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# Groups with dense nearly normal subgroups

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**Abstract.** This paper studies groups in which the set of nearly normal subgroups is dense in the lattice of all subgroups. In partuclar it is provided a characterization for locally finite groups and non periodic groups.

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## 1 Introduction

Let $(\mathcal{L}, \leq)$ be a partially ordered set. A subset $I$ of $\mathcal{L}$ is said to be dense in $\mathcal{L}$ if every open non empty interval of $\mathcal{L}$ contains an element of $I$. If $\chi$ is a property pertaining to subgroups, a group $G$ is said to have *dense* $\chi$*-subgroups* if the set of $\chi$-subgroups of $G$ is dense in the lattice $\mathcal{L}(G)$ of all subgroups of $G$. Groups with dense $\chi$-subgroups have been studied for many properties $\chi$: being a normal subgroup [4], a pronormal subgroup [7], a normal-by-finite subgroup [2]. A subgroup $H$ of a group $G$ is said to be *nearly normal* in $G$ if $H$ has finite index in its normal closure $H^G$. Groups in which each subgroup has finite index in its normal closure have been characterized by B.H. Neumann [5]; he proved that each subgroup of the group $G$ is nearly normal if and only if the derived subgroup $G'$ is finite. In [1] Franciosi and de Giovanni have investigated groups in which every infinite subgroup has finite index in its normal closure and groups satisfying the minimal condition on subgroups having infinite index in their normal closure. The aim of this paper is to study groups for which the set of nearly normal subgroups is dense in the lattice of all subgroups. In particular it will be proved that locally finite groups with dense nearly normal subgroups have finite derived subgroup, and non periodic groups with dense nearly normal subgroups will be characterized. Most of our notation is standard and can be found in [6].

Recall that an element $x$ of a group $G$ is said *$FC$-central* if it has finitely many conjugates in $G$. The *$FC$-centre* of a group $G$ is the subgroup of all $FC$-central elements of $G$, and $G$ is said to be an *$FC$-group* if it coincides with its $FC$-centre.

## 2 The periodic case

**Lemma 1.** *Let $G$ be a group with dense nearly normal subgroups and let $x$ be an element of infinite order of $G$. Then the index $|\langle x\rangle^G : \langle x\rangle|$ is finite and in particular $x$ belongs to the $FC$-centre of $G$.*

Proof. For each prime $p$ there exists a nearly normal subgroup $H_p$ of $G$ such that $\langle x^{p^2}\rangle < H_p < \langle x\rangle$. If $p$ and $q$ are two different primes, then $\langle x\rangle = \langle H_p, H_q\rangle$, so that $\langle x\rangle$ has finite index in $\langle x\rangle^G$. QED

**Lemma 2.** *Let $G$ be a locally (soluble-by-finite) group with dense nearly normal subgroups. If $G$ contains an infinite periodic subgroup then $G$ is an $FC$-group.*

Proof. Let $M$ be a maximal infinite periodic subgroup of $G$. Then $M$ is locally finite. Let $p$ be a prime and let $x$ be an element of $M$ of period a power of $p$. Assume by contradiction that $x$ has infinitely many conjugates in $G$. Then the index $|\langle x\rangle^G : \langle x\rangle|$ is infinite and so, if $F$ is a finite subgroup of $G$ containing $\langle x\rangle$, $\langle x\rangle$ is maximal in $F$. By Lemma 2 of [3] it follows that either $F$ is a $p$-group and $|F : \langle x\rangle| \leq p$ or $\pi(F) = \{p, q\}$ and $|F| \leq q^{|\langle x\rangle|}|\langle x\rangle|$. Let $y_1, y_2, \ldots, y_n$ be $n$ elements of $M$ such that $y_1 \notin \langle x\rangle$ and for each $i > 1$ $y_i \notin \langle x, y_1, \ldots, y_{i-1}\rangle$. Then there exists a positive integer $m$ such that $M = \langle x, y_1, \ldots, y_m\rangle$ and $M$ is finite. This contradiction shows that for every $p \in \pi(M)$, the $p$-elements of $M$ belong to the $FC$-centre $FC(G)$ of $G$ and so $M \leq FC(G)$. Therefore $M$ is the torsion subgroup of $FC(G)$ and in particular $M$ is normal in $G$. It follows that every periodic element of $G$ belongs to $FC(G)$ and then by Lemma 2.1 $G$ is an $FC$-group. QED

