Note on strongly Lie nilpotent rings

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Abstract. This note contains a few introductory results on strongly Lie nilpotent rings and, in particular, an analogue of a well known theorem of P. Hall on nilpotent groups.

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1 Introduction

Let $R$ be an associative ring. For all $a, b \in R$ we set $a \circ b = ab - ba$. It is well-known that $(R, +, \circ)$ is a Lie ring. For all $A, B \subseteq R$, the additive subgroup of $R$ generated by all Lie products $a \circ b$ $(a \in A, b \in B)$ is denoted by $A \circ B$.

Now we put $\gamma_1(R) = R$ and for any $n \in \mathbb{N}$, $n > 1$, $\gamma_n(R) = \gamma_{n-1}(R) \circ R$. If there exists $c \in \mathbb{N}$ such that $\gamma_{c+1}(R) = 0$, then $R$ is called a Lie nilpotent ring.

We define the Lie powers $R^{(n)}(n \in \mathbb{N})$ as follows: $R^{(1)} = R$, and for all $n \in \mathbb{N}$, $n > 1$, $R^{(n)}$ is the ideal of $R$ generated by $R^{(n-1)} \circ R$. If there exists $c \in \mathbb{N}$ such that $R^{(c+1)} = 0$, then $R$ is called a strongly Lie nilpotent ring (see [7]).

Clearly, $\gamma_n \subseteq R^{(n)}$ for all $n \in \mathbb{N}$, thus a strongly Lie nilpotent ring is Lie-nilpotent.

There are many results on strongly Lie nilpotent group rings, see for example Bovdi’s paper [2].

The 2nd section of this note contains a few developments in the spirit of Jennings’ paper [4]. In the 3rd section, an analogue of a well known theorem of P. Hall on nilpotent groups for strongly Lie nilpotent rings is obtained.

2 Central series of ideals

We recall that if $I$ and $J$ are ideals of a ring $R$ and $I \subseteq J$, then $J/I$ is called a central factor if $J \circ R \subseteq I$ or, equivalently, $J/I$ belongs to the centre $Z(R/I)$ of the ring $R/I$. 
A chain \((J^{(\lambda)})\) of ideals of a ring \(R\) is called a central series of \(R\) if every factor \(J^{(\lambda+1)}/J^{(\lambda)}\) is central (see [4]).

The lower central series of a ring \(R\) is the descending series whose terms \(R^{(\alpha)}\) are defined by setting: \(R^{(1)} = R\) and, for \(\alpha > 1\), \(R^{(\alpha)} = \bigcap_{\beta < \alpha} R^{(\beta)}\) if \(\alpha\) is a limit ordinal and \(R^{(\alpha)}\) is the ideal of \(R\) generated by \(R^{(\alpha-1)} \circ R\), otherwise.

Following an idea of Jennings [4], we now define the upper central series of an arbitrary ring.

If \(B\) is an additive subgroup of a ring \(R\), then the set \(M := \{x| x \in B, \ R x \subseteq B\}\) is the largest left ideal of \(R\) which is contained in \(B\). Moreover, the set \(F := \{y| y \in M, \ yR \subseteq M\}\) is the largest ideal of \(R\) which is contained in \(B\).

It is easy to see that \(F(R) = \{y| y \in Z(R), \ yR \subseteq Z(R)\}\) is the largest ideal of \(R\) which is contained in the centre \(Z(R)\) of \(R\). The ideal \(F(R)\) is called the strong centre of \(R\). We remark that the annihilator of a ring \(R\) is contained in \(F(R)\).

The upper central series of a ring \(R\) is the ascending series whose terms \(F^{(\alpha)}(R)\) are defined by setting \(F^{(0)}(R) = \{0\}\) and, for \(\alpha > 0\), \(F^{(\alpha)}(R) = \bigcup_{\beta < \alpha} F^{(\beta)}(R)\) if \(\alpha\) is a limit ordinal and \(F^{(\alpha+1)}(R)/F^{(\alpha)}(R) = F(R/F^{(\alpha)}(R))\) otherwise. In particular, \(F^{(1)}(R)\) is the strong centre of \(R\).

Moreover, for any positive integer \(k\)

\[
F^{(k)}(R) = \{x| x \in R, \ \forall r, s \in R \ x(1 + r) \circ s \in F^{(k-1)}(R)\} \tag{1}
\]

The following result gives some relationship between the lower central series and the upper central series of arbitrary ring \(R\).

