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Generalisations of T-groups

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Abstract. This paper discusses work with Adolfo Ballester-Bolinches, James Beidleman, M.C. Pedraza-Aguilera, and M. F. Ragland. Let f be a subgroup embedding functor such that for every finite group G, $f(G)$ contains the set of normal subgroups of G and is contained in the set of Sylow-permutable subgroups of G. We say $H\ f\ G$ if H is an element of $f(G)$. Given such an f, let fT denote the class of finite groups in which $H\ f\ G$ if and only if H is subnormal in G; because Sylow-permutable subgroups are subnormal, this is the class in which f is a transitive relation. Thus if $f(G)$ is, respectively, the set of normal subgroups, permutable subgroups, or Sylow-permutable subgroups of G, then fT is, respectively, the class of T-groups, PT-groups, or PST-groups. Let $\mathcal{F}$ be a formation of finite groups. A subgroup M of a finite group G is said to be $\mathcal{F}$-normal in G if $G/Core_G(M)$ belongs to $\mathcal{F}$. A subgroup U of a finite group G is called a K-$\mathcal{F}$-subnormal subgroup of G if either $U = G$ or there exist subgroups $U = U_0 \leq U_1 \leq \cdots \leq U_n = G$ such that U_{i-1} is either normal or $\mathcal{F}$-normal in U_i, for $i = 1, 2, \ldots, n$. We call a finite group G an $fT_{\mathcal{F}}$-group if every K-$\mathcal{F}$-subnormal subgroup of G is in $f(G)$. When $\mathcal{F}$ is the class of all finite nilpotent groups, the $fT_{\mathcal{F}}$-groups are precisely the fT-groups. We analyse the structure of $fT_{\mathcal{F}}$-groups for certain classes of formations, particularly where the fT-groups are the T-, PT-, and PST-groups.

Keywords: T-groups, formations

MSC 2000 classification: 20D99

This paper includes work done with A. Ballester-Bolinches, M.C. Pedraza-Aguilera, M. Ragland, and J. Beidleman. See [1] for results on the situation in which $f(G)$ is the set of normal subgroups of G and [2] for results about T-, PT-, and PST-groups.

All groups treated are finite.

Definitions

A subgroup H is *subnormal* in G if $H = G$ or there exists a chain of subgroups $H = H_0 < H_1 < H_2 < ... < H_k = G$ such that H_{i-1} is normal in H_i for $1 \leq i \leq k$. Clearly subnormality is transitive: If H is subnormal in J and J is subnormal in G, then H is subnormal in G.

A *subgroup embedding functor* is a function f that associates a set of subgroups $f(G)$ to each group G such that if ι is an isomorphism from G onto G', then $H \in f(G)$ if and only if $\iota(H) \in f(G')$.

If f is a subgroup embedding functor and H is a subgroup of G, we say H $f\ G$ if $H \in f(G)$.

We assume f contains n, where $n(G)$ is the set of normal subgroups of G, and is contained in pS, where $pS(G)$ is the set of Sylow permutable subgroups of G – these are the subgroups H of G such that $HP = PH$ for every Sylow subgroup P of G.

Let $p(G)$ be the set of permutable subgroups of G, i.e. those subgroups H such that $HK = KH$ for all subgroups K of G.

We define fW to be the class of groups such that $H \leq G$ implies $H\ f\ G$, and fT to be the class of groups such that $H\ sn\ G$ implies $H\ f\ G$. Thus fT contains fW.

If $f = n$, then nW is the class of Dedekind groups, i.e. the groups such that all subgroups are normal, while nT is the class of T-groups, the groups in which every subnormal subgroup is normal. Hence the $(n)T$-groups are those in which normality is transitive.

If $f = p$, then pW is the class of Iwasawa groups, i.e. the groups such that all subgroups are permutable, while pT is the class of PT-groups, the groups in which every subnormal subgroup is permutable. Because normal implies permutable implies subnormal, the PT-groups are those in which permutability is transitive.

If $f = pS$, then pSW is the class of nilpotent groups, while pST is the class of PST-groups, the groups in which every subnormal subgroup is Sylow permutable. Because normal implies Sylow permutable implies subnormal, the PST-groups are those in which Sylow permutability is transitive.

