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Totally umbilical CMC hypersurfaces of a conformally recurrent manifold

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Abstract. It has been shown that a non-degenerate totally umbilical constant mean curvature hypersurface of a conformally recurrent pseudo-Riemannian manifold is conformally recurrent.

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MSC 2000 classification: 53B20, 53B25

Introduction

By a conformally recurrent manifold, we mean (see Adati and Miyazawa [1]) a pseudo-Riemannian manifold with a non-degenerate metric g satisfying $\nabla C = p \otimes C$, where p is a 1-form, ∇ the Levi-Civita connection and C the Weyl conformal curvature tensor of g. Such manifolds of dimension 4 and with Lorentzian metric g were completely classified within the framework of Lorentzian geometry by McLenaghan and Leroy [6] as a subclass of the class of complex recurrent spacetimes (defined by the condition that the self-dual part of C is recurrent with a complex recurrent vector). Complex recurrent (in particular, conformally recurrent) manifolds also seem to be important in the study of Huygens' principle [5]. A conformally recurrent manifold with $p = 0$ is known as a conformally symmetric manifold and was studied by Chaki and Gupta [3] and also by Sharma [7] assuming the existence of a 1-parameter group of proper conformal motions. The purpose of this paper is to prove the following result and state its consequences.

Theorem 1. *Let M (dim. > 3) be a totally umbilical hypersurface of a pseudo- Riemannian conformally recurrent manifold. If M has constant mean curvature (CMC), then it is conformally recurrent with recurrence vector as the tangential component of the recurrence vector of the ambient space.*

An odd-dimensional Riemannian manifold M is said to be a Sasakian manifold if it admits a global unit Killing vector field satisfying

$$R(x,y)\xi = g(y,\xi)x - g(x,\xi)y$$

where x, y denote arbitrary vector fields on M and R the curvature tensor of the Riemannian metric g (see Blair [2] for details).

Corollary 1. *Under the hypothesis of the above theorem, if M is a Sasakian manifold, then it is locally isometric to a unit sphere.*

Corollary 2. *Let M (dim. > 3) be a closed orientable hypersurface of an orientable Riemannian conformally recurrent manifold $\overline{M}$ with a homothetic vector field V which is nowhere tangential to M. If M has CMC and the Ricci curvature of $\overline{M}$ along the normal vector field is non-negative on M, then M is conformally recurrent and the tangential component of V is Killing on M.*

1 Preliminaries

We assume both M and $\overline{M}$ orientable and denote their inner product by $<,>$. Let N denote the unit normal vector field such that $<N,N>=\epsilon=\pm 1$. Then the Gauss' and Weingarten's formulas are

$$\overline{\nabla}_x y = \nabla_x y + B(x,y)N, \quad \overline{\nabla}_x N = -\epsilon A x$$

where x, y are arbitrary vector fields tangent to M; $\nabla, \overline{\nabla}$ denote Levi-Civita connections of M and $\overline{M}$ respectively, B the second fundamental form and A the Weingarten operator such that $B(x,y) = <Ax, y>$. Let R, S, Q, r and C denote the curvature tensor, Ricci tensor, Ricci operator, scalar curvature and Weyl conformal curvature tensor of M and the same letters with overbars denote the corresponding objects of $\overline{M}$. By hypothesis, $B(x,y) = k < x, y >$, for a constant k. The Gauss' and Codazzi equations are therefore

$$\overline{R}(x,y,z,w) = R(x,y,z,w) - \epsilon k^2(<y,z><x,w> - <x,z><y,w>)$$

$$\overline{R}(x,y)N = 0, \quad \overline{S}(x,N) = 0$$

$$\overline{S}(x,y) - \epsilon <\overline{R}(N,x)y, N> = S(x,y) + \epsilon k(k-n+1) <x,y>$$