**Proposition 1.** Let \(R\) be a ring, and let \(k\) and \(l\) be positive integers. (1)

\[
\begin{align*}
(1) \quad & R^{(k)} \cdot R^{(l)} \subseteq R^{(k+l-1)} \\
(2) \quad & R^{(k)} \circ R^{(l)} \subseteq R^{(k+l)} \\
(3) \quad & (R^{(k)})^{(l)} \subseteq R^{(kl)} \\
(4) \quad & R^{(k)} \cdot F^{(l)}(R) \subseteq F^{(l-k+1)}(R) \text{ se } k \leq l \\
(5) \quad & F^{(l)}(R) \cdot R^{(k)} \subseteq F^{(l-k+1)}(R) \text{ se } k \leq l \\
(6) \quad & R^{(k)} \circ F^{(l)}(R) \subseteq F^{(l-k)}(R) \text{ se } k \leq l \\
(7) \quad & F^{(k)}(R/F^{(l)}(R)) = F^{(k+l)}(R)/F^{(l)}(R)
\end{align*}
\]

**Proof.** For (1), (2) see [4], Theorem 3.3 e Theorem 3.4. We prove our assertions by induction. First, (3) is trivial for \(l = 1\). If \(l > 1\), then, by (2), we have

\[
(R^{(k)})^{(l-1)} \circ R^{(k)} \subseteq R^{k(l-1)} \circ R^{(k)} \subseteq R^{(k(l-1)+k)} = R^{(kl)}
\]

for all positive integer \(k\). Hence \((R^{(k)})^{(l)} \subseteq R^{(kl)}\).
(4): If \( k = 1 \), then, for all \( l \in \mathbb{N} \)
\[
R^{(k)} F^{(l)}(R) = R^{(1)} F^{(l)}(R) \subseteq F^{(l)}(R) \subseteq F^{(l-k+1)}(R)
\]
Now let \( k > 1 \). For all \( a \in R^{(k-1)}, \ b \in R \) and \( c \in F^{(l)}(R) \), the inductive hypothesis implies that
\[
(a \circ b)c = ac \circ b - a(c \circ b) \in F^{(l-k+1)}(R),
\]
as desired.

(5): Analogously to (4).

(6): If \( k = 1 \), then, for all \( l \in \mathbb{N} \)
\[
R^{(k)} \circ F^{(l)}(R) = R \circ F^{(l)}(R) \subseteq F^{(l-1)}(R) = F^{(l-k)}(R)
\]
Now let \( k > 1 \). For all \( a \in R^{(k-1)}, \ b \in R, \ r \in R \) and \( c \in F^{(l)}(R) \), inductively we have
\[
(a \circ b) \circ c = b \circ (c \circ a) + a \circ (b \circ c) \in F^{(l-k)}(R)
\]
Hence, by (5), we have
\[
(a \circ b)r \circ c = (a \circ b) \circ rc + r \circ (c(a \circ b)) \in F^{(l-k)}(R)
\]

(7): If \( k = 1 \), then, for all \( l \in \mathbb{N} \)
\[
F^{(k)}(R/F^{(l)}(R)) = F(R/F^{(l)}(R)) = F^{(l+1)}(R)/F^{(l)}(R) = F^{(k+1)}(R)/F^{(l)}(R)
\]
Now let \( k > 1 \). For all \( l \in \mathbb{N} \) and for all \( y \in R \) we have
\[
y + F^{(l)}(R) \in F^{(k)}(R/F^{(l)}(R)) \iff
\]
\[
\iff \forall a, b \in R \quad (y(1 + a) \circ b) + F^{(l)}(R) \in F^{(k-1)}(R/F^{(l)}(R))
\]
\[
\iff \forall a, b \in R \quad (y(1 + a) \circ b) + F^{(l)}(R) \in F^{(k-1+l)}(R)/F^{(l)}(R)
\]
\[
\iff \forall a, b \in R \quad y(1 + a) \circ b \in F^{(k-1+l)}(R) \iff y \in F^{(k+1)}(R)
\]
which completes the proof. \( \qed \)

**Corollary 1.** If \( R \) is a ring and \( k \) is a positive integer, then
\[
\text{char } R/F^{(k)}(R) = \text{char } R^{(k+1)}.
\]
Proof. Let \( k \in \mathbb{N} \) and let \( m := \text{char} \ F^{(k)}(R) \neq 0 \). For all \( a \in R^{(k)}, \ r \in R \), we have

\[
m(a \circ r) = a \circ mr \in R^{(k)} \circ F^{(k)}(R) = 0,
\]
by Prop. 1 (6). Since \( R^{(k+1)} \) is the ideal of \( R \) generated by \( R^{(k)} \circ R \), it follows that \( \text{char} R^{(k+1)} \) divides \( m \).

