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Some Trends in the Theory of Groups with Restricted Conjugacy Classes

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Abstract. This survey article is an expanded version of the lectures given at the conference "Advances in Group Theory and Applications 2011", concerning the effect of finiteness conditions on infinite (generalized) soluble groups.

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To Cristiano and Reinhold

1 Introduction

This survey article is an expanded version of the lectures that I delivered on the occasion of the conference "Advances in Group Theory and Applications 2011", which took place in Porto Cesareo. The aim of my lectures was to describe the state of knowledge on the effect of the imposition of finiteness conditions on a (generalized) soluble group, with special attention to conjugacy classes, generalized normal subgroups and lattice properties. My interest in this topic started about thirtyfive years ago when - as a young student - I was attending the lectures of Mario Curzio, and my ideas about mathematics were strongly influenced by Federico Cafiero. It was an exciting time, especially because of the impressive development of the theory produced by the schools founded by Reinhold Baer, Philip Hall and Sergei N. Černikov. Since then I never stopped working on these topics, and even though the fashions have changed, I am absolutely convinced that the theory of groups is full of fascinating properties still waiting to be discovered and exciting results that are waiting to be proved (and of course I mean here the theory of infinite soluble groups). I hope that these short notes may be able to transmit - especially to young people - the passion for this discipline, which in turn was forwarded to me by mathematicians much better than me.

Most notation is standard and can be found in [31].

This paper is dedicated to my youngest sons Cristiano and Reinhold, who were born during the organization of this conference.

2 Generalized normal subgroups

In 1955, a relevant year for the development of the theory of infinite groups, Bernhard H. Neumann [29] proved the following important result. It shows that the imposition of suitable normality conditions to all subgroups can be used to characterize group classes generalizing the class of abelian groups.

Theorem 1. (B.H. Neumann [29]) *Let G be a group. Then each subgroup of G has only finitely many conjugates if and only if the centre $Z(G)$ has finite index in G.*

It follows in particular from Neumann's theorem that if a group G has finite conjugacy classes of subgroups, then there is an upper bound for the orders of such classes.

A group G is called an *FC-group* if every element of G has only finitely many conjugates, or equivalently if the index $|G : C_G(x)|$ is finite for each element x of G. Finite groups and abelian groups are obvious examples of groups with the property FC; moreover, it is also clear that all central-by-finite groups belong to the class of FC-groups. Since for each element x of a group G the centralizer of the normal subgroup $\langle x\rangle^G$ coincides with the core of the centralizer $C_G(x)$, it follows easily that a group is an FC-group if and only if it has finite conjugacy classes of cyclic subgroups.

If x is any element of a group G, we have $x^g = x[x, g]$ for each element g of G, and hence the conjugacy class of x is contained in the coset xG', where G' is the commutator subgroup of G. Thus groups with finite commutator subgroup are FC-groups.

The FC-property for groups have been introduced seventy years ago, and since their first appearance in the literature important contributions have been given by several authors. A special mention is due here to R. Baer, Y.M. Gorcakov, P. Hall, L.A. Kurdachenko, B.H. Neumann, M.J. Tomkinson for the particular relevance of their works.

The direct product of an arbitrary collection of finite groups has clearly the property FC, and so in particular the conjugacy classes of elements of an FC-group can have unbounded orders, in contrast to the behavior of groups with finite conjugacy classes of subgroups. In fact, groups with boundedly finite conjugacy classes form a very special class of FC-groups, as the following theorem shows.

Theorem 2. (B.H. Neumann [28]) *A group G has boundedly finite conju-*

gacy classes of elements if and only if its commutator subgroup G' is finite.

The proof of Theorem 2 essentially depends on (and contains as a special case) a celebrated result of I. Schur concerning the relation between the size of the centre and that of the commutator subgroup of a group. This is one of the most important theorems in the theory of infinite groups.

