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Note on strongly Lie nilpotent rings

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Abstract. This note contains a few introductory results on strongly Lie nilpotent rings and, in particular, an analogue of a well known theorem of P. Hall on nilpotent groups.

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1 Introduction

Let R be an associative ring. For all $a, b \in R$ we set $a \circ b = ab - ba$. It is well-known that $(R, +, \circ)$ is a Lie ring. For all $A, B \subseteq R$, the additive subgroup of R generated by all Lie products $a \circ b \quad (a \in A, b \in B)$ is denoted by $A \circ B$.

Now we put $\gamma_1(R) = R$ and for any $n \in \mathbf{N}, \quad n > 1, \quad \gamma_n(R) = \gamma_{n-1}(R) \circ R$. If there exists $c \in \mathbf{N}$ such that $\gamma_{c+1}(R) = 0$, then R is called a *Lie nilpotent ring.*

We define the Lie powers $R^{(n)}(n \in \mathbf{N})$ as follows: $R^{(1)} = R$, and for all $n \in \mathbf{N}, \quad n > 1$, $R^{(n)}$ is the ideal of R generated by $R^{(n-1)} \circ R$. If there exists $c \in \mathbf{N}$ such that $R^{(c+1)} = 0$, then R is called a *strongly Lie nilpotent ring* (see [7]).

Clearly, $\gamma_n \subseteq R^{(n)}$ for all $n \in \mathbf{N}$, thus a strongly Lie nilpotent ring is Lie-nilpotent.

There are many results on strongly Lie nilpotent group rings, see for example Bovdi's paper [2].

The 2nd section of this note contains a few developments in the spirit of Jennings' paper [4]. In the 3rd section, an analogue of a well known theorem of P. Hall on nilpotent groups for strongly Lie nilpotent rings is obtained.

2 Central series of ideals

We recall that if I and J are ideals of a ring R and $I \subseteq J$, then J/I is called a *central factor* if $J \circ R \subseteq I$ or, equivalently, J/I belongs to the centre $Z(R/I)$ of the ring R/I.

A chain $(J^{(\lambda)})$ of ideals of a ring R is called a *central series* of R if every factor $J^{(\lambda+1)}/J^{(\lambda)}$ is central (see [4]).

The *lower central series* of a ring R is the descending series whose terms $R^{(\alpha)}$ are defined by setting: $R^{(1)} = R$ and, for $\alpha > 1, \quad R^{(\alpha)} = \bigcap_{\beta<\alpha} R^{(\beta)}$ if α is a limit ordinal and $R^{(\alpha)}$ is the ideal of R generated by $R^{(\alpha-1)} \circ R$, otherwise.

Following an idea of Jennings [4], we now define the upper central series of an arbitrary ring.

If B is an additive subgroup of a ring R, then the set $M := \{x | x \in B, \quad Rx \subseteq B\}$ is the largest left ideal of R which is contained in B. Moreover, the set $F := \{y | y \in M, \quad yR \subseteq M\}$ is the largest ideal of R which is contained in B.

It is easy to see that $F(R) = \{y | y \in Z(R), \quad yR \subseteq Z(R)\}$ is the largest ideal of R which is contained in the centre $Z(R)$ of R. The ideal $F(R)$ is called the *strong centre* of R. We remark that the annihilator of a ring R is contained in $F(R)$.

The *upper central series* of a ring R is the ascending series whose terms $F^{(\alpha)}(R)$ are defined by setting $F^{(0)}(R) = \{0\}$ and, for $alpha > 0$, $F^{(\alpha)}(R) = \bigcup_{\beta<\alpha} F^{(\beta)}(R)$ if α is a limit ordinal and $F^{(\alpha+1)}(R)/F^{(\alpha)}(R) = F(R/F^{(\alpha)}(R))$ otherwise. In particular, $F^{(1)}(R)$ is the strong centre of R.

Moreover, for any positive integer k

$$F^{(k)}(R) = \{x | x \in R, \quad \forall r, s \in R \quad x(1+r) \circ s \in F^{(k-1)}(R)\} \tag{1}$$

The following result gives some relationship between the lower central series and the upper central series of arbitrary ring R.

