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Lifting quasifibrations–II. Non-normalizing Baer involutions

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Abstract. A characterization of lifted quasifibrations admitting certain non-normalizing Baer involutions is given.

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1 Introduction

Recently, the second author studied those quasifibrations in $PG(3, F = K[\gamma])$ which may be lifted from quasifibrations in $PG(3, K)$ where K is a field and F is a quadratic extension of K. Such lifted quasifibrations admit elation and Baer groups E and C, respectively, where C normalizes E such that $[E, C] \neq \langle 1 \rangle$, which characterize the quasifibrations.

Furthermore, the more general question was considered of whether the form of a quasifibration, resembling the lifted quasifibrations, could be obtained merely by assuming the existence of an elation group E as in the lifted case and an F-linear Baer involution which normalizes but does not centralize E. However, a case was overlooked and the statement of the result in [2] is not complete. Here we provide the necessary correction which allows for a second possibility for such quasifibrations. Also, there is a hypothesis which was left out of one of the theorems of [2] and we use this opportunity to make the necessary addition to this theorem.

In addition, we give examples which satisfy both of the two possibilities exclusively and examples which satisfy the two possibilities simultaneously. The key to the class of examples is to ask if there could be a Baer involution which does not normalize the elation group of a lifted quasifibration.

More generally, we consider here the following situation: There is a quasifibration in $PG(3,F)$ admitting an elation group E whose component orbits union the axis of E define derivable nets with partial spreads in $PG(3,F)$. Furthermore, there is a non-trivial F-linear Baer group B acting as a collineation of the quasifibration which normalizes E and an F-linear Baer group which C which does not normalize E. We investigate the possible Baer-elation configuration groups assuming that C leaves invariant the axis of E, which is always true in the finite case, under the assumption that the quasifibration (spread) is a lifted quasifibration. (In the finite case, if C would not leave the axis of E invariant then the plane would admit a collineation group isomorphic to $SL(2,F)$ that is generated by elations, implying the plane is Desarguesian.)

Our main results are as follows:

The normalizing case:

Theorem 1. *(to replace Theorem (3.1) of Johnson [2]) Let $\mathcal{Q}$ be a quasifibration in $PG(3,F)$, where F is a field.*

If the associated translation net admits an elation group E whose orbits are derivable nets and a nontrivial F-linear Baer group B which normalizes but does not centralize E then the quasifibration has one of the following two forms:

(1)

$$x=0, y=x\begin{bmatrix} u & H(t) \\ t & u^{\rho} \end{bmatrix}, \forall u,t\in F,$$

for H a function on F and ρ an automorphism of F.

In this case, there is a Baer group of the following form:

$$\left\langle \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & e & 0 & 0 \\ 0 & 0 & e & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}; e^{1+\rho}=1 \right\rangle .$$

Or, we have

(2)

$$x=0, y=x\begin{bmatrix} u+G(t) & H(t) \\ t & u^{\rho} \end{bmatrix}, \forall u,t\in F \text{ and } \rho^2=1,$$

where

$$\begin{aligned} H(b^{-1}e^{-1}H(t)) &= bet, \\ G(b^{-1}e^{-1}H(t)) &= -(b^{-1}eG(t))^{\rho}, \ \forall t\in F. \end{aligned}$$

Moreover, the τ-fixed components exterior to the derivable net have the following form:

$$\begin{aligned} H(t) &= bet, \\ G(t) &= be^{-1}u^{\rho}-u, \text{ for } \rho^2=1 \end{aligned}$$

In this case, the order of B is 2 and no Baer group properly containing the Baer involution can fix $FixB$ pointwise. Moreover, the form of B is

$$\left\langle \begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e^{-1} & 0 \end{bmatrix} \right\rangle .$$

(3) The translation net admits both types of groups if and only if $\rho^2 = 1$, $G(t) = 0$ and $H(b^{-1}e^{-1}H(t)) = ebt$ for all $t \in F$.

(4) In part (1), assume that the fixed point field of ρ is K, and that $K[\theta] = F$ is a quadratic extension of K, then the quasifibration is lifted from a quasifibration S in $PG(3, K)$ if and only if $\rho^2 = 1$ and $(v - s)(H(v) - H(s)) \notin K$ for all $v, s \in F$, $v \neq s$. (In particular, this is true if $K = F^{\rho+1}$.)

(5)

(a) In part (4), if $\rho^2 = 1$ but $\rho \neq 1$ (assuming that $Fix\rho = K$) and the retracted quasifibration S is a spread then the lifted quasifibration is a spread.

(b) In part (4), if $\rho^2 = 1$ but $\rho \neq 1$ (assuming that $Fix\rho = K$) and $K = K[\theta]^{\rho+1}$ and the lifted quasifibration is a spread then the retracted quasifibration S is a spread.

