z. Zhang: *Multiple commutator formulas for unitary groups*, (2012), 1–22, to appear.

[40] R. Hazrat, Z. Zhang: *Generalized commutator formulas*, Comm. Algebra, **39** (2011), n. 4, 1441–1454.

[41] R. Hazrat, Z. Zhang: *Multiple commutator formula*, Israel J. Math. (2012), 1–25, to appear.

[42] E. K. Hinson: *Word length in elementary matrices*, J. Algebra, **142** (1991), n. 1, 76–80.

[43] F. Ischebeck: *Hauptidealringe mit nichttrivialer* SK_1*-Gruppe*, Arch. Math., **35** (1980), 138–139.

[44] W. van der Kallen: $\mathrm{SL}_3(\mathbb{C}[x])$ *does not have bounded word length*, Springer Lecture Notes Math., **966** (1982), 357–361.

[45] W. van der Kallen: *A module structure on certain orbit sets of unimodular rows*, J. Pure Appl. Algebra, **57** (1989), 281–316.

[46] M. Kassabov, N. Nikolov: *Universal lattices and property tau*, Invent. Math., **165** (2006), 209–224.

[47] M.-A. Knus: Quadratic and hermitian forms over rings, Springer Verlag, Berlin et al., 1991

[48] V. I. Kopeiko: *The stabilization of symplectic groups over a polynomial ring*, Math. U.S.S.R. Sbornik, **34** (1978), 655–669.

[49] M. Larsen, A. Shalev: *Word maps and Waring type problems*, J. Amer. Math. Soc., **22** (2009), 437–466.

[50] M. Larsen, A. Shalev, Pham Huu Tiep: *The Waring problem for finite simple groups*, Ann. Math., **174** (2011), 1885–1950.

[51] M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep: *The Ore conjecture*, J. Europ. Math. Soc., **12** (2010), 939–1008.

[52] M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep: *Commutators in finite quasisimple groups*, Bull. London Math. Soc., **43** (2011), 1079–1092.

[53] M. Liebeck, E. A. O'Brien, A. Shalev, Pham Huu Tiep: *Products of squares in finite simple groups*, Proc. Amer. Math. Soc., **43** (2012), n. 3, 1079–1092.

[54] B. Liehl: *Beschränkte Wortlänge in* SL_2, Math. Z., **186** (1984), 509–524.

[55] D. Loukanidis, V. K. Murty: *Bounded generation for* SL_n $(n \geq 2)$ *and* Sp_n $(n \geq 1)$, Preprint.

[56] A. W. Mason: *A note on subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators*, J. London Math. Soc., **11** (1974), 509–512.

[57] A. W. Mason: *On subgroup of* $\mathrm{GL}(n, A)$ *which are generated by commutators*, II, J. reine angew. Math., **322** (1981), 118–135.

[58] A. W. Mason: *A further note on subgroups of* $\mathrm{GL}(n, A)$ *which are generated by commutators*, Arch. Math., **37** (1981), n. 5, 401–405.

[59] A. W. Mason, W. W. Stothers: *On subgroup of* $\mathrm{GL}(n, A)$ *which are generated by commutators*, Invent. Math., **23** (1974), 327–346.

[60] H. Matsumoto: *Sur les sous-groupes arithmétiques des groupes semi-simples déployés*, Ann. Sci. École Norm. Sup. ser. 4, **2** (1969), 1–62.

[61] D. W. Morris: *Bounded generation of* $\mathrm{SL}(n, A)$ *(after D. Carter, G. Keller, and E. Paige)*, New York J. Math., **13** (2007), 383–421.

[62] V. K. Murty: *Bounded and finite generation of arithmetic groups*, Number Theory (Halifax, 1994), Amer. Math. Soc., Providence, RI, **15** (1995), 249–261.

[63] V. A. Petrov: *Odd unitary groups*, J. Math. Sci., **130** (2003), n. 3, 4752–4766.

[64] E. Plotkin: *Stability theorems for* K*-functors for Chevalley groups*, Proc. Conf. Nonassociative Algebras and Related Topics (Hiroshima — 1990), World Sci.London et al. (1991), 203–217.

[65] E. B. Plotkin: *Surjective stabilization for* K_1*-functor for some exceptional Chevalley groups*, J. Soviet Math., **64** 1993, 751–767.

[66] E. Plotkin: *On the stability of the* K_1*-functor for Chevalley groups of type* E_7, J. Algebra, **210** (1998), 67–85.

[67] R. Ree: *Commutators in semi-simple algebraic groups*, Proc. Amer. Math. Soc., **15** (1964), 457–460.

[68] A. Shalev: *Commutators, words, conjugacy classes, and character methods*, Turk. J. Math., **31** (2007), 131–148.

[69] A. Shalev: *Word maps, conjugacy classes, and a noncommutative Waring-type theorem*, Ann. Math., **170** (2009), n. 3, 1383–1416.

[70] Y. Shalom: *Bounded generation and Kazhdan property (T)*, Inst. Hautes Études Sci. Publ. Math., **90** (1999), 145–168.

[71] Y. Shalom: *The algebraisation of Kazhdan property (T)*, Intern. Congress Mathematicians, **II** (2006), 1283–1310.