Theorem 3. (I. Schur [40]) *Let G be a group whose centre has finite index. Then the commutator subgroup G' of G is finite.*

An easy application of such result proves that if G is a group such that the factor group $G/Z(G)$ is locally finite, then the commutator subgroup G' of G is locally finite. Since a finitely generated group has the FC-property if and only if it is finite over its centre, it follows also that the commutator subgroup of any FC-group is locally finite; in particular, the elements of finite order of an arbitrary FC-group form a subgroup. Thus also Dietzmann's Lemma, stating that periodic FC-groups are covered by their finite normal subgroups, can be seen as a direct consequence of the theorem of Schur.

Schur's theorem has been extended by Baer to any term of the upper central series (with finite ordinal type) in the following way.

Theorem 4. (R. Baer [1]) *Let G be a group in which the term $Z_i(G)$ of the upper central series has finite index for some positive integer i. Then the $(i+1)$-th term $\gamma_{i+1}(G)$ of the lower central series of G is finite.*

Although finitely generated finite-by-abelian groups are central-by-finite, the consideration of any infinite extraspecial group shows that the converse of Schur's theorem is false in the general case. On the other hand, Philip Hall was able to obtain a relevant and useful partial converse of Theorem 4; of course, it provides as a special case a partial converse for the theorem of Schur.

Theorem 5. (P. Hall [23]) *Let G be a group such that the $(i+1)$-th term $\gamma_{i+1}(G)$ of the lower central series of G is finite. Then the factor group $G/Z_{2i}(G)$ is finite.*

Combining the theorems of Baer and Hall, we have the following statement.

Corollary 1. *A group G is finite-by-nilpotent if and only if it is finite over some term (with finite ordinal type) of its upper central series.*

If G is any group, the last term of its (transfinite) upper central series is called the *hypercentre* of G, and the group G is *hypercentral* if it coincides with its hypercentre.

The consideration of the locally dihedral 2-group $D(2^\infty)$ shows that Baer's theorem cannot be extended to terms with infinite ordinal type of the upper central series. In fact,

$$Z_{\omega+1}(D(2^\infty)) = D(2^\infty)$$

but

$$\gamma_2(D(2^\infty)) = \gamma_3(D(2^\infty)) = Z(2^\infty).$$

Similarly, free non-abelian groups show that Hall's result does not hold for terms with infinite ordinal type of the lower central series. In fact, if F is any free non-abelian group, then $\gamma_\omega(F) = Z(F) = \{1\}$. On the other hand, the statement of Corollary 1 can be generalized as follows.

Theorem 6. (M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak [13]) *Let G be a group. The hypercentre of G has finite index if and only if G contains a finite normal subgroup N such that the factor group G/N is hypercentral.*

For other types of generalizations of Schur's theorem see for instance [19].

It was mentioned above that any direct product of finite groups has the FC-property. Such direct products are obviously also periodic and residually finite, and it was conversely proved by P. Hall [24] that every periodic countable residually finite FC-group is isomorphic to a subgroup of a direct product of a collection of finite groups (recall here that a group is called *residually finite* if the intersection of all its subgroups of finite index is trivial). An easy example shows that this property does not hold in general for uncountable groups. The result of P. Hall was extended by L.A. Kurdachenko [27] to the case of metabelian groups with countable centre. Moreover, many authors have determined conditions under which an FC-group is isomorphic at least to a section of a direct product of finite groups. On this problem we mention here only the following interesting result.

Theorem 7. (M.J. Tomkinson [42]) *Let G be an FC-group. Then the commutator subgroup G' of G is isomorphic to a section of the direct product of a collection of finite groups.*

Clearly, a subgroup X of a group G has only finitely many conjugates if and only if it is normal in a subgroup of finite index of G. One may consider other generalized normality properties, in which the obstruction to normality is represented by a finite section of the group. In particular, another theorem of B.H. Neumann deals with the case of groups in which every subgroup has finite index in a normal subgroup.