The normalizing, non-normalizing situation is considered only for lifted quasifibrations, although a more general study is certainly feasible. Note for each derivable net, the Baer subline structure on the axis of the elation group is the same. We assume that this structure is permuted by a Baer group which does not normalize the elation group.

Theorem 2. *Let π be a quasifibration in $PG(3, K[\theta] = F)$ which has been lifted from a quasifibration in $PG(3, K)$.*

(a) If $\operatorname{char} K = 2$ and if π admits an F-linear Baer involution τ which leaves invariant the axis of the elation group E whose orbits union the axis are derivable nets then τ cannot centralize E. Furthermore, if τ leaves invariant the Baer subline structure of any fixed component then there does exist an F-linear Baer involution τ^* which does normalize E and hence, the quasifibration is of types (1) and (2) of the main theorem.

(b) For arbitrary characteristic, assume that there exists a nontrivial F-linear Baer group C which does not normalize the elation group E but leaves the axis of E invariant. If for some C-fixed component, C permutes the sublines of the Baer subplanes on some derivable net then the quasifibration has the following form:

$$x = 0, y = x \begin{bmatrix} u & b^2 t^\sigma \\ t & u^\sigma \end{bmatrix} \ \forall u, t \in F, \text{ and } \sigma^2 = 1,\ \sigma \neq 1,$$

and where $b^2 \in F - K$.

(c) The quasifibration of (b) is a spread in $PG(3, F)$ which retracts to a spread in $PG(3, K)$.

2 Retraction

In Johnson [2], it was shown that quasifibrations lift to quasifibrations and spreads lift to spreads. However, it is not necessarily the case that the retraction of a 'spread' lifted from a quasifibration is also a spread; this depends on the associated fields. The following is exactly Theorem (2.19) in Johnson [2] with the exception of the addition that $K = K[\theta]^{\sigma+1}$ in the hypothesis of part (3).

Theorem 3. *Let S be a quasifibration in $PG(3, K)$, for K a field, that admits a quadratic extension $K[\theta]$.*

(1) *Then there is a set of quasifibrations in $PG(3, K[\theta])$ called the quasifibrations lifted from S.*

(2) *If S is a spread then all of the lifted quasifibrations are spreads.*

(3) *If any lifted quasifibration is a spread and $K = K[\theta]^{\sigma+1}$ then S is a spread and all lifted quasifibrations are spreads.*

We note the proof given in Johnson proves (3) but not the stronger result that S is a spread when $K - K[\theta]^{\sigma+1} \neq \phi$.

3 Derivable Partial Spreads in $PG(3, K)$

In the finite case, any derivable net with partial spread in $PG(3, K \simeq GF(q))$ admits two Baer subplanes incident with the zero vector which are K-subspaces. However, in the infinite case, it was pointed out in Jha and Johnson [1], there are infinite derivable nets in $PG(3, K)$, for K an infinite field of characteristic two which is not perfect, which have exactly one K-invariant Baer subplane. Hence, the question is: In an arbitrary derivable net, in general, is there even one K-invariant Baer subplane?

Furthermore, in the original paper on lifting quasifibrations, it was assumed without comment that in the context of elation groups and Baer groups indicated in that paper, there were always two K-invariant Baer subplanes.

In this section, we answer the first posed question by showing that there is always one K-invariant Baer subplane in any derivable net and provide the proof that there are always two K-invariant Baer subplanes if there are suitable elation and Baer groups.

Let Q be a quasifibration in $PG(3, K)$, for K a field, whose translation net π admits an elation group E whose orbits union the axis define derivable nets.

Let B be a nontrivial Baer group which normalizes E but $[E, B] \neq \langle 1 \rangle$. Let Z denote the skewfield such that a given derivable net D is a pseudo-regulus net with respect to $PG(3, Z)$. Then, consider a component L of D and note that L is both a Z-space and a K-space.

Assume that K-fixes no 1-dimensional Z-subspaces. Note that K must centralize Z, by the form of D as a pseudo-regulus net. Thus, since K is irreducible by assumption and $K \leq GL(2, Z)$, then the centralizer of K contains K and Z within $GL(2, Z)$ is a division ring $\mathcal{D}$ containing both Z and K.

Assume that $\mathcal{D}$ properly contains K and Z. Since $\mathcal{D}$ fixes L, it follows that L is a $\mathcal{D}$-vector space of dimension 1 as $\mathcal{D}$ is a K-subspace of dimension at least two and hence, exactly two. So, $\mathcal{D}$ is a quadratic extension of K and a quadratic extension of Z. Now re-consider the net D which is a Z-pseudo-regulus net. Derive the net to again have a Z-pseudo-regulus net D^* where the Desarguesian Baer subplanes with respect to Z are also Pappian Baer subplanes with respect to K. Hence, it follows that K is isomorphic to Z restricted to L. However, K and Z are faithful on L.