Now let \( n := \text{char} R^{(k+1)} \neq 0 \). For each \( r, r_1, \ldots, r_k, s_1, \ldots, s_k \in R \) we have

\[
\cdots (((nr(1 + r_1) \circ s_1)(1 + r_2) \circ s_2) \cdots )(1 + r_k) \circ s_k =
\]

\[
= n((1 + r_1) \circ s_1)(1 + r_2) \circ s_2) \cdots )(1 + r_k) \circ s_k = 0
\]

By (1), it follows that \( nr \in F^{(k)}(R) \). Hence \( \text{char} R/F^{(k)}(R) \) divides \( n \).

It follows immediately that \( \text{char} R/F^{(k)}(R) = 0 \) if and only if \( \text{char} R^{(k+1)} = 0 \). \( \blacksquare \)

The following proposition gives a relation between the characteristic of the factors of the upper central series of a ring and that of its strong centre.

Proposition 2. If \( R \) is a ring such that \( \text{char} F(R) \neq 0 \), then the characteristic of \( F^{(k+1)}(R)/F^{(k)}(R) \) divides the characteristic of \( F(R) \), for each non-negative integer \( k \).

Proof. Let \( n := \text{char} F(R) \neq 0 \). We show by induction on \( k \) that \( nx \in F^{(k)}(R) \), for all \( x \in F^{(k+1)}(R) \) and \( k \in \mathbb{N}_0 \).

For \( k = 0 \), there is nothing to prove. Let \( k \geq 1 \) and assume that \( ny \in F^{(k-1)}(R) \) for each \( y \in F^{(k)}(R) \). Let \( x \in F^{(k-1)}(R) \). For all \( r, s \in R \) we have \( x(1+r)os \in F^{(k)}(R) \). Inductively, \( nx(1+r)os \in F^{(k-1)}(R) \). Hence \( (nx)(1+r)os \) belongs to \( F^{(k-1)}(R) \) and \( nx \in F^{(k)}(R) \), by (1). \( \blacksquare \)

3 Analogue of a theorem of P. Hall

In [4], Jennings proves that a ring is strongly Lie nilpotent if and only if it has a finite central series. Moreover, we have

Proposition 3. Let \( R \) be a ring. If \( c \in \mathbb{N} \) and \( 0 = I_0 \subset \ldots \subset I_c = R \) is a central series of \( R \), then

\[
R^{(c-k+1)} \subseteq I_k \subseteq F^{(k)}(R)
\]

for each \( k \in \{0, 1, \ldots, c\} \).

Proof. The first inclusion holds by [4] (Theorem 2.1). We prove, by induction on \( k \), that \( I_k \subseteq F^{(k)}(R) \). For \( k = 0 \), there is nothing to prove. Let \( k \geq 1 \)
and assume that $I_{k-1} \subseteq F^{(k-1)}(R)$. Let $z \in I_k$. Since $I_k/I_{k-1}$ is a central factor, we have inductively

$$z(1 + r) \circ s \in I_{k-1} \subseteq F^{(k-1)}(R)$$

for all $r, s \in R$. Hence $z \in F^{(k)}(R)$, by (1).

The proposition shows that the lower and upper central series of any strongly Lie nilpotent ring $R$ have the same length $c$. This length $c$ is called the strongly Lie nilpotent class of $R$.

The following result is analogous to one obtained for nilpotent rings (see [5], 1.2.6).

**Proposition 4.** If $R$ is a strongly Lie nilpotent ring, then $\text{char } R = 0$ if and only if $\text{char } F(R) = 0$.

**Proof.** If $\text{char } F(R) = 0$, then clearly $\text{char } R = 0$. Conversely, let $\text{char } R = 0$ and assume that $\text{char } F(R) = m \neq 0$. If $c$ is the strongly nilpotent class of $R$, then $R^{(c)} \subseteq F(R)$. Hence $\text{char } R^{(c)} \neq 0$. Let $i := \min\{j | j \in \mathbb{N}, \text{ char } R^{(j)} \neq 0\}$ and let $n := \text{char } R^{(i)}$. Then there is an element $x \in R^{(i-1)}$ such that $mnx \neq 0$.

For all $y, z \in R$, we have

$$nx(1 + y) \circ z = n(x(1 + y) \circ z) = n(x \circ z + xy \circ z) = 0$$

By (1), $nx \in F(R)$, therefore $mnx = 0$, a contradiction to the choice of $x$.

The results above are examples of a strong analogy between the theories of nilpotent groups and strongly Lie nilpotent rings.

In particular, we recall the well-known theorem of P. Hall for nilpotent groups: if $N$ is a normal subgroup of a group $G$ and $N, G/N'$ are nilpotent, then $G$ is nilpotent (see [6]). A version of this theorem for Lie algebras is contained, for example, in [1].

We give a version of the theorem of P. Hall for strongly Lie nilpotent rings.

**Lemma 1.** Let $R$ be a ring, $I$ an ideal of $R$ such that its strong centre $F(I)$ is an ideal of $R$ and $M$ the largest ideal of $R$ contained in $I \circ I$.