**Lemma 3.** *Let $G$ be an $FC$ group with dense nearly normal subgroups. Then the commutator subgroup $G'$ of $G$ is finite.*

Proof. Let $H = \langle x_1, \ldots, x_n\rangle$ be a finitely generated subgroup of $G$. By Dietzman's lemma and Lemma 2.1 it follows that for each $i \in \{1, \ldots, n\}$ the index $|\langle x_i\rangle^G : \langle x_i\rangle|$ is finite. Then $H$ is nearly normal in $G$. Let $L$ be a subgroup of $G$ and let $T$ be its finite residual. If $T$ has finite index in $L$, $T$ has no proper subgroups of finite index and therefore $T$ is contained in the centre of $G$. In particular $T$ is normal in $G$ and $L = TS$, where $S$ is a finitely generated subgroup of $G$. Thus $L$ has finite index in its normal closure. On the other hand if the index $|L : T|$ is infinite, $L$ contains a subgroup of finite index which is not maximal. By hypothesis there exists a nearly normal subgroup $M$ of $G$ such that $M < L$ and $|L : M|$ is finite. As before it follows that $L$ is nearly normal in $G$. Since each subgroup of $G$ has finite index in its normal closure, the derived subgroup $G'$ of $G$ is finite. QED

The above lemmas have the following immediate consequence.

**Corollary 1.** *Let $G$ be a locally finite group with dense nearly normal subgroups. Then the commutator subgroup $G'$ of $G$ is finite.*

## 3 The non periodic case

We shall consider now non periodic groups with dense nearly normal subgroups. The infinite dihedral group shows that such groups could have infinite derived subgroup. The next results prove that the infinite dihedral group is involved in the description of non periodic groups with the prescribed property.

**Lemma 4.** *Let $G$ be a non periodic group with dense nearly normal subgroups. Then $G$ is finite-by-abelian-by-finite.*

Proof. Let $a$ be an element of infinite order of $G$ and let $x$ be an element of $C_G(a)$. Then the abelian subgroup $\langle a, x\rangle$ is generated by elements of infinite order, and hence by Lemma 2.1 is contained in the $FC$-centre of $G$. In particular $C_G(a)$ is an $FC$-group with dense nearly normal subgroups and then $(C_G(a))'$ is finite by Lemma 2.3. Since the index $|G : C_G(a)|$ is finite, it follows that $G$ is finite-by-abelian-by-finite. QED

**Theorem 1.** *Let $G$ be a non periodic group. Then $G$ has dense nearly normal subgroups if and only if $G$ satisfies one of the following conditions:*

(a) *The commutator subgroup $G'$ of $G$ is finite.*

(b) *$G$ contains a finite normal subgroup $E$ such that $G/E$ is infinite dihedral and for every open non empty interval $]H, K[$ of subgroups of $G$, with $K$ finite, $H$ is contained in $E$.*