Hence, Z is a field isomorphic to K and $\mathcal{D}$ is a field if Z is not identical to K. Since any component has $\mathcal{D}$-dimension 1, it follows that the Z-regulus net is also a K-regulus net; the unique opposite net is both a Z and a K-regulus net. Hence, it follows that $K = Z$.

Hence, we obtain:

Theorem 4. *In every derivable net in $PG(3, K)$, for K a field, there is a K-invariant Baer subplane incident with the zero vector. Furthermore the derivable net is a Z-regulus net where K is isomorphic to Z.*

Proof. K^* acts faithfully and fixed-point-free on some 1-dimensional subspace X_L for each component L of the derivable net D. It follows that K^* is a kernel homology for the spread defined by a set of such 1-dimensional subspaces. (That is, K^* permutes the Baer subplanes of the net and hence some such Baer subplane is a 2-dimensional K-subspace.) But, Z^* is also a kernel homology group for such a subplane π_o. Hence, K acting on π_o is Z acting on π_o. Hence, it follows, that K is isomorphic to Z. $\boxed{QED}$

Corollary 1. *Let D denote a derivable net with partial spread in $PG(3, K)$ for K a field. Then, D may be represented by one of the following two forms:*

(1)

$$x = 0, y = x \begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix} \forall u \in K$$

where σ is an automorphism of K or

(2) K is infinite of characteristic two and the net is represented as follows:

$$x = 0, y = x \begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix} \forall u \in K$$

where

$$A(u) = Wu + uW$$

where W is a linear transformation of K over the prime field.

Proof. We have seen that there is at least one Baer subplane which is a K-subspace. If there are two, it is an easy matter to choose a representation for these two K-subspaces so that the form becomes that of case (1).

Hence, assume that $x_2 = y_2 = 0$ defines a Baer subplane of D where D contains $x = 0, y = 0, y = x$.

It follows that the form for D is:

$$x = 0, y = x \begin{bmatrix} u & A(u) \\ 0 & f(u) \end{bmatrix} \forall u \in K$$

for some functions f, A on u. Since f must additive and multiplicative and bijective, it is clear that $f(u) = u^\sigma$ for σ an automorphism of K. Since the set of matrices must form a field isomorphic to K, it follows that:

$$vA(u) + A(v)u^\sigma = v^\sigma A(u) + A(v)u = A(uv)$$

for all $u, v \in K$. Assume that $\sigma \neq 1$. Then, for some u_o such that $u_o^\sigma \neq u_o$, we obtain:

$$A(v) = (v^\sigma - v)A(u_o)/(u_o^\sigma - u_o)$$

for all $v \in K$. Thus, either A is identically zero, and we are back to the situation (1) or $A(v) = (v^\sigma - v)k$ for $k \neq 0$.

In this case, a basis change

$$\begin{bmatrix} 1 & k & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & k \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

will change the form to that of situation (1).

Hence, $\sigma = 1$.

We now claim that

$$A(u) = Wu + uW$$

where M is some linear transformation over K and the characteristic of K is 2.

We know that we can represent the derivable net as a regulus in some projective space. Hence, we may change bases over the prime field leaving invariant $x=0$, $y=0$, $y=x$, leaving invariant the K-space Baer subplane and transform the representation into standard form:

$$x=0, y=x\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$$

for all T in a field Z isomorphic to K.

Hence, there exists a matrix such that

$$\begin{bmatrix} C^{-1} & -C^{-1}EC \\ 0 & C^{-1} \end{bmatrix}\begin{bmatrix} u & A(u) \\ 0 & u \end{bmatrix}\begin{bmatrix} C & E \\ 0 & C \end{bmatrix}=\begin{bmatrix} T & 0 \\ 0 & T \end{bmatrix}$$

for some T in $C^{-1}KC$ where C and E are bijective linear transformations of K over the prime field. After a bit of computation, we obtain:

$$A(u)=-(uC^{-1}D+C^{-1}Du)$$

for all $u\in K$. Let $W=C^{-1}D$, and using the above representation, we obtain:

$$v(Wu+uW)+(Wv+vW)u=(vA(u)+A(v)u=A(uv)=W(uv)+(uv)W.$$

$$\text{Hence, } -2vWu=2uC^{-1}Dv=0 \text{ for all } u,v\in D.$$

This is a contradiction unless $2=0$. This proves the result. $\boxed{QED}$

We now show that under the assumption of elation and Baer groups as above, there are always two K-invariant Baer subplanes.

Theorem 5. *Assume that there exists a K-linear Baer group B which normalizes E but does not centralize E.*

Then, K^* leaves invariant at least two Baer subplanes of any derivable net defined by an orbit of E union the axis of E; there are at least two Baer subplanes which are K-subspaces.