The *nilpotent residual* of a group G is the unique smallest normal subgroup X of G such that the quotient group G/X is nilpotent. This nilpotent residual is denoted $G^{\mathfrak{N}}$; here $\mathfrak{N}$ denotes the class of finite nilpotent groups. (This residual exists because if X and Y are normal subgroups of G such that G/X and G/Y are nilpotent, then $G/X \cap Y$ is nilpotent, also.)

Theorem 1. *(Gaschütz, Zacher, Agrawal) [2] If $f = n$, p, or pS, then G is a finite soluble fT-group if and only if $G^{\mathfrak{N}}$ is abelian of odd order; $G^{\mathfrak{N}}$ and $G/G^{\mathfrak{N}}$ are of relatively prime order; $G/G^{\mathfrak{N}} \in fW$; and every subgroup of $G^{\mathfrak{N}}$ is normal in G.*

H is *pronormal* in G if for each $g \in G$, H and its conjugate H^g are conjugate in the join $< H, H^g >$, i.e $H^g = H^x$, where $x \in < H, H^g >$.

It is also possible to show that H is pronormal in G if and only if for each $g \in G$, H and H^g are conjugate via an element of $< H, H^g >^{\mathfrak{N}}$.

Examples:

Sylow p-subgroups are pronormal; so are maximal subgroups.

A subgroup that is both subnormal and pronormal is normal.

A *formation* $\mathfrak{F}$ is a class of groups such that:

(1) If $G \in \mathfrak{F}$ and X is a normal subgroup of G, then $G/X \in \mathfrak{F}$.

(2) If G/X, $G/Y \in \mathfrak{F}$ for X and Y normal subgroups in G, then $G/X \cap Y \in \mathfrak{F}$.

Here (2) is the property of $\mathfrak{N}$ guaranteeing the existence of the $\mathfrak{N}$-residual $G^{\mathfrak{N}}$. We can define $G^{\mathfrak{F}}$ similarly.

Let $\mathfrak{F}$ be a formation of finite groups containing all nilpotent groups such that any normal subgroup of any fT-group in $\mathfrak{F}$ and any subgroup of any soluble fT-group in $\mathfrak{F}$ belongs to $\mathfrak{F}$. We say such an $\mathfrak{F}$ has *Property* $f*$.

A subgroup M of a finite group G is said to be $\mathfrak{F}$-*normal* in G if $G/Core_G(M)$ belongs to $\mathfrak{F}$. A subgroup U of a finite group G is called a K-$\mathfrak{F}$-*subnormal* subgroup of G if either $U = G$ or there exist subgroups $U = U_0 \leq U_1 \leq \cdots \leq U_n = G$ such that U_{i-1} is either normal or $\mathfrak{F}$-normal in U_i, for $i = 1, 2, \ldots, n$.

We call a finite group G an $fT_{\mathfrak{F}}$-group if every K-$\mathfrak{F}$-subnormal subgroup of G is in $f(G)$. When $\mathfrak{F} = \mathfrak{N}$, the $fT_{\mathfrak{N}}$-groups are precisely the fT-groups. (This is because an $\mathfrak{N}$-normal subgroup is subnormal, so K-$\mathfrak{N}$-subnormal is the same as subnormal.)

H is $\mathfrak{F}$-*pronormal* in G if for each $g \in G$, H and H^g are conjugate via an element of $< H, H^g >^{\mathfrak{F}}$.

Just as K-$\mathfrak{N}$-subnormality is the same as subnormality, $\mathfrak{N}$-pronormality is the same as pronormality.

Results

Theorem 2. *[3] If $\mathfrak{F}$ is a subgroup-closed saturated formation containing $\mathfrak{N}$, a soluble group is in $\mathfrak{F}$ if and only if each of its subgroups is $\mathfrak{F}$-subnormal. (This generalises the well known fact for $\mathfrak{N}$.)*

If $\mathfrak{F}_1 \supseteq \mathfrak{F}_2$, every K-$\mathfrak{F}_2$-subnormal subgroup is K-$\mathfrak{F}_1$-subnormal, and every $\mathfrak{F}_1$-pronormal subgroup is $\mathfrak{F}_2$-pronormal.