Theorem 8. (B.H. Neumann [29]) *A group G has finite commutator subgroup if and only if the index $|X^G : X|$ is finite for each subgroup X of G.*

Note that it is also easy to prove that a group G has the property FC if and only if each cyclic subgroup of G has finite index in its normal closure.

For our purposes, it is convenient to introduce the following definitions.

Let G be a group, and let X be a subgroup of G. Then

- X is *almost normal* in G if it has finitely many conjugates in G, or equivalently if the index $|G : N_G(X)|$ is finite,

- X is *nearly normal* in G if it has finite index in its normal closure X^G,
- X is *normal-by-finite* in G if the core X_G has finite index in X.

Using this terminology, we can describe FC-groups in the following way:
For a group G the following properties are equivalent:

- *G is an FC-group,*
- *all cyclic subgroups of G are almost normal,*
- *all cyclic subgroups of G are nearly normal.*

Clearly, any finite subgroup of an arbitrary group is normal-by-finite, and so the imposition of this latter condition to (cyclic) subgroups does not force the group to have the FC-property. However, it follows from Dietzmann's Lemma that if all subgroups of an FC-group G are normal-by-finite, then every subgroup of G is also almost normal, and hence $G/Z(G)$ is finite.

Although almost normality and near normality are equivalent for cyclic (and so even for finitely) subgroups, easy examples show that these two concepts are usually incomparable for arbitrary subgroups. On the other hand, it can be proved that each almost normal subgroup of an FC-group is nearly normal, and that nearly normal subgroups of finite rank are always almost normal (see [21]). Moreover, combining Neumann's results with the theorem of Schur, we obtain:

Corollary 2. *Let G be a group in which all subgroups are almost normal. Then every subgroup of G is nearly normal.*

The above corollary has been extended in [21], proving that if G is a group in which every abelian subgroup is either almost normal or nearly normal, then the commutator subgroup of G is finite, and so all subgroups of G are nearly normal. Observe also that the hypotheses in the statements of Theorem 1 and Theorem 8 can be weakened, requiring that only the abelian subgroups are almost normal or nearly normal, respectively.

Theorem 9. (I.I. Eremin [18]) *Let G be a group in which all abelian subgroups are almost normal. Then the factor group $G/Z(G)$ is finite.*

Theorem 10. (M.J. Tomkinson [43]) *Let G be a group in which all abelian subgroups are nearly normal. Then the commutator subgroup G' of G is finite.*

A group G is called a *BCF-group* if all its subgroups are normal-by-finite and have bounded order over the core. Thus a group is BCF if and only if there exists a positive integer k such that $|X : X_G| \leq k$ for each subgroup X of G.

Theorem 11. (J. Buckley - J.C. Lennox - B.H. Neumann - H. Smith - J. Wiegold [4]) *Let G be a locally finite BCF-group. Then G contains an abelian subgroup of finite index.*

The above theorem has been later extended to the case of locally graded BCF-groups by H. Smith and J. Wiegold [41].

The structure of groups in which all non-abelian subgroups are either almost normal or nearly normal has been investigated by M. De Falco, F. de Giovanni, C. Musella and Y.P. Sysak [12]. The basic situation in this case is that of groups whose non-abelian subgroups are normal. Such groups are called *metahamiltonian* and have been introduced by G.M. Romalis and N.F. Sesekin in 1966. Of course, Tarski groups (i.e. infinite simple groups whose proper non-trivial subgroups have prime order) are metahamiltonian. On the other hand, within the universe of (generalized) soluble groups, any metahamiltonian group has (boundedly) finite conjugacy classes. In fact:

Theorem 12. (G.M. Romalis - N.F. Sesekin [34],[35],[36]) *Let G be a locally graded metahamiltonian group. Then the commutator subgroup G' of G is finite of prime-power order.*

Recall here that a group G is *locally graded* if every finitely generated non-trivial subgroup of G has a proper subgroup of finite index. It is easy to see that any locally (soluble-by-finite) group is locally graded. Thus locally graded groups form a large class of generalized soluble groups, and the assumption for a group to be locally graded is enough to avoid Tarski groups and other similar pathologies.