2 Proof of the Theorem

Differentiating (1) along the hypersurface gives

$$(\overline{\nabla}_v \overline{R})(x,y,z,w) = (\nabla_v R)(x,y,z,w) \tag{1}$$

for an arbitrary vector field v tangent to M. Next, differentiating the expression for Weyl tensor $\overline{C}$ of $\overline{M}$ along v and using the conformal recurrence hypothesis, we find

$$\begin{aligned}(\overline{\nabla}_v\overline{R})(X,Y,Z,W) = p(v)\overline{C}(X,Y,Z,W) + \frac{1}{n-2}[(\overline{\nabla}_v\overline{S})(Y,Z) < X,W >\\ - (\overline{\nabla}_v\overline{S})(X,Z) < Y,W > + (\overline{\nabla}_v\overline{S})(X,W) < Y,Z >\\ - (\overline{\nabla}_v\overline{S})(Y,W) < X,Z > - \frac{v\overline{r}}{n-1}(< Y,Z >< X,W >\\ - < X,Z >< Y,W >)] \quad (2)\end{aligned}$$

where $n-1$ is the dimension of M and X,Y,Z,W denote arbitrary vector fields on $\overline{M}$. From (4) and (5) one obtains

$$\begin{aligned}(\nabla_v R)(x,y,z,w) = p(v)R(x,y,z,w) + \frac{1}{n-2}[T(y,z) < x,w > -\\ T(x,z) < y,w > + T(x,w) < y,z > - T(y,w) < x,z >] -\\ (f + \epsilon k^2 p(v))(< y,z >< x,w > - < x,z >< y,w >) \quad (3)\end{aligned}$$

where $T = \overline{\nabla}_v\overline{S} - p(v)\overline{S}$ and $f = \frac{v\overline{r}-p(v)\overline{r}}{(n-1)(n-2)}$. Now equations (1) and (4) yield

$$\begin{aligned}(\nabla_v R)(x,y,z,w) &= p(v)R(x,y,z,w) + (\overline{\nabla}_v\overline{R})(x,y,z,w) - p(v)(\overline{R}(x,y,z,w)\\ &\quad + \epsilon k^2(< y,z >< x,w > - < x,z >< y,w >)). \quad (4)\end{aligned}$$

At this point we let (e_i) denote a local orthonormal basis of the tangent space of M. Hence (e_i, N) is a local orthonormal basis of the tangent space of $\overline{M}$ at points of M. Substituting $y = z = e_i$, in (7), multiplying by $(e_i, e_i) = \epsilon_i$, summing over i, using (5) with $X = x, W = w, Y = Z = N$, and (3) gives

$$\begin{aligned}(n-2)t(x,w) &= (n-3)T(x,w) - \epsilon T(N,N) < x,w > +\\ &\quad [(n-2)f - \epsilon(n-2)^2 p(V)k^2] < x,w > \quad (5)\end{aligned}$$

where $t = \nabla_v S - p(v)S$. Substituting $x = w = e_i$, multiplying by ϵ_i, summing over i, gives

$$2\epsilon T(N,N) = p(v)r - vr + (n-1)(n-2)(f - \epsilon p(v)k^2) \quad (6)$$

using this in (8) provides

$$\begin{aligned}(n-3)T(x,w) &= (n-2)t(x,w) + \frac{1}{2} < x,w > [(n-2)(n-3)f\\ &\quad - vr + p(v)r + \epsilon(n-2)(n-3)p(v)k^2] \quad (7)\end{aligned}$$

Finally, using this in (6) we obtain $\nabla C = p \otimes C$, completing the proof.

Corollary 1 follows from the following result of Ghosh and Sharma [4]: *A conformally recurrent Sasakian manifold is locally isometric to a unit sphere.* Corollary 2 follows from the following result of Yano [9]: *Let M be a closed orientable hypersurface of an orientable Riemannian manifold $\overline{M}$ with a homothetic vector field which is nowhere tangent to M. If M has constant mean curvature and the Ricci curvature of $\overline{M}$ along the normal N is non-negative on M, then M is totally umbilical*, and the well-known fact that *a homothetic vector field on a compact orientable manifold without boundary, is necessarily Killing* (see Yano [8]).

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