Proof. Since B is Baer and normalizes E, B fixes the axis $x=0$ and some component $y=0$ of a derivable net D which B must leave invariant. Thus, B permutes the set of Baer subplanes of the net D and fixes any Baer subplane which contains nonzero points fixed by B.

Case 1. B fixes exactly one Baer subplane of D incident with the zero vector.

In this case the Baer subplane π_o contains all of the fixed points of B on $x=0$ and on $y=0$. However, this implies that $FixB\cap(x=0)$ and $FixB\cap(y=0)$ is π_o as well as $FixB$. In this case, B centralizes E. Hence, case 1 does not occur.

Case 2. B fixes exactly two Baer subplanes of D incident with the zero vector.

Let the two Baer subplanes be π_o and π_1 and note that since it must be the case that $(FixB \cap \pi_o \cap (x=0)) \cup (FixB \cap \pi_1 \cap (x=0)) = FixB \cap (x=0)$, we see that $FixB \cap (x=0)$ is, without loss of generality, $\pi_o \cap (x=0)$. Thus, $\pi_o \cap (x=0)$ is a K-subspace, implies immediately that π_o is a K-subspace. Similarly, on the other component $y=0$, it follows that π_1 is a K-subspaces as B does not centralize E. Hence, there are two Baer subplanes which are K-subspaces.

Case 3. B fixes at least three Baer subplanes of D.

Since there is at least one Baer subplane π_o which is a K-subspace, we are finished unless B fixes π_o.

Now realize D as a Z-regulus net with components:

$$x = 0, y = x \begin{bmatrix} A & 0 \\ 0 & A \end{bmatrix}; \forall A \in Z.$$

Then, K^* is a subgroup of $GL(2, Z)$ and hence elements of K^* have the general form:

$$Diag \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where $A, B, C, D \in Z$.

Now B is in $\Gamma L(2, Z)$ when restricted to either $x = 0$ or $y = 0$. And, B commutes with K so that B is in $GL(2, K)$ when restricted to either $x = 0$ or $y = 0$.

Now assume, without loss of generality, that B fixes π_0, represented as $\{(0, x_2, 0, y_2)\}$ with respect to Z and also fixes $\{(x_1, 0, y_1, 0)\}$ and $(\alpha, \alpha, \beta, \beta)\}$ with respect to Z (all entries are in Z). Then,

restricted to $x = 0$ then the elements of B have the following form:

$$(y_1, y_2) \longmapsto (y_1^{\rho} a, y_2^{\rho} a)$$

where ρ is an automorphism and a is an element of Z. Note that if $\rho = 1$ then there can be non-zero fixed points on $x = 0$ if and only if $a = 1$ which implies that the element is an affine homology. Hence, $\rho \neq 1$. Note that, under this assumption, if $K = Z$ on $x = 0$, it would follow that $\rho = 1$.

Since K^* commutes with B (as B is K-linear), it follows that the elements

$$Diag \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

are such that $A^\rho = A, B^\rho = B, C^\rho = C, D^\rho = D$. Since K is isomorphic to Z, this can only occur when $\rho^2 = 1$ and K and Z are both quadratic extensions of $Fix\rho = Z^-$.

Since K is a quadratic extension of Z^-, we may realize any component L of D as a Pappian spread over K which then contains Z^- regulus nets. The Z-subspaces are also Z^- subspaces so B fixes at least three 'components' on L of the Z^--regulus within the Pappian spread over K. Since the elements of B are all Z^--linear, it follows that B fixes on L all components of the Z^--regulus defined by three B-fixed components on L (1-dimensional Z-subspaces fixed by B on L). Hence, the representation given as above is valid with K^* represented as above.

Now let the elements of B be denoted as follows:

$$Diag \begin{bmatrix} U + AT & BT \\ T & U \end{bmatrix}$$

for all $U, T \in Z^-$ and A, B constants such that

$$X^2 + AX - B$$

is irreducible over Z^-.

Now, let $\pi_{c,d} = \{(c\alpha, d\alpha, c\beta, d\beta); \alpha, \beta \in Z\}$ denote a Baer subplane of D written as a Z-regulus.

Since K^* must fix $\pi_o = \pi_{0,1}$, in this notation, it follows that the only way that K^* could fix π_o would be that $T = 0$, a contradiction.

Hence, there are at least two Baer subplanes of D which are K-subspaces. $\boxed{QED}$

Corollary 2. *Let Q be a quasifibration in $PG(3, K)$, for K a field, and assume that the associated translation net π admits an elation group E whose orbits union the axis are derivable nets. Assume that there is exactly one Baer subplane π_o in any such derivable net which is a K-subspace.*

If there exists a non-trivial K-linear Baer collineation σ which normalizes E, then this collineation fixes π_o pointwise.