Neumann's theorems suggest that the behavior of normalizers has a strong influence on the structure of a group. In fact, groups with few normalizers of subgroups with a given property are of a very special type.

Theorem 13. (Y.D.Polovickiĭ [30]) *Let G be a group with finitely many normalizers of abelian subgroups. Then the factor group $G/Z(G)$ is finite.*

This theorem is a special case of a later result by F. De Mari and F. de Giovanni [15] concerning groups with few normalizer subgroups.

The *norm* $N(G)$ of a group G is the intersection of all normalizers of subgroups of G. Thus the norm of a group consists of all elements which induces by conjugation a power automorphism (recall that an automorphism of a group G is called a *power automorphism* if it maps each subgroup of G onto itself). As power automorphisms are central (see [6]), it follows that the norm of an arbitrary group G is contained in the second centre $Z_2(G)$ of G. Moreover, as a consequence of Polovickii's theorem, we have that that if the index $|G : N(G)|$ is finite, then also the centre $Z(G)$ has finite index in G, confirming that the section $N(G)/Z(G)$ is usually small.

The last part of this section is devoted to the study of certain finiteness conditions that are strictly related to the property FC, at least within the universe of locally (soluble-by-finite) groups. On this type of problems see also [22] and [16].

A group G is said to have the property $\mathfrak{K}$ if for each element x of G the set

$$\{[x, H] \mid H \leq G\}$$

is finite. Similarly, a group G has the property $\mathfrak{K}_\infty$ if the set

$$\{[x, H] \mid H \leq G, \quad H \text{ infinite}\}$$

is finite for every element x of G. As the commutator subgroup of any FC-group is locally finite, it is easy to prove that all FC-groups have the property $\mathfrak{K}$. Although also Tarski groups have the property $\mathfrak{K}$, the situation is completely clear in the case of locally (soluble-by-finite) groups.

Theorem 14. (M. De Falco - F. de Giovanni - C. Musella [9]) *A group G is an FC-group if and only if it is locally (soluble-by-finite) and has the property $\mathfrak{K}$.*

Also groups in the class $\mathfrak{K}_\infty$ are not too far from having the FC-property.

Theorem 15. (M. De Falco - F. de Giovanni - C. Musella [9]) *A soluble-by-finite group G has the property $\mathfrak{K}_\infty$ if and only if it is either an FC-group or a finite extension of a group of type p^∞ for some prime number p.*

Finally, we say that a group G has the property $\mathfrak{N}$ if the set

$$\{[X, H] \mid H \leq G\}$$

is finite for each subgroup X of G. It turns out that for soluble groups the property $\mathfrak{N}$ is equivalent to the property BFC.

Theorem 16. (M. De Falco - F. de Giovanni - C. Musella [9]) *Let G be a soluble group with the property $\mathfrak{N}$. Then the commutator subgroup G' of G is finite.*

3 Lattice properties

If G is any group, the set $\mathfrak{L}(G)$ of all subgroups of G is a complete lattice with respect to the ordinary set-theoretic inclusion. In this lattice, the operations $\wedge$ and $\vee$ are given by the rules

$$X \wedge Y = X \cap Y$$

and

$$X \vee Y = \langle X, Y \rangle$$

for each pair (X, Y) of subgroups of G. There is a very large literature on the relations between the structure of a group and that of its subgroup lattice (see for instance the monograph [39], but also the papers [7],[8],[10],[11],[20] for recent developments). For instance, one of the first and significant results was proved by O. Ore, and shows that a group has distributive subgroup lattice if and only if it is locally cyclic.