Proof. Suppose that $G$ has dense nearly normal subgroups and that $G'$ is infinite. By Lemma 3.1, $G$ is finite-by-abelian-by-finite and so, in order to prove that $G$ is finite-by-infinite dihedral, it can be assumed that $G$ is abelian-by-finite. Let $T$ be the largest periodic normal subgroup of $G$. If $T$ is infinite $G$ is an $FC$-group by Lemma 2.2 and $G'$ is finite by Lemma 2.3. Therefore $T$ is finite and replacing $G$ by $G/T$, it can be assumed that $G$ has no non trivial periodic normal subgroups. Then the Baer radical $B$ of $G$ is torsion-free and so is contained in the $FC$-centre of $G$ by Lemma 2.1. In particular $B$ is a torsion-free $FC$-group and hence is abelian. By Lemma 2.3 $G$ is not an $FC$-group and then there exists a periodic element $g$ of $G$ with infinitely many conjugates. Clearly $\langle g\rangle \cap B = 1$. If $B$ contains two non trivial elements $b_1$ and $b_2$ such that $\langle b_1\rangle \cap \langle b_2\rangle = 1$, then $\langle b_1\rangle^G \cap \langle b_2\rangle^G$ is finite and therefore $\langle b_1\rangle^G \cap \langle b_2\rangle^G = 1$. Put $A_1 = \langle b_1\rangle^G$ and $A_2 = \langle b_2\rangle^G$ and let $p, q$ two different primes. Since $\langle g\rangle \ltimes {A_1}^{p^2}$ and $\langle g\rangle \ltimes {A_1}^{q^2}$ are not maximal in $\langle g\rangle \ltimes A_1$, there exist two nearly normal subgroups $L$ and $S$ of

$G$ such that $\langle g\rangle \ltimes {A_1}^{p^2} < L < \langle g\rangle \ltimes A_1$ and $\langle g\rangle \ltimes {A_1}^{q^2} < S < \langle g\rangle \ltimes A_1$. Since $\langle g\rangle \ltimes A_1 = \langle L, S\rangle$, then $\langle g\rangle \ltimes A_1$ has finite index in its normal closure. Similarly $\langle g\rangle \ltimes A_2$ is nearly normal in $G$. It follows that $\langle g\rangle = (\langle g\rangle \ltimes A_1) \cap (\langle g\rangle \ltimes A_2)$ has finite index in $\langle g\rangle^G$, a contradiction. Therefore $B$ is locally cyclic. Moreover $C_G(B)$ is central-by-finite, and hence abelian, since $G$ has no non-trivial periodic normal subgroups. Thus $C_G(B) \leq B$ and $G/B$ is abelian. Let $b$ be a non trivial element of $B$. By Lemma 2.2 and Lemma 2.3 the derived subgroup $G'\langle b\rangle^G/\langle b\rangle^G$ of $G\langle b\rangle^G/\langle b\rangle^G$ is finite and $G'$ is cyclic-by-finite. Since $G' \leq B$, it follows that $G'$ is cyclic. The centralizer $C_G(G')$ is a nilpotent normal subgroup of $G$, so that $C_G(G') \leq B$ and $G/B$ has order at most 2. Since $G$ is not abelian, $G$ contains an element $g$ acting on $B$ as the inversion. Since $g$ acts trivially on $B/G'$, it follows that $B/G'$ is an elementary abelian 2-group of rank 1, and thus has order at most 2 and $G$ is a finite extension of an infinite cyclic group. Since $G'$ is infinite, $G$ contains a finite normal subgroup $E$ such that $G/E$ is infinite dihedral. Now let $H$ and $K$ be finite subgroups of $G$ such that the interval $]H,K[$ is not empty. If $HE/E \neq 1$, then $2 = |HE/E| = |KE/E| = |LE/E|$, for each subgroup $L$ in the interval $]H,K[$. It follows that, for every $L$ in $]H,K[$, the normal closure $(LE/E)^{G/E} = L^G E/E$ is infinite, so that $L^G$ is infinite. This contradiction shows that $G$ satisfies $(b)$. Conversely assume that $G$ satisfies $(b)$ and let $H$, $K$ be subgroups of $G$ such that $]H,K[$ is not empty. Suppose that $K$ is finite and let $L$ be an element of $]H,K[$. By hypothesis $HE/E$ is trivial. If $LE/E = 1$, then $L$ is nearly normal in $G$. If $LE/E \neq 1$ then $LE = KE$ and $K = KE \cap K = LE \cap K = L(E \cap K)$. Therefore $H < K \cap E < K$ and $K \cap E$ has finite index in its normal closure. Assume now that $K$ is infinite. Then there exists an infinite subgroup $L$ in $]H,K[$. Since $L$ has finite index in $G$, $L$ is nearly normal in $G$. It follows that $G$ has dense nearly normal subgroups. QED

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