Proof. Since σ normalizes E and is Baer, σ leaves invariant some derivable net D. Within D there is a unique Baer subplane π_o which is a K-subspace. Moreover, the above theorem shows that σ must centralize E. Hence, the fixed point space of σ is a Baer subplane that lies within the derivable net D and since this is a K-subspace, it follows that $Fix\sigma = \pi_o$. $\boxed{QED}$

4 The Main Results

In the original paper on lifting, everything is correct except for the characterization of quasifibrations admitting elation groups whose orbits define derivable nets and F-linear Baer involutions τ, the assumption that if τ leaves a derivable net invariant and fixes at least three Baer subplanes of the net (incident with the zero vector) then the collineation fixes all Baer subplanes is not correct, as we shall see. For example, in the finite case, when K is isomorphic to $GF(q^2)$, it might be possible for τ to fix $q+1$ Baer subplanes of a derivable net — assuming that the Baer subplanes are not K-subspaces — without the collineation forced to fix all Baer subplanes. This possibility actually occurs.

The reader is referred to Johnson [2] for any necessary background information not explicitly given. In particular, since the following corrects a particular case of the proof of the result mentioned above, the reader is directed to the proof of Theorem (3.1) of [2].

Let π be a translation plane with spread in $PG(3,F)$, for F a field.

Assume that there exists an elation group E whose orbits union the axis $x=0$ are derivable nets.

Assume that there exists an F-linear Baer involution τ which normalizes E so leaves $x=0$ invariant. Assume that $[E,\tau]\neq\langle 1\rangle$.

Since τ is Baer, τ must fix one of these nets, say D, τ must fix a second component which we may take as $y=0$ (as this is the reason that τ fixes the net). Now the derivable net, since defined by a field, is a regulus net with respect to a field Z. In this sense, we may choose the regulus net to have the following form:

$$x=0, y=x\begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}\ \forall U\in Z.$$

We may choose coordinates so that D has the following form:

$$x=0, y=x\begin{bmatrix} u & 0 \\ 0 & u^{\rho} \end{bmatrix}\ \forall u\in F$$

where ρ is an automorphism of F. Let Z denote the field so that the derivable net D defines a Z-regulus in $PG(3,Z)$.

The proof given in [2] when τ fixes at least three Z-subspaces on $x=0$ assumes that when τ fixes at least three Z-subspaces on $x=0$, it fixes them all —this is not correct. The following provides the variation.

There is no problem if $\rho=1$ as then $Z=F$, and since τ is F-linear, once three F-subspaces are fixed, τ must fix all such F-subspaces. Hence, we may assume that $\rho\neq 1$.

The subplanes $\pi_{a,b}$ incident with the zero vector have the following form:

$$\pi_{a,b} = \{(a\alpha, b\alpha^{\rho}, a\beta, b\beta^{\rho}); \alpha, \beta \in F\}.$$

There are exactly two F-subspaces among the $\pi_{a,b}$; $\pi_{1,0}$ and $\pi_{0,1}$. Hence, either τ interchanges these two subspaces or leaves them invariant.

If τ leaves both of these invariant then τ has the following form:

$$\tau := \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & w & 0 \\ 0 & 0 & 0 & z \end{bmatrix}$$

where $u^2 = v^2 = w^2 = z^2 = 1$. Also, note that

$$\begin{aligned} & \begin{bmatrix} u^{-1} & 0 \\ 0 & v^{-1} \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{\rho} \end{bmatrix} \begin{bmatrix} w & 0 \\ 0 & z \end{bmatrix} \\ = & \begin{bmatrix} u^{-1}tw & 0 \\ 0 & v^{-1}t^{\rho}z \end{bmatrix} \end{aligned}$$

implies that $(v^{-1}z)^{\rho} = vz = u^{-1}w = uw$, since all elements u, v, w, z are ± 1. Hence, we obtain

$$\tau := \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & u & 0 \\ 0 & 0 & 0 & v \end{bmatrix},$$

if $uw = 1$ and

$$\tau := \begin{bmatrix} u & 0 & 0 & 0 \\ 0 & v & 0 & 0 \\ 0 & 0 & -u & 0 \\ 0 & 0 & 0 & -v \end{bmatrix},$$

if $uw = -1$.

In the first case, τ commutes with E. Hence, we must have the second case.

If τ fixes at least three Z-1-spaces on $x = 0$ (including $\pi_{1,0} \cap (x = 0)$ and $\pi_{0,1} \cap (x = 0)$), assume that τ fixes some $\pi_{a,b}$ for $ab \neq 0$.

Hence, $(a\alpha, b\alpha^{\rho}, a\beta, b\beta^{\rho})\tau = (au\alpha, bv\alpha^{\rho}, -au\beta, -bv\beta^{\rho})$.