Let G and G^* be groups. A *projectivity* from G onto G^* is an isomorphism from the lattice $\mathfrak{L}(G)$ of all subgroups of G onto the subgroup lattice $\mathfrak{L}(G^*)$ of G^*; if there exists such a map, G^* is said to be a *projective image* of G. A group class $\mathfrak{X}$ is *invariant under projectivities* if all projective images of groups in $\mathfrak{X}$ are likewise $\mathfrak{X}$-groups. Relevant examples of group classes invariant under projectivities are the following:

- the class of all finite groups
- the class of all periodic groups
- the class of all soluble groups (B. Yakovlev, 1970)
- the class of all groups with finite Prüfer rank
- the class of all soluble minimax groups (R. Baer, 1968)

Moreover, it is clear that any group class defined by a lattice-theoretic property is invariant under projectivities; in particular, it follows from Ore's theorem quoted above that the class of locally cyclic groups is invariant under projectivities. On the other hand, the class of all abelian groups does not have such property. In fact, the elementary abelian group of order 9 and the symmetric group of degree 3 have isomorphic subgroup lattices.

It is easy to understand that the main obstacle in the study of projective images of abelian groups is the fact that normality is not preserved under projectivities; actually, the behavior of images of normal subgroups under projectivities plays a central role in the investigations concerning projectivities of groups.

Let $\mathfrak{L}$ be any lattice. An element a of $\mathfrak{L}$ is said to be *modular* if

$$(a \vee x) \wedge y = a \vee (x \wedge y)$$

for all $x, y \in \mathfrak{L}$ such that $a \leq y$ and

$$(a \vee x) \wedge y = x \vee (a \wedge y)$$

for all $x, y \in \mathfrak{L}$ such that $x \leq y$. The lattice $\mathfrak{L}$ is *modular* if all its elements are modular, i.e. if the identity

$$(x \vee y) \wedge z = x \vee (y \wedge z)$$

holds in $\mathfrak{L}$, whenever x, y, z are elements such that $x \leq z$.

If N is any normal subgroup of a group G, and φ is a projectivity from G onto another group G^*, it follows from the Dedekind's modular law that the image N^φ of N is a modular element of the lattice $\mathfrak{L}(G^*)$. In particular, any projective image of an abelian group has modular subgroup lattice, and groups with modular subgroup lattice can be considered as suitable lattice approximations of abelian groups.

Locally finite groups with modular subgroup lattice have been completely classified by K. Iwasawa [25] almost seventy years ago, while a full description of periodic groups with modular subgroup lattice has been obtained by

R. Schmidt [38] in 1986. Moreover, it turns out that the commutator subgroup of any group with modular subgroup lattice is periodic, and hence a torsion-free group has modular subgroup lattice if and only if it is abelian.

In the study of finiteness conditions from a lattice point of view, the following relevant result by G. Zacher is crucial; it was independently proved also by I.A. Rips.

Theorem 17. (G. Zacher [45]) *Let φ be a projectivity from a group G onto a group G^*, and let H be a subgroup of finite index of G. Then the subgroup H^φ has finite index in G^*.*

As a direct consequence, we have:

Corollary 3. *The class of residually finite groups is invariant under projectivities.*

A few years later, R. Schmidt [37] obtained a lattice theoretic description of the finiteness of the index of a subgroup, which of course allows to recognize subgroups of finite index within the lattice of subgroups. On the other hand, the index of a subgroup is not preserved under projectivities, as for instance all groups of prime order obviously have isomorphic subgroup lattice. The following result clarifies this situation.

Theorem 18. (M. De Falco - F. de Giovanni - C. Musella - R. Schmidt [10]) *Let G be a group and let X be a subgroup of finite index of G. Then the number $\pi(|G:X|)$ of prime factors (with multiplicity) of the index $|G:X|$ can be described by means of lattice properties. In particular, $\pi(|G:X|)$ is invariant under projectivities.*

It is easy to show that there exists a group G with an infinite class of conjugate elements such that the subgroup lattice of G is isomorphic to that of an abelian group. Therefore the class of FC-groups is not invariant under projectivities. Obviously, the same example also proves that neither the class of central-by-finite groups is invariant under projectivities nor that of finite-by-abelian groups. On the other hand, it is possible to study lattice analogues of both central-by-finite groups and finite-by-abelian groups.

