(ω)topological connectedness and hyperconnectedness

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Abstract. In this paper, the notions of connectedness and hyperconnectedness in (ω)topological spaces are introduced and studied.

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1 Introduction

A set X equipped with an increasing sequence \(\{J_n\}\) of topologies is called an (ω)topological space [2]. It is denoted by \((X,\{J_n\})\) or, simply, by X if there is no scope for confusion. Separation axioms, compactness and paracompactness of (ω)topological spaces were studied in [2]. In [3], we proved Michael’s theorem (Theorem 1, [10]) and Stone’s theorem [13] on paracompactness in (ω)topological spaces. In [14], Thomas proved some results on maximal connected spaces. Mathew [9] studied hyperconnected topological spaces (Steen and Seebach [12]).

In this paper, we introduce (ω)connected and (ω)hyperconnected spaces. We also introduce (ω)semiopen sets as an analogue of semiopen sets (Levine [7]). Along with other results, we prove (i) a set lying between an (ω)connected set and its (ω)closure is (ω)connected, (ii) if an (ω)topological space X is maximal (ω)hyperconnected, then the class of all nonempty (ω)open sets is an ultrafilter.

The aim of studying (ω)topological spaces is to develop a framework for studying an increasing (evolving) sequence of topologies on a set. In literature,
we get occurrence of infinite sequence of evolving topologies and topological spaces, a few examples are given below.

In (Datta and Roy Choudhuri [4]) and (Raut and Datta [11]) authors studied a non-archimedean extension of the real number system R involving nontrivial infinitesimally small elements which are modelled as hierarchically structured local p-adic fields. This approach aims at offering a natural framework for dealing with an infinite sequence of topologically distinct spaces in a complex evolutionary process.

In digital topology and evolving infinite networks (Fan, Chen and Ko [5]) change of topologies with dynamical consequences are considered. Such topological notions have applications in computer science, infinite graphs and related areas. In dynamical system theory, emergence of chaos in a deterministic system relates to an interplay of finite or infinite number of different topologies in the underlying set.

Another motivation of ($\omega$)topology is that if $\{J_n\}$ be an increasing sequence of topological spaces on $X$ and $J = \bigcup_n J_n$ then $(X, J)$ is not a topological space and even it is not an Alexandroff space [1] which is a generalization of a topological space requiring only countable union of open sets to be open. In fact, an arbitrary (or countable) union of sets $\in J$ may not belong to $J$. But taking advantage of the topologies $J_n$ we can, however, get many properties of $(X, \{J_n\})([2],[3])$, close to that of a topological space which is not necessarily possessed by an Alexandroff space.

2 Preliminaries

For an ($\omega$)topological space $(X, \{J_n\})$, a set $G \in \bigcup_n J_n$ is called an ($\omega$)open set. A set $F$ is ($\omega$)closed if its complement $F^c = X - F$ is ($\omega$)open. The union and intersection of a finite number of ($\omega$)open sets is ($\omega$)open. However, the countable union of ($\omega$)open sets may not be ($\omega$)open. These sets are called ($\sigma\omega$)open sets. Since the arbitrary union of ($J_n$)open sets is ($J_n$)open, the union of an arbitrary number of ($\omega$)open sets is also ($\sigma\omega$)open. Similarly, ($\delta\omega$)closed sets are defined as the intersection of a countable number of ($\omega$)closed sets. The intersection of all ($\omega$)closed sets containing a set $A$ is called the ($\omega$)closure of $A$ and is denoted by ($\omega$)cl$A$. Obviously, it is a ($\delta\omega$)closed set.

It is clear that the class $\mathcal{T}$ of all ($\sigma\omega$)open sets in $X$ forms a topology.

3 ($\omega$)connectedness

Definition 1. An ($\omega$)topological space $(X, \{J_n\})$ is said to be ($\omega$)connected, if $X$ cannot be expressed as the union of two disjoint nonempty ($\omega$)open sets.
Obviously, when $J_n = J$ for all $n$, the space $(X, \{J_n\})$ is an $(\omega)$connected space iff the topological space $(X, J)$ is connected. Further, an $(\omega)$topological space $(X, \{J_n\})$ is $(\omega)$connected iff $(X, J_n)$ is connected for all $n$.

**Definition 2.** A subset $Y$ of an $(\omega)$topological space $(X, \{J_n\})$ is said to be $(\omega)$connected, if the $(\omega)$topological space $(Y, \{J_n\}|Y)$ is $(\omega)$connected.

**Theorem 1.** If the space $(X, T)$ is connected, then the $(\omega)$topological space $(X, \{T_n\})$ is $(\omega)$connected.

**Proof.** Since every $(\omega)$open set is $(\sigma\omega)$open, the result follows. \[QED\]

We now give an example to show that the converse of the theorem is not true.