Proposition 1. *Let R be a ring, and let k and l be positive integers. (1) $R^{(k)} \cdot R^{(l)} \subseteq R^{(k+l-1)}$*

(2) $R^{(k)} \circ R^{(l)} \subseteq R^{(k+l)}$

(3) $(R^{(k)})^{(l)} \subseteq R^{(kl)}$

(4) $R^{(k)} \cdot F^{(l)}(R) \subseteq F^{(l-k+1)}(R)$ se $k \leq l$

(5) $F^{(l)}(R) \cdot R^{(k)} \subseteq F^{(l-k+1)}(R)$ se $k \leq l$

(6) $R^{(k)} \circ F^{(l)}(R) \subseteq F^{(l-k)}(R)$ se $k \leq l$

(7) $F^{(k)}(R/F^{(l)}(R)) = F^{(k+l)}(R)/F^{(l)}(R)$

Proof. For (1), (2) see [4], Theorem 3.3 e Theorem 3.4. We prove our assertions by induction. First, (3) is trivial for $l = 1$. If $l > 1$, then, by (2), we have

$$(R^{(k)})^{(l-1)} \circ R^{(k)} \subseteq R^{k(l-1)} \circ R^{(k)} \subseteq R^{(k(l-1)+k)} = R^{(kl)}$$

for all positive integer k. Hence $(R^{(k)})^{(l)} \subseteq R^{(kl)}$.

(4): If $k = 1$, then, for all $l \in \mathbf{N}$

$$R^{(k)}F^{(l)}(R) = R^{(1)}F^{(l)}(R) \subseteq F^{(l)}(R) \subseteq F^{(l-k+1)}(R)$$

Now let $k > 1$. For all $a \in R^{(k-1)}$, $\quad b \in R$ and $c \in F^{(l)}(R)$, the inductive hypothesis implies that

$$(a \circ b)c = ac \circ b - a(c \circ b) \in F^{(l-k+1)}(R),$$

as desired.

(5): Analogously to (4).

(6): If $k = 1$, then, for all $l \in \mathbf{N}$

$$R^{(k)} \circ F^{(l)}(R) = R \circ F^{(l)}(R) \subseteq F^{(l-1)}(R) = F^{(l-k)}(R)$$

Now let $k > 1$. For all $a \in R^{(k-1)}$, $\quad b \in R$, $\quad r \in R$ and $c \in F^{(l)}(R)$, inductively we have

$$(a \circ b) \circ c = b \circ (c \circ a) + a \circ (b \circ c) \in F^{(l-k)}(R)$$

Hence, by (5), we have

$$(a \circ b)r \circ c = (a \circ b) \circ rc + r \circ (c(a \circ b)) \in F^{(l-k)}(R)$$

(7): If $k = 1$, then, for all $l \in \mathbf{N}$

$$F^{(k)}(R/F^{(l)}(R)) = F(R/F^{(l)}(R)) = F^{(l+1)}(R)/F^{(l)}(R) = F^{(k+l)}(R)/F^{(l)}(R)$$

Now let $k > 1$. For all $l \in \mathbf{N}$ and for all $y \in R$ we have

$$y + F^{(l)}(R) \in F^{(k)}(R/F^{(l)}(R)) \Longleftrightarrow$$

$$\Longleftrightarrow \forall a, b \in R \quad \big(y(1+a) \circ b\big) + F^{(l)}(R) \in F^{(k-1)}(R/F^{(l)}(R))$$

$$\Longleftrightarrow \forall a, b \in R \quad \big(y(1+a) \circ b\big) + F^{(l)}(R) \in F^{(k-1+l)}(R)/F^{(l)}(R)$$

$$\Longleftrightarrow \forall a, b \in R \quad y(1+a) \circ b \in F^{(k-1+l)}(R) \Longleftrightarrow y \in F^{(k+l)}(R)$$

which completes the proof. $\boxed{QED}$

Corollary 1. *If R is a ring and k is a positive integer, then*

$$\operatorname{char} R/F^{(k)}(R) = \operatorname{char} R^{(k+1)}.$$

Proof. Let $k \in \mathbf{N}$ and let $m := \operatorname{char} R/F^{(k)}(R) \neq 0$. For all $a \in R^{(k)}, \quad r \in R$, we have

$$m(a \circ r) = a \circ mr \in R^{(k)} \circ F^{(k)}(R) = 0,$$

by Prop. 1 (6). Since $R^{(k+1)}$ is the ideal of R generated by $R^{(k)} \circ R$, it follows that $\operatorname{char} R^{(k+1)}$ divides m.