A subgroup M of a group G is said to be *modularly embedded* in G if the lattice $\mathfrak{L}(\langle x, M\rangle)$ is modular for each element x of G. This concept was introduced by P.G. Kontorovic and B.I. Plotkin [26] in order to characterize torsion-free nilpotent groups by their subgroup lattices. Of course, any subgroup of the centre of a group G is modularly embedded in G, and actually the modular embedding seems to be the best translation of centrality into the subgroup lattice. In fact, using modularly embedded subgroups, the following lattice interpretation of Schur's theorem can be obtained.

Theorem 19. (M. De Falco - F. de Giovanni - C. Musella [8]) *Let G be a* *group containing a modularly embedded subgroup of finite index. Then G has a finite normal subgroup N such that the subgroup lattice $\mathfrak{L}(G/N)$ is modular.*

Suitable iterations of the above concepts allow to introduce lattice analogues of nilpotent groups, and can probably be used in order to translate Theorem 4 and Theorem 5 in terms of lattice properties.

Also Neumann's theorems have been investigated from a lattice point of view. In order to describe the corresponding results we need the following definitions, which introduce certain relevant types of generalized normal subgroups.

Let G be a group, and let X be a subgroup of G.

- X is called *almost modular* if there exists a subgroup H of G containing X such that the index $|G : H|$ is finite and X is a modular subgroup of H

- X is called *nearly modular* if there exists a modular subgroup H of G containing X such hat the index $|H : X|$ is finite

It follows from Schmidt's lattice characterization of the finiteness of the index of a subgroup that the above definitions are purely lattice theoretic, and so they can be given in any (complete) lattice. A lattice $\mathfrak{L}$ is called *almost modular* (respectively, *nearly modular*) if all its elements are almost modular (respectively, nearly modular).

Groups with almost modular subgroup lattice can be seen as lattice analogues of central-by-finite groups, while groups with nearly modular subgroup lattice correspond in this context to finite-by-abelian groups.

Theorem 20. (F. de Giovanni - C. Musella - Y.P. Sysak [20]) *Let G be a periodic group. The subgroup lattice $\mathfrak{L}(G)$ is almost modular if and only if $G = M \times K$, where M is a group with modular subgroup lattice, K is an abelian-by-finite group containing a finite normal subgroup N such that the lattice $\mathfrak{L}(K/N)$ is modular and $\pi(M) \cap \pi(K) = \emptyset$.*

Since a group is central-by-finite if and only if it is both abelian-by-finite and finite-by-abelian, the above result provides a lattice corresponding of Theorem 1, at least in the case of periodic groups. The next statement provides a lattice translation of Theorem 8, again within the universe of periodic groups.

Theorem 21. (M. De Falco - F. de Giovanni - C. Musella - Y.P. Sysak [11]) *Let G be a periodic group. The subgroup lattice $\mathfrak{L}(G)$ is nearly modular if and only if G contains a finite normal subgroup N such that the lattice $\mathfrak{L}(G/N)$ is modular.*

The following natural problem is still unsolved.

Question A *Is it possible to obtain lattice translations of the theorems of Eremin and Tomkinson on groups whose abelian subgroups are almost normal or nearly normal (at least inside the class of locally finite groups)?*

We mention finally that also some results concerning the class of BCF-groups have been translated into the theory of subgroup lattices.

A group G is called a BMF*-group* if there exists a positive integer k such that $\pi(|X : core^*(X)|) \leq k$ for each subgroup X of G, where $core^*(X)$ denotes the largest modular subgroup of G which is contained in X.

Theorem 22. (M. De Falco - F. de Giovanni - C. Musella [7]) *Let G be a locally finite BMF-group. Then G contains a subgroup M of finite index such that the lattice $\mathfrak{L}(M)$ is modular.*

Question B *Is it possible to extend Theorem 22 to the case of locally graded BMF-groups?*

4 Inertial properties

A subgroup X of a group G is said to be *inert* if the index $|X : X \cap X^g|$ is finite for each element g of G. Clearly, every normal-by-finite subgroup is inert, so that in particular normal subgroups and finite subgroups of arbitrary groups are inert. A group is *inertial* if all its subgroups are inert. Thus CF-groups (i.e. groups in which all subgroups are normal-by-finite) are inertial. The first result of this section shows that within the universe of locally finite groups there are no infinite simple inertial groups.