Hence, we must have:

$$u = \delta \text{ and } v = \delta^{\rho}.$$

This implies that $u = v$. However, this forces τ to be an affine homology.

So, assume that τ inverts $\pi_{0,1}$ and $\pi_{1,0}$. Note that the previous proof shows that if any nontrivial F-linear Baer group fixes at least three Z-subspaces and fixes $\pi_{1,0}$ and $\pi_{0,1}$ then we have a contradiction. Hence, if an F-linear Baer group W interchanges $\pi_{1,0}$ and $\pi_{0,1}$ and fixes at least three Z-subspaces on $x=0$ then $W_{\pi_{1,0}} = \langle 1 \rangle$. Hence, the only problem arises when there is a Baer involution interchanging $\pi_{1,0}$ and $\pi_{0,1}$, and fixing at least three Z-subspaces.

It now follows that

$$\tau := \begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & e \\ 0 & 0 & e^{-1} & 0 \end{bmatrix}.$$

Since τ normalizes E, we must have:

$$\begin{aligned} & \begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix} \begin{bmatrix} t & 0 \\ 0 & t^{\rho} \end{bmatrix} \begin{bmatrix} 0 & e \\ e^{-1} & 0 \end{bmatrix} \\ = & \begin{bmatrix} be^{-1}t^{\rho} & 0 \\ 0 & b^{-1}et \end{bmatrix}. \end{aligned}$$

Hence, we must have

$$(be^{-1}t^{\rho})^{\rho} = b^{-1}et$$

for all t. Letting $t=1$, we have:

$$(be^{-1})^{\rho} = b^{-1}e \text{ or rather: } (be^{-1})^{\rho+1} = 1.$$

Furthermore, clearly $\rho^2 = 1$. Let K denote the fixed field of ρ then F is a quadratic extension of K. Hence, $F = K[\theta]$. This implies that ρ is the unique non-identity element of $Gal_K F$ as $\rho \neq 1$.

Now let the components of π be

$$x = 0, y = x \begin{bmatrix} u + G(t) & H(t) \\ t & u^{\sigma} \end{bmatrix}, \forall u, t \in F.$$

The group is $\langle \tau, E \rangle$.

Applying τ to the above matrix set implies the following relations:

$$\begin{aligned} H(b^{-1}e^{-1}H(t)) &= bet, \\ G(b^{-1}e^{-1}H(t)) &= -(b^{-1}eG(t))^{\sigma}, \forall t \in F. \end{aligned}$$

Moreover, the τ-fixed components exterior to the derivable net have the following form:

$$\begin{aligned} H(t) &= bet, \\ G(t) &= be^{-1}u^{\sigma} - u. \end{aligned}$$

Hence, we have these exterior components of the form:

$$y = x \begin{bmatrix} be^{-1}u^{\sigma} & bet \\ t & u^{\sigma} \end{bmatrix}.$$

Now if there is a Baer group B^{+} properly containing τ, and fixing at least three Baer subplanes of D and interchanging $\pi_{1,0}$ and $\pi_{0,1}$, we have seen above that this cannot occur.

5 Lifted Quasifibrations and Non-Normalizing Involutions

In this section, we shall show that it is possible that a quasifibration could simultaneously be of both types (1) and (2) of the main theorem of the previous section. The key point revolves around the Baer involutions. We assume the hypotheses of the second theorem listed in the introduction. We shall give the proof as follows:

We assume that the Baer group C fixes both the axis $x = 0$ and a component which we may take without loss of generality as $y = 0$ and which we assume preserves the Baer subline structure of the standard net D on $y = 0$. There is an analogous argument when the group preserves the Baer subline structure on the axis. Hence, every element g of C fixes or interchanges the two F-subspaces on $\pi_{1,0}$ and $\pi_{0,1}$ on $y = 0$. We initially consider that C contains a Baer involution which does not normalize E.

First assume that $\operatorname{char} F = 2$. Assume that τ centralizes the elation group E. Then, it follows that τ leaves each component invariant of some E-orbit and, of course, τ leaves the axis of E invariant. Since τ is F-linear, it follows easily that Fix τ may be taken to be $\pi_{0,1}$. So, τ has the following form:

$$\begin{bmatrix} a & b & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & a & b \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ for some elements } a, b \in F.$$

However, $\tau^2 = 1$ so we have

$$a^2 = 1 \text{ and } ab + b = 0, \text{ so that } b \neq 0.$$

Since $\operatorname{char} F = 2$, the mapping $x \longmapsto x^2$ is an injective homomorphism. Hence, $a = 1$.