Now let $n := \operatorname{char} R^{(k+1)} \neq 0$. For each $r, r_1, \dots r_k, s_1, \dots s_1, \dots s_k \in R$ we have

$$(\cdots((((nr(1+r_1) \circ s_1)(1+r_2) \circ s_2) \cdots)(1+r_k) \circ s_k =$$

$$= n\big((\cdots((((r(1+r_1) \circ s_1)(1+r_2) \circ s_2) \cdots)(1+r_k) \circ s_k\big) = 0$$

By (1), it follows that $nr \in F^{(k)}(R)$. Hence $\operatorname{char} R/F^{(k)}(R)$ divides n

It follows immediately that $\operatorname{char} R/F^{(k)}(R) = 0$ if and only if $\operatorname{char} R^{(k+1)} = 0$. $\boxed{QED}$

The following proposition gives a relation between the characteristic of the factors of the upper central series of a ring and that of its strong centre.

Proposition 2. *If R is a ring such that $\operatorname{char} F(R) \neq 0$, then the characteristic of $F^{(k+1)}(R)/F^{(k)}(R)$ divides the characteristic of $F(R)$, for each nonnegative integer k.*

Proof. Let $n := \operatorname{char} F(R) \neq 0$. We show by induction on k that $nx \in F^{(k)}(R)$, for all $x \in F^{(k+1)}(R)$ and $k \in \mathbf{N_0}$.

For $k = 0$, there is nothing to prove. Let $k \geq 1$ and assume that $ny \in F^{(k-1)}(R)$ for each $y \in F^{(k)}(R)$. Let $x \in F^{(k-1)}(R)$. For all $r, s \in R$ we have $x(1+r)\circ s \in F^{(k)}(R)$. Inductively, $n(x(1+r)\circ s) \in F^{(k-1)}(R)$. Hence $(nx)(1+r)\circ s$ belongs to $F^{(k-1)}(R)$ and $nx \in F^{(k)}(R)$, by (1). $\boxed{QED}$

3 Analogue of a theorem of P. Hall

In [4], Jennings proves that a ring is strongly Lie nilpotent if and only if it has a finite central series. Moreover, we have

Proposition 3. *Let R be a ring. If $c \in \mathbf{N}$ and $0 = I_0 \subset \ldots \subset I_c = R$ is a central series of R, then*

$$R^{(c-k+1)} \subseteq I_k \subseteq F^{(k)}(R)$$

for each $k \in \{0, 1, \dots c\}$

Proof. The first inclusion holds by [4] (Theorem 2.1). We prove, by induction on k, that $I_k \subseteq F^{(k)}(R)$. For $k = 0$, there is nothing to prove. Let $k \geq 1$

and assume that $I_{k-1} \subseteq F^{(k-1)}(R)$. Let $z \in I_k$. Since I_k/I_{k-1} is a central factor, we have inductively

$$z(1+r) \circ s \in I_{k-1} \subseteq F^{(k-1)}(R)$$

for all $r, s \in R$. Hence $z \in F^{(k)}(R)$, by (1). $\boxed{QED}$

The proposition shows that the lower and upper central series of any strongly Lie nilpotent ring R have the same length c. This length c is called the *strongly Lie nilpotent class* of R

The following result is analogous to one obtained for nilpotent rings (see [5], 1.2.6).

Proposition 4. *If R is a strongly Lie nilpotent ring, then* $\operatorname{char} R = 0$ *if and only if* $\operatorname{char} F(R) = 0$.

Proof. If $\operatorname{char} F(R) = 0$, then clearly $\operatorname{char} R = 0$. Conversely, let $\operatorname{char} R = 0$ and assume that $\operatorname{char} F(R) = m \neq 0$. If c is the strongly nilpotent class of R, then $R^{(c)} \subseteq F(R)$. Hence $\operatorname{char} R^{(c)} \neq 0$. Let $i := \min\{j | j \in \mathbf{N}, \quad \operatorname{char} R^{(j)} \neq 0\}$ and let $n := \operatorname{char} R^{(i)}$. Then there is an element $x \in R^{(i-1)}$ such that $mnx \neq 0$.

For all $y, z \in R$, we have

$$nx(1+y) \circ z = n(x(1+y) \circ z) = n(x \circ z + xy \circ z) = 0$$

By (1), $nx \in F(R)$, therefore $mnx = 0$, a contradiction to the choice of x. $\boxed{QED}$

The results above are examples of a strong analogy between the theories of nilpotent groups and strongly Lie nilpotent rings.

In particular, we recall the well-known theorem of P. Hall for nilpotent groups: if N is a normal subgroups of a group G and $N, G/N'$ are nilpotent, then G is nilpotent (see [6]). A version of this theorem for Lie algebras is contained, for example, in [1].

We give a version of the theorem of P. Hall for strongly Lie nilpotent rings.