Theorem 23. (V.V. Belyaev - M. Kuzucuoglu - E. Seckin [3]) *Let G be a simple locally finite group. If G is inertial, then it is finite.*

The latter theorem has recently been extended to the case of simple locally graded groups by M.R. Dixon, M. Evans and A. Tortora [17]. The structure of soluble groups in which all subgroups are inert has been investigated by D.J.S. Robinson [33]; in particular, he characterized finitely generated soluble inertial groups and soluble minimax inertial groups.

A subgroup X of a group G is said to be *strongly inert* if it has finite index in $\langle X, X^g \rangle$ for each element g of G, and the group G is called *strongly inertial* if all its subgroups are strongly inert. Thus nearly normal subgroups (and in particular all subgroups of finite index) are strongly inert; it is also clear that finite subgroups of locally finite groups are strongly inert. It is easy to prove that any strongly inert subgroup is also inert, and hence strongly inertial groups are inertial. Clearly, the subgroups of order 2 of the infinite dihedral group D_∞

are not strongly inert, although they are inert in D_∞. Thus strongly inertial groups have no infinite dihedral sections. It is easy to show that inert and strongly inert subgroups of arbitrary groups have some inheritance properties. In fact, we have:

Let G be a group, and let X and Y be subgroups of G such that $Y \leq X$ and the index $|X : Y|$ is finite.

- *X is inert in G if and only if Y is inert in G;*
- *if X is strongly inert in G, then Y is strongly inert in G.*

Moreover, abelian strongly inert subgroups have the following useful behavior.

Lemma 1. *Let X be an abelian strongly inert subgroup of a group G. Then the subgroup $[X, X^g]$ is finite for each element g of G.*

Proof. As the indices $|\langle X, X^g\rangle : X|$ and $|\langle X, X^g\rangle : X^g|$ are finite, the subgroup $X \cap X^g$ has finite index in $\langle X, X^g\rangle$. Moreover, $X \cap X^g$ is contained in $Z(\langle X, X^g\rangle)$ since X is abelian, and so it follows from Schur's theorem that the commutator subgroup $\langle X, X^g\rangle' = [X, X^g]$ is finite. $\boxed{QED}$

Next result shows that all groups with finite conjugacy classes are strongly inertial, and so also inertial.

Lemma 2. *Every FC-group G is strongly inertial.*

Proof. Let X be any subgroup of G, and $H = \langle X, g\rangle$ for some g in G. The centralizer $C = C_H(\langle g\rangle^H)$ has finite index in H, so that the index $|X : X \cap C|$ is likewise finite. Moreover, $(X \cap C)^g = X \cap C$ and $X \cap C$ is normal in H. Application of Dietzmann's Lemma yields that the normal closure of $X/X \cap C$ in $H/X \cap C$ is finite. Therefore X has finite index in X^H and so X is strongly inert in G. $\boxed{QED}$

It is also possible to prove that in an arbitrary group G the elements of finite order form a locally finite subgroup if and only if all finite subgroups of G are strongly inert. In particular, the elements of finite order of any strongly inertial group form a locally finite subgroup.

Finitely generated strongly inertial groups are characterized by the following result. In particular, it turns out that in this case strongly inertial groups coincide with groups having the FC-property.

Theorem 24. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a finitely generated strongly inertial group. Then the factor group $G/Z(G)$ is finite.*

Corollary 4. *The commutator subgroup of any strongly inertial group is locally finite.*

In particular, any perfect strongly inertial group is locally finite, and so Theorem 23 has the following consequence.