However, then

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u & H(t) \\ t & u^\sigma \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} u+bt & H(t)+u^\sigma+(u+bt)b \\ t & tb+u^\sigma \end{bmatrix}$$

which can only be in the spread provided $tb+u^\sigma = v^\sigma$ and $u+bt = v$, implying that $b^\sigma t^\sigma = bt$ for all t in F, a contradiction.

Hence, τ does not centralize E when $\operatorname{char} F = 2$.

Let E^+ denote the full elation group with axis $x=0$; the axis of E. Then τ must normalize E^+. Moreover, since τ is Baer, we may assume that τ fixes $x=0$ and $y=0$. If τ fixes another component of D, then it follows fairly easily that τ must normalize E. Hence, $\mathrm{Fix}\tau$ shares at most one component, apart from $x=0$, of each of the derivable nets. Therefore, $\mathrm{Fix}\tau$ shares $x=0$ and exactly one other component with each derivable net. Hence, no element of $E-\{1\}$ normalizes τ.

We are assuming that τ fixes or interchange $\pi_{1,0}\cap(y=0)$ and $\pi_{0,1}\cap(y=0)$. —If the first case τ restricted to $y=0$ is $\begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix}$ implying that $d^2=1$ so that $d=1$, for even characteristic implying that τ fixes $y=0$ pointwise.

Hence, we must have the second case and here, τ restricted to $y=0$ is $\begin{bmatrix} 0 & b \\ b^{-1} & 0 \end{bmatrix}$.

If we let τ map $y=x$ onto $y=x\begin{bmatrix} u_o & H(t_o) \\ t_o & u_o^\sigma \end{bmatrix}$, it follows that

$$\tau := \begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & bt_o & bu_o^\sigma \\ 0 & 0 & b^{-1}u_o & b^{-1}H(t_o) \end{bmatrix}.$$

Now if $t_o = 0$ then τ normalizes E. Hence, $t_o \neq 0$. Since $\tau^2 = 1$, by calculation, it follows that:

$$H(t_o) = b^2 t_o.$$

Now τ maps $y = x\begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}$ onto $y = x\begin{bmatrix} u^\sigma u_o & u^\sigma H(t_o) = H(ut_o) \\ ut_o & u_o^\sigma u \end{bmatrix}$, for each $u \in K$. Thus, we have:

$$H(v) = H(vt_o^{-1}t_o) = (vt_o^{-1})^\sigma b^2 t_o = b^2 t_o^{1-\sigma} v^\sigma$$

for all $v \in F$. We note that in this situation, we have:

$$H(b^{-2}H(v)) = H(t_o^{1-\sigma}v^\sigma) = b^2 t_o^{1-\sigma}(t_o^{1-\sigma}v^\sigma)^\sigma = b^2 v.$$

Hence, we have situation given as above with $b = e$. Note that this implies that we have a Baer involution of the required type which does, in fact, normalize E; namely

$$\tau := \begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{bmatrix}.$$

Hence, either the spread is completely determined as in case (b) or there is, initially, a Baer involution τ which does normalize the specific elation group E but does not centralize it. In the latter case, we have situation (1) or (2) of the main structure theorem.

In case (b) above, we note that G_δ: $G_\delta(t) = b^2 t^\sigma - \delta t^\sigma$ for any $\delta \in K$ is bijective. This condition suffices by a proposition in [2] (2.5) to conclude that the quasifibration is a spread whose retraction is also a spread which gives part (c).

Now assume that $\operatorname{char} F \neq 2$ and that τ does not normalize E. We follow the general outline of the proof for the even characteristic case.

Now since we are assuming that τ does not normalize E, then it does not centralize it. If τ fixes three components on D, it follows that τ has one of the following two forms:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

or

$$\begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & b \\ 0 & 0 & b^{-1} & 0 \end{bmatrix}$$

when τ fixes or interchanges $\pi_{1,0}$ and $\pi_{0,1}$ respectively. However, both of these forms normalize E.

Hence, τ does not leave D invariant and moves all components of D other than $x = 0$ and $y = 0$. Furthermore, restricted to $y = 0$, τ has one of the above two forms:

In the first case,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & u_o & H(t_o) \\ 0 & 0 & -t_o & -u_o^\sigma \end{bmatrix}$$

for some $t_o \neq 0$. In this case, it follows that

$$\begin{aligned} y &= x\begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix} \text{ maps to } y = x\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} u & 0 \\ 0 & u^\sigma \end{bmatrix}\begin{bmatrix} u_o & H(t_o) \\ -t_o & -u_o^\sigma \end{bmatrix} \\ &= \begin{bmatrix} u_o u & uH(t_o) \\ u^\sigma t_o & u^\sigma u_o^\sigma \end{bmatrix} \text{ for all } u \in F. \end{aligned}$$

This implies that

$$\begin{aligned} H(u^\sigma t_o) &= uH(t_o) \text{ so} \\ H(v) &= v^\sigma t_o^{-\sigma} H(t_o). \end{aligned}$$

Note that since τ has order 2, we obtain the following conditions on t_o and u_o:

$$u_o^\sigma = u_o \text{ and } u_o^2 - t_o H(t_o) = 1.$$

since $u_o \in K$, it follows that $t_o H(t_o)$ is in K. Then, noting that $t_o^{-1-\sigma}$ is in $F^{\sigma+1} \subseteq K$, we have that $t_o^{-\sigma} H(t_o) = c \in K$.