If there is a finite central series of $R$ between $F(I)$ and $I$, then there is a finite central series of $R$ between $0$ and $M$.

**Proof.** Let $t \in \mathbb{N}$ and

$$F(I) = I_0 \subseteq I_1 \subseteq \cdots \subseteq I_t = I$$

(2)

a finite central series of $R$ between $F(I)$ and $I$.

For each $i \in \mathbb{N}$, $i \leq 2t$, let $B_i$ the additive subgroup of $R$ generated by $\bigcup_{h+k=i} I_h \circ I_k$, and let $\overline{B}_i$ be the ideal $R$ generated by $B_i$. 
Evidently
\[ 0 = \mathcal{B}_1 \subseteq \mathcal{B}_2 \subseteq \cdots \subseteq \mathcal{B}_{2t} \]  
(3)

We show that (3) is a central series of \( R \).

It is sufficient to prove that, for all \( a \in I_h, \ b \in I_k \) such that \( h + k = i \) and for all \( r, s, v \in R \) we have
\[
\begin{align*}
    a \circ b \circ v & \in \mathcal{B}_{i-1} \\
    (a \circ b) \circ r \circ v & \in \mathcal{B}_{i-1} \\
    r(a \circ b) \circ v & \in \mathcal{B}_{i-1} \\
    r(a \circ b) \circ s \circ v & \in \mathcal{B}_{i-1}
\end{align*}
\]

Since (2) is a central series, by the Jacobi identity, we have
\[
(a \circ b) \circ v = a \circ v \circ b + v \circ b \circ a \in B_{i-1} \subseteq \mathcal{B}_{i-1}
\]

Moreover (cfr. [3], Lemma 2)
\[
(a \circ b)(r \circ v) = v(a \circ r) \circ b \\
- r(a \circ b \circ v) \\
- a \circ r \circ bv + a \circ br \circ v \\
- a \circ b \circ v \circ r + a \circ r \circ b \circ v \in \mathcal{B}_{i-1}.
\]

Hence
\[
(a \circ b)r \circ v = (a \circ b \circ v)r + (a \circ b)(r \circ v) \in \mathcal{B}_{i-1}.
\]

It follows that
\[
r(a \circ b) \circ v = -(a \circ b \circ r) \circ v + (a \circ b)r \circ v \in \mathcal{B}_{i-1}.
\]

Finally,
\[
s(a \circ b)r \circ v = s((a \circ b)r \circ v) + (s \circ v)(a \circ b)r \in \mathcal{B}_{i-1}.
\]

Hence for all \( i \in \mathbb{N}, \ 1 < i \leq 2t \) we have
\[
(B_i \cap M) \circ R \subseteq (B_i \circ R) \cap (M \circ R) \subseteq B_{i-1} \cap M.
\]

Therefore
\[
0 = B_1 \cap M \subseteq \cdots \subseteq B_{2t} \cap M = M
\]
is a finite central series of \( R \) between 0 and \( M \).

**Theorem 1.** Let \( R \) be a ring, \( I \) an ideal of \( R \) such that its strong centre \( F(I) \) is an ideal of \( R \), and let \( M \) be the largest ideal of \( R \) contained in \( I \circ I \).

If \( I \) and \( R/M \) are strongly Lie nilpotent rings, then \( R \) is strongly Lie nilpotent.
Proof. We proceed by induction on the strongly Lie nilpotent class \( c \) of \( I \). If \( c = 1 \), then \( I = F(I) \) and \( I \circ I = 0 \). It follows that \( M = 0 \). Hence \( R \) is strongly Lie nilpotent.

If \( c = 2 \), then \( I \circ I \subseteq F(I) \). Hence \( M \subseteq F(I) \). As \( R/M \) is strongly nilpotent, it follows that \( R/F(I) \) is strongly Lie nilpotent. Now, \( I/F(I) \) is an ideal of \( R/F(I) \), and therefore there is a finite central series of \( R \) between \( F(I) \) and \( I \).

By \( 1 \), there is a finite central series of \( R \) between \( 0 \) and \( M \). It follows that \( R \) is strongly Lie nilpotent.

If \( c > 3 \) and \( \overline{M} \) is the largest ideal of \( R/F(I) \) contained in \( I/F(I) \circ I/F(I) \), then \( F(I) \subseteq M \) and \( \overline{M} = M/F(I) \). Since \( (R/F(I))/\overline{M} \cong R/M \), we have that \( (R/F(I))/\overline{M} \) is strongly Lie nilpotent. Now \( I/F(I) \) is strongly Lie nilpotent of class \( c - 1 \) and, inductively \( R/F(I) \) is strongly Lie nilpotent.

Proceeding as in the case of \( c = 2 \), we complete our proof.

References