Lemma 1. *Let R be a ring, I an ideal of R such that its strong centre $F(I)$ is an ideal of R and M the largest ideal of R contained in $I \circ I$.*

If there is a finite central series of R between $F(I)$ and I, then there is a finite central series of R between 0 and M.

Proof. Let $t \in \mathbf{N}$ and

$$F(I) = I_0 \subset I_1 \subset \cdots \subset I_t = I \tag{2}$$

a finite central series of R between $F(I)$ and I.

For each $i \in \mathbf{N}, \quad i \leq 2t$, let B_i the additive subgroup of R generated by $\bigcup_{h+k=i} I_h \circ I_k$, and let $\overline{B}_i$ be the ideal R generated by B_i.

Evidently

$$0 = \overline{B}_1 \subseteq \overline{B}_2 \subseteq \cdots \subseteq \overline{B}_{2t} \tag{3}$$

We show that (3) is a central series of R.

It is sufficient to prove that, for all $a \in I_h, \quad b \in I_k$ such that $h + k = i$ and for all $r, s, v \in R$ we have

$$a \circ b \circ v \in \overline{B}_{i-1}$$
$$(a \circ b)r \circ v \in \overline{B}_{i-1}$$
$$r(a \circ b) \circ v \in \overline{B}_{i-1}$$
$$r(a \circ b)s \circ v \in \overline{B}_{i-1}$$

Since (2) is a central series, by the Jacobi identity, we have

$$a \circ b \circ v = a \circ v \circ b + v \circ b \circ a \in B_{i-1} \subseteq \overline{B}_{i-1}$$

Moreover (cfr. [3], Lemma 2)

$$\begin{aligned}(a \circ b)(r \circ v) = {} & v(a \circ r) \circ b \\ & - r(a \circ b \circ v) \\ & -a \circ r \circ bv + a \circ br \circ v \\ & -a \circ b \circ v \circ r + a \circ r \circ b \circ v \in \overline{B}_{i-1}.\end{aligned}$$

Hence

$$(a \circ b)r \circ v = (a \circ b \circ v)r + (a \circ b)(r \circ v) \in \overline{B}_{i-1}.$$

It follows that

$$r(a \circ b) \circ v = -(a \circ b \circ r) \circ v + (a \circ b)r \circ v \in \overline{B}_{i-1}.$$

Finally,

$$s(a \circ b)r \circ v = s((a \circ b)r \circ v) + (s \circ v)(a \circ b)r \in \overline{B}_{i-1}.$$

Hence for all $i \in \mathbf{N}, \quad 1 < i \leq 2t$ we have

$$(\overline{B}_i \cap M) \circ R \subseteq (\overline{B}_i \circ R) \cap (M \circ R) \subseteq \overline{B}_{i-1} \cap M.$$

Therefore

$$0 = \overline{B}_1 \cap M \subseteq \cdots \subseteq \overline{B}_{2t} \cap M = M$$

is a finite central series of R between 0 and M. $\boxed{QED}$

Theorem 1. *Let R be a ring, I an ideal of R such that its strong centre $F(I)$ is an ideal of R, and let M be the largest ideal of R contained in $I \circ I$.*

If I and R/M are strongly Lie nilpotent rings, then R is strongly Lie nilpotent.

Proof. We proceed by induction on the strongly Lie nilpotent class c of I. If $c = 1$, then $I = F(I)$ and $I \circ I = 0$. It follows that $M = 0$. Hence R is strongly Lie nilpotent.

If $c = 2$, then $I \circ I \subseteq F(I)$. Hence $M \subseteq F(I)$. As R/M is strongly nilpotent, it follows that $R/F(I)$ is strongly Lie nilpotent. Now, $I/F(I)$ is an ideal of $R/F(I)$, and therefore there is a finite central series of R between $F(I)$ and I. By 1, there is a finite central series of R between 0 and M. It follows that R is strongly Lie nilpotent.

If $c > 3$ and $\overline{M}$ is the largest ideal of $R/F(I)$ contained in $I/F(I) \circ I/F(I)$, then $F(I) \subseteq M$ and $\overline{M} = M/F(I)$. Since $(R/F(I))/\overline{M} \cong R/M$, we have that $(R/F(I))/\overline{M}$ is strongly Lie nilpotent. Now $I/F(I)$ is strongly Lie nilpotent of class $c-1$ and, inductively $R/F(I)$ is strongly Lie nilpotent.

Proceeding as in the case of $c = 2$, we complete our proof. $\boxed{QED}$

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