It then follows that

$$H(t) = ct^\sigma \text{ where } c = t_o^{-\sigma} H(t_o) \in K.$$

Now the only way that we can obtain a quasifibration is that K is not $F^{\sigma+1}$. However, since the quasifibration is lifted from a quasifibration in $PG(3, K)$, we must also have:

$$vH(v) = vcv^\sigma \text{ is never in } K.$$

However, since c and $v^{\sigma+1}$ are both in K, we never obtain a retraction quasifibration. Hence, this case does not occur. Note that if we merely assume that π is a quasifibration with an elation group E whose orbits are derivable nets and τ is a non-normalizing F-linear Baer involution, the above argument applies without assuming that π has been lifted from a quasifibration. In this setting $\sigma = \rho$ is simply an arbitrary non-trivial automorphism of F.

So, in this more general case,

$$H(v) = cv^{\rho^{-1}}, \text{ where } c = t_o^{-\rho^{-1}-1} t_o H(t_o) \text{ for } t_o H(t_o) \in Fix\rho$$

where the derivable net has components

$$x = 0, y = x\begin{bmatrix} u & 0 \\ 0 & u^\rho \end{bmatrix} \ \forall u \in F.$$

This situation certainly could arise under different hypotheses.

So, assume we have the second situation that τ has the following form:

$$\tau := \begin{bmatrix} 0 & b & 0 & 0 \\ b^{-1} & 0 & 0 & 0 \\ 0 & 0 & bt_o & bu_o^{\sigma} \\ 0 & 0 & b^{-1}u_o & b^{-1}H(t_o) \end{bmatrix}, \text{ for } t_o \neq 0.$$

Then, we may follow the proof of the previous case for char $F = 2$ to show that we have the same form as above.

Now assume that the group C does not contain an involution which does not normalize E. Then, either g or g^2 has the form given above which fixes the two F-subspaces of $\pi_{1,0}$ or $\pi_{0,1}$ on $y = 0$. Also, the element on $y = 0$ commutes with $\begin{bmatrix} u & 0 \\ 0 & u^{\sigma} \end{bmatrix}$ for all u and the proof above provides the same results, unless possibly $g^2 = 1$. If g inverts the two F-subspaces in question on $y = 0$, then g must normalize E. This is situation (2) of the main structure theorem. If there exists a nonidentity element of C which does not invert the two F-subspaces and does not normalize E then we have the structure exactly as indicated. Hence, all elements of C normalize E or we have the structure completely determined. This completes the proof except possibly for the group preserving the Baer subline structure on the axis $x = 0$ of E. However, here the proof is almost identical to the $y = 0$ situation and is left to the reader.

5. 1 Examples

Consider the possible set of additive quasifibrations of the following form:

$$x = 0, y = x \begin{bmatrix} u + ct^{\alpha} & dt^{\sigma} \\ t & u^{\sigma} \end{bmatrix} \forall u, t \in f$$

where α is an automorphism of F and σ is an automorphism of F such that $\sigma^2 = 1$. To obtain at least a quasifibration, we require that

$$u^{\sigma+1} + ct^{\alpha}u - dt^{\sigma+1} \neq 0 \ \forall u, t \in F, \ (u, t) \neq (0, 0).$$

Assume also that $c = -c^{\sigma}$.

We note that, if $d = b^2$ then

$$\begin{aligned} H(b^{-2}H(t)) &= H(t^{\sigma}) = b^2 t^{\sigma^2} = b^2 t \text{ and} \\ G(b^{-2}H(t)) &= G(t^{\sigma}) = ct^{\alpha\sigma} = -G(t)^{\sigma} = -(ct^{\alpha})^{\sigma} \\ \text{if and only if } c &= -c^{\sigma}. \end{aligned}$$

Thus, provided there is a quasifibration, we obtain an example of (2) and if $c \neq 0$, the spread is not of type (1). However, if $c = 0$, the example is of type

(1) and (2). When $c = 0$ and d is not a square in K, then the quasifibration is not a lifted quasifibration.

References

[1] V. Jha, N. L. Johnson: *Infinite Baer nets*, Journal of Geometry **68** (2000), 114–141.

[2] N. L. Johnson: *Lifting quasifibrations*, Note di Mat. **16** (1996) 25–41.