[72] A. Sivatski, A. Stepanov: *On the word length of commutators in* $\mathrm{GL}_n(R)$, K-theory, **17** (1999), 295–302.

[73] A. Smolensky, B. Sury, N. Vavilov: *Gauss decomposition for Chevalley groups revisited*, Intern. J. Group Theory, **1** (2012), n. 1, 3–16.

[74] M. R. Stein: *Generators, relations and coverings of Chevalley groups over commutative rings*, Amer. J. Math., **93** (1971), n. 4, 965–1004.

[75] M. R. Stein: *Stability theorems for* K_1, K_2 *and related functors modeled on Chevalley groups*, Japan. J. Math., **4** (1978), n. 1, 77–108.

[76] A. Stepanov: *Structure of Chevalley groups over rings via universal localization*, J. K-theory (2013), to appear.

[77] A. Stepanov, N. Vavilov: *Decomposition of transvections: a theme with variations*, K-theory, **19** (2000), 109–153.

[78] A. Stepanov, N. Vavilov: *On the length of commutators in Chevalley groups*, Israel J. Math., **185** (2011), 253–276.

[79] Sury B.: The congruence subgroup problem, Hindustan Book Agency, New Delhi, (2003).

[80] A. A. Suslin: *On the structure of the special linear group over the ring of polynomials*, Izv. Akad. Nauk SSSR, Ser. Mat., **141** (1977), n. 2, 235–253.

[81] A. A. Suslin, V. I. Kopeiko: *Quadratic modules and orthogonal groups over polynomial rings*, J. Sov. Math., **20** (1985), n. 6, 2665–2691.

[82] A. A. Suslin, M. S. Tulenbaev: *Stabilization theorem for Milnor's K_2-functor*, J. Sov. Math., **17** (1981), 1804–1819.

[83] G. Taddei, *Normalité des groupes élémentaires dans les groupes de Chevalley sur un anneau*, Contemp. Math., **55** (1986), 693–710.

[84] O. I. Tavgen: *Bounded generation of Chevalley groups over rings of S-integer algebraic numbers*, Izv. Acad. Sci. USSR **54** (1990), n. 1, 97–122.

[85] O. I. Tavgen: *Bounded generation of normal and twisted Chevalley groups over the rings of S-integers*, Contemp. Math., **131** (1992), n. 1, 409–421.

[86] J. Tits: *Systèmes générateurs de groupes de congruence*, C. R. Acad. Sci. Paris, Sér A, **283** (1976), 693–695.

[87] L. N. Vaserstein: *Bass's first stable range condition*, J. Pure Appl. Algebra, **34** (1984), nn. 2–3, 319–330.

[88] L. N. Vaserstein: *On normal subgroups of Chevalley groups over commutative rings*, Tôhoku Math. J., **36** (1986), n. 5, 219–230.

[89] L. N. Vaserstein: *Polynomial parametrization for the solution of Diophantine equations and arithmetic groups*, Ann. Math., bf 171 (2010), n. 2, 979–1009.

[90] L. N. Vaserstein, E. Wheland: *Factorization of invertible matrices over rings of stable rank one*, J. Austral. Math. Soc., Ser. A, **48** (1990), 455–460.

[91] L. N. Vaserstein, E. Wheland: *Commutators and companion matrices over rings of stable rank* 1, Linear Algebra Appl., **142** (1990), 263–277.

[92] N. Vavilov: *Structure of Chevalley groups over commutative rings*, Proc. Conf. Nonassociative Algebras and Related Topics (Hiroshima, 1990), World Sci. Publ., London et al., 1991, 219–335.

[93] N. Vavilov: *A third look at weight diagrams*, Rend. Sem. Mat. Univ. Padova., **104** (2000), n. 1, 201–250.

[94] N. A. Vavilov: *Can one see the signs of the structure constants*?, St. Petersburg Math. J., **19** (2008), n. 4, 519–543.

[95] N. Vavilov, E. Plotkin: *Chevalley groups over commutative rings* I*: Elementary calculations*, Acta Applic. Math., **45** (1996), 73–113.

[96] N. A. Vavilov, A. V. Smolensky, B. Sury: *Unitriangular factorisations of Chevalley groups*, J. Math. Sci., **183** (2012), no. 5, 584–599.

[97] N. A. Vavilov, A. V. Stepanov: *Standard commutator formula*, Vestnik St.-Petersburg Univ., ser.1, **41** (2008), n. 1, 5–8.

[98] N. A. Vavilov, A. V. Stepanov: *Standard commutator formulae, revisited*, Vestnik St.-Petersburg State Univ., ser.1, **43** (2010), no. 1, 12–17.

[99] M. Vsemirnov: *Short unitriangular factorisations of* $\mathrm{SL}_2\left(\mathbb{Z}\left[\frac{1}{p}\right]\right)$, Quart. J. Math. Oxford, (2012), 1–15, to appear.

[100] Hong You: *On subgroups of Chevalley groups which are generated by commutators*, J. Northeast Normal Univ., **2** (1992), 9–13.

[101] Hong You: *Commutators and unipotents in symplectic groups*, Acta Math. Sinica, New Ser., **10** (1994), 173–179.