Corollary 5. *Simple strongly inertial groups are finite.*

Recall that a group G is *minimax* if it has a series of finite length whose factors satisfy either the minimal or the maximal condition on subgroups. If G is any soluble-by-finite minimax group, then its finite residual J is the largest divisble subgroup, and J is the direct product of finitely many Prüfer subgroups. Moreover, the Fitting subgroup F/J of G/J is nilpotent and G/F is finitely generated and abelian-by-finite. If G is any soluble-by-finite minimax group, the set of all prime numbers p such that G has a section of type p^∞ is an invariant, called the *spectrum* of G. A soluble-by-finite minimax group is called *p-primary* if its finite residual is a p-group. For detailed informations on the structure of soluble-by-finite minimax groups see [31], Chapter 10.
It is easy to prove the following result.

Lemma 3. *Let G be a minimax residually finite group. If G is strongly inertial, then the factor group $G/Z(G)$ is finite.*

In the above lemma, the hypothesis that the minimax group G is residually finite cannot be dropped out, even in the non-periodic case. To see this, consider the semidirect product $G = \langle x\rangle \ltimes P$, where the normal subgroup P is of type p^∞ for some prime number p and $\langle x\rangle$ an infinite cyclic subgroup such that $a^x = a^{1+p}$ for each a in P. Then G is a minimax strongly inertial group and its centre has order p.

Let Q be a torsion-free abelian minimax group, and let D be a divisible abelian p-group of finite rank (where p is a prime number). Consider D as a trivial Q-module. Since $Ext(Q, D) = \{0\}$, it follows from the Universal Coefficients Theorem that the cohomology group $H^2(Q, D)$ is isomorphic to the homomorphism group $Hom(M(Q), D)$, where $M(Q)$ is the Schur multiplier of Q (in this case $M(Q)$ is the exterior square $Q \bigwedge Q$ of Q). Let δ be an element of infinite order of $Hom(M(Q), D)$, and for all subgroups $E < D$ and $R < Q$ denote by δ_E and δ_R the homomorphisms naturally induced by δ from $M(Q)$ to D/E and from $M(R)$ to D, respectively. Assume also that the image $(R \wedge y)^{\delta_E}$ is finite for all elements y of Q, whenever $\delta_{E,R} = 0$. In this situation a central extension of D by Q with cohomology class δ is called a *central extension of type IIa* (see [33]).

Theorem 25. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *A soluble-by-finite p-primary minimax group is strongly inertial if and only if it satisfies one of the following conditions:*

(a) *The factor group $G/Z(G)$ is finite.*

(b) *G is a Černikov group and all subgroups of its finite residual are normal in G.*

(c) *G contains a finite normal subgroup N such that the group $H = G/N$ is the semidirect product of a divisible abelian normal p-subgroup D by a subgroup Q with finite commutator subgroup such that Q/Q' is torsion-free, all subgroups of D are normal in H and p does not belong to the spectrum of the centralizer of D in Q.*

(d) *G contains a finite normal subgroup N such that G/N is a central extension of type IIa.*

Corollary 6. *A soluble-by-finite minimax group is strongly inertial if and only if it is inertial and has no infinite dihedral sections.*

Next two results describe the behavior of groups in which all cyclic subgroups are strongly inert and those in which all infinite subgroups are strongly inert.

Theorem 26. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a locally (soluble-by-finite) group. Then all cyclic subgroups of G are strongly inert if and only if every finitely generated subgroup of G is central-by-finite.*

Theorem 27. (M. De Falco - F. de Giovanni - C. Musella - N. Trabelsi [14]) *Let G be a locally (soluble-by-finite) group. Then all infinite subgroups of G are strongly inert if and only if G is either strongly inertial or a finite extension of an infinite cyclic subgroup.*

Groups in which all subnormal subgrouops are nearly normal have been investigated by C. Casolo [5]. We leave here as an open question the following corresponding problem.

Question C *Describe the structure of finitely generated soluble groups in which all subnormal subgroups are strongly inert.*

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