**Example 1.** Let $P\{1, 2, 3, \ldots, n\}$ denote the power set of the set $\{1, 2, 3, \ldots, n\}$. We define an increasing sequence $\{T_n\}$ of topologies on $N$ as follows:

$$T_n = \{N\} \cup P\{1, 2, 3, \ldots, n\}.$$ Then the $(\omega)$topological space $(N, \{T_n\})$ is $(\omega)$connected. However, the topology of all $(\sigma\omega)$open sets of the above $(\omega)$topology is not connected. Since, the set of all even positive integers and odd positive integers are two disjoint $(\sigma\omega)$open sets whose union is $N$.

If $X$ is not $(\omega)$connected, then there exist two disjoint nonempty $(\omega)$open sets $A$ and $B$ such that $X = A \cup B$. In this case, $X$ is said to be $(\omega)$disconnected and we write $X = A|B$. We call it an $(\omega)$separation of $X$. Since the two $(\omega)$open sets $A$ and $B$ belong to some $J_n$, it is clear that if $X$ is not $(\omega)$connected then for some $n$, the topological space $(X, J_n)$ is not connected. As a consequence we get the following theorem.

**Theorem 2.** If $C$ is an $(\omega)$connected subset of an $(\omega)$topological space $X$ which has the $(\omega)$separation $X = A|B$, then either $C \subset A$ or $C \subset B$.

**Corollary 1.** If any two points of $Y \subset X$ are contained in some $(\omega)$connected subset of $Y$, then $Y$ is $(\omega)$connected.

**Corollary 2.** The union of a family of $(\omega)$connected sets having nonempty intersection is $(\omega)$connected.

**Corollary 3.** If $C$ is an $(\omega)$connected set in $X$ and $C \subset E \subset (\omega)clC$, then $E$ is $(\omega)$connected.

**Proof.** If $E$ is not $(\omega)$connected, then it has an $(\omega)$separation $E = A|B$. By Theorem 2, $C \subset A$ or $C \subset B$. Let us assume $C \subset A$. Suppose $A, B \in J_n|E$. \[QED\]
This is a contradiction and so $E$ is $(\omega)$connected.

**Definition 3.** $X$ is said to be an $(\omega)T_0$-space if for every pair of distinct points $x$ and $y$ of $X$, there exists an $(\omega)$open set $G$ such that $x \in G$ and $y \notin G$.

**Definition 4.** $X$ is said to be an $(\omega)T_1$-space if for every pair of distinct points $x$ and $y$ of $X$, there exists a $n$ such that for some $U, V \in J_n$, we have $x \in U$, $y \in V$, $y \notin U$ and $x \notin V$.

**Definition 5.** An $(\omega)$topology $\{J'_n\}$ on $X$ is said to be stronger (resp. weaker) than an $(\omega)$topology $\{J_n\}$ on $X$ if $\bigcup_n J'_n \subset \bigcup_n J_n$ (resp. $\bigcup_n J'_n \subset \bigcup_n J_n$).

**Definition 6.** An $(\omega)$topological space $(X, \{J_n\})$ with property $P$ is said to be maximal (resp. minimal) if for any other $(\omega)$topology $\{J'_n\}$ strictly stronger (resp. strictly weaker) than $\{J_n\}$, the space $(X, \{J'_n\})$ cannot have this property.

**Theorem 3.** If $X$ is maximal $(\omega)$connected, then $X$ is $(\omega)T_0$.

**Proof.** Suppose, if possible $X$ is not $(\omega)T_0$. Then there exist $x, y \in X$, $x \neq y$ such that $x \in (\omega)cl\{y\}$ and $y \in (\omega)cl\{x\}$. Let $J'_n$ be the topology generated by $J_n \cup \{y\}$. Then the $(\omega)$topological space $(X, \{J'_n\})$ is not $(\omega)connected$. Let $X = A|B$ be an $(\omega)$separation of $(X, \{J'_n\})$. Then either $\{x, y\} \subset A$ or $\{x, y\} \subset B$. Suppose $\{x, y\} \subset A$. Then there exists a set $U \in J_n$ with $x \in U$. But $U$ also contains $y$. Since for any point $z \in A$ with $z \neq y$, there is a $(J_n)$open neighborhood $G \subset A$ of $z$, it follows that $A \in J_n$. Clearly $B \in J_n$. Thus $A|B$ is an $(\omega)$separation of $(X, \{J_n\})$ which is a contradiction.

Below we provide examples to show that a maximal $(\omega)$connected space may or may not be $(\omega)T_1$.

**Example 2.** Let us consider the $(\omega)$topological space defined in Example 1. Clearly $(N, \{J_n\})$ is $(\omega)T_1$. Also this space is $(\omega)$connected and hence, can be extended to a maximal $(\omega)$connected space.