Thus all our $fT_{\mathfrak{F}}$-groups are fT-groups, because K-$\mathfrak{N}$-subnormal subgroups are K-$\mathfrak{F}$-subnormal.

Theorem 3. *[3] If $\mathfrak{F}$ is a subgroup-closed saturated formation containing $\mathfrak{N}$, then a soluble group is a $T_{\mathfrak{F}}$-group if and only if each of its subgroups is $\mathfrak{F}$-pronormal.*

If $\mathfrak{F}$ contains $\mathfrak{N}$, then $G \in \mathfrak{F}$ is a $T_{\mathfrak{F}}$-group if and only if G is Dedekind.

Theorem 4. *If $\mathfrak{F}$ contains $\mathfrak{U}$, the formation of supersoluble groups, then the soluble $T_{\mathfrak{F}}$-groups are just the Dedekind groups.*

Proof. Each soluble $T_{\mathfrak{F}}$-group, being a soluble T-group, is in $\mathfrak{U}$, which is contained in $\mathfrak{F}$. Thus by Theorem 3, such a group is Dedekind.

Let $\mathfrak{O}$ be the set of ordered pairs (p, q) where p and q are prime numbers such that q divides $p - 1$, and for (p, q) in $\mathfrak{O}$, denote by $X_{(p,q)}$ a non-abelian group of order pq.

Let $\mathfrak{X}$ be the class consisting of every group that is isomorphic to $X_{(p,q)}$ for some $(p,q) \in \mathfrak{O}$ and denote by $\mathfrak{X}_{\mathfrak{F}}$ the class $\mathfrak{X} \cap \mathfrak{F}$.

Let $\mathfrak{Y}$ be the class of non-abelian simple groups, and let $\mathfrak{Y}_{\mathfrak{F}}$ be the class $\mathfrak{Y} \cap \mathfrak{F}$, and denote by $\mathfrak{S}$ the class of finite soluble groups.

Definition.

A group G is said to be an $fR_{\mathfrak{F}}$-group if G is an fT-group and

[i] No section of $G/G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{X}_{\mathfrak{F}}$.

[ii] No chief factor of $G^{\mathfrak{S}}$ is isomorphic to an element of $\mathfrak{Y}_{\mathfrak{F}}$.

Theorem 5. *If G is a group and $\mathfrak{F}$ has Property $f*$, then $G \in fT_{\mathfrak{F}}$ if and only if $G \in fR_{\mathfrak{F}}$.*

Theorem 6. *Let G be a group and $\mathfrak{F}$ be a formation containing $\mathfrak{N}$. If G is a soluble $fT_{\mathfrak{F}}$-group then Conditions (i), (ii), and (iii) below hold, and if (i), (ii) and (iii) hold and $\mathfrak{S} \cap \mathfrak{F}$ has Property $f*$ where $f = n$, p, or pS, then G is a soluble $fT_{\mathfrak{F}}$-group.*

[i] $G^{\mathfrak{F}}$ is a normal abelian Hall subgroup of G with odd order;

[ii] $X/X^{\mathfrak{F}}$ is an fW-group for every X sn G;

[iii] Every subgroup of $G^{\mathfrak{F}}$ is normal in G.

Definition. $he(G)$ is the set of hypercentrally embeddded subgroups of G, i.e. the set of subgroups H such that $H/H_G \leq Z_\infty(G/H_G)$, the hypercentre of G/H_G.

Lemma 1. *For all G, $p(G)$ is contained in $he(G)$, which is contained in $pS(G)$. However, these subgroup embedding functors are all distinct.*

Theorem 7. *If $\mathfrak{F}$ is a formation, then $\mathfrak{S} \cap \mathfrak{F}$ satisfies $pS*$ if and only if it satisfies $he*$. If G is a soluble group and $\mathfrak{S} \cap \mathfrak{F}$ possesses this property, then $G \in pST_{\mathfrak{F}}$ if and only if $G \in heT_{\mathfrak{F}}$.*

Thus it is possible for distinct functors f and g to yield the same generalisations $fT_{\mathfrak{F}}$ and $gT_{\mathfrak{F}}$, leading to the following:

Question. What other possibilities for f lead to new fT and fW and therefore potentially new $fT_{\mathfrak{F}}$?

References

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