**Example 3.** Let us define an $(\omega)$topological space $(N, \{J_n\})$ as follows:

$$J_n = \{\phi\} \cup \{E \subset N \mid 1 \in E\} \text{ for all } n.$$ 

Then clearly $(N, \{J_n\})$ is a maximal $(\omega)$connected space. However, $(N, \{J_n\})$ is not $(\omega)T_1$. 

$$B = B \cap (\omega)clC \subset B \cap (\omega|E)clA = \phi \text{ (since } A \cap B = \phi).$$
Theorem 4. Let \((X, \{\mathcal{J}_n\})\) be maximal \((\omega)\)connected and \(G\) be an \((\omega)\)open \((\omega)\)connected subset of \(X\). Then \((G, \{\mathcal{J}_n|G\})\) is maximal \((\omega)\)connected.

Proof. If possible, suppose \((G, \{\mathcal{J}_n|G\})\) is not maximal \((\omega)\)connected. Let \(\mathcal{P}_n\) be an \((\omega)\)topology on \(G\) strictly stronger than \(\{\mathcal{J}_n|G\}\) and \((G, \{\mathcal{P}_n\})\) is \((\omega)\)connected. Let \(H \subset G\) be such that \(H \in \mathcal{P}_{n_0} - \mathcal{J}_{n_0}|G\) for some \(n_0\). If \(\mathcal{Q}_n\) is a topology on \(G\) generated by \(\mathcal{J}_n|G\cup \{H\}\), then \(\mathcal{Q}_n\) is an \((\omega)\)connected \((\omega)\)topology on \(G\). Also if \(\mathcal{S}_n\) is the topology on \(X\) generated by \(\mathcal{J}_n \cup \{H\}\), then the \((\omega)\)topology \(\{\mathcal{S}_n\}\) on \(X\) is strictly stronger than \(\{\mathcal{J}_n\}\) and so \((X, \{\mathcal{S}_n\})\) is not \((\omega)\)connected. Let \(X = A\bigcup B\) be an \((\omega)\)separation of \((X, \{\mathcal{S}_n\})\). Then either \(G \subset A\) or \(G \subset B\), since, otherwise, \((G \cap A) \cup (G \cap B)\) is an \((\omega)\)separation of \((G, \{\mathcal{Q}_n\})\). Suppose \(G \subset A\). Since \(G \in \bigcup_n \mathcal{J}_n\), it follows that \(A \in \bigcup_n \mathcal{J}_n\). But obviously \(B \in \bigcup_n \mathcal{J}_n\). Therefore \(X = A\bigcup B\) is an \((\omega)\)separation of \((X, \{\mathcal{J}_n\})\) which is a contradiction. QED

Definition 7. Let \(x \in X\). The component \(C(x)\) of \(x\) in \(X\) is the union of all \((\omega)\)connected subsets of \(X\) containing \(x\).

From Corollary 2, it follows that \(C(x)\) is \((\omega)\)connected.

Theorem 5. In an \((\omega)\)topological space \((X, \{\mathcal{J}_n\})\), (i) each component \(C(x)\) is a maximal \((\omega)\)connected set in \(X\), (ii) the set of all distinct components in \(X\) forms a partition of \(X\), (iii) each \(C(x)\) is \((\delta \omega)\)closed in \(X\).

Proof. Straightforward. QED

4 \((\omega)\)hyperconnectedness

Definition 8. \(X\) is said to be \((\omega)\)hyperconnected if for any two nonempty \((\omega)\)open sets \(U\) and \(V\), \(U \cap V \neq \phi\).

Therefore for any nonempty \((\omega)\)open set \(U\), \((\omega)\)cl\(U\) = \(X\), since otherwise \(V_1 = X - (\omega)\)cl\(U\) is a nonempty \((\sigma \omega)\)open set and \(U \cap V_1 = \phi\) which implies that for any nonempty \((\omega)\)open set \(V \subset V_1\), we have \(U \cap V = \phi\).

Theorem 6. \((X, \{\mathcal{J}_n\})\) is \((\omega)\)hyperconnected iff the topological space \((X, \mathcal{T})\) is hyperconnected.

Proof. Suppose \((X, \{\mathcal{J}_n\})\) is \((\omega)\)hyperconnected. Let \(A\) and \(B\) be two nonempty \((\mathcal{T})\)open sets. Then \(A = \bigcup_{i=1}^\infty A_i\), \(B = \bigcup_{j=1}^\infty B_j\) where \(A_i\) and \(B_j\) are nonempty \((\omega)\)open sets. Now \(A \cap B \supset A_i \cap B_j \neq \phi\). Thus \((X, \mathcal{T})\) is hyperconnected.

Conversely, since each \((\omega)\)open set is \((\sigma \omega)\)open set. The hyperconnectedness of the space \((X, \mathcal{T})\), implies that the \((\omega)\)topological space \((X, \{\mathcal{J}_n\})\) is \((\omega)\)hyperconnected. QED
**Definition 9.** A set $A \subset X$ is said to be $(\omega)$semiopen if there exists an $n$ such that for some $U \in \mathcal{J}_n$, we have

$$U \subset A \subset (\omega)clU.$$ 

Let $SO_\omega(X, \{\mathcal{J}_n\})$ or, simply, $SO_\omega(X)$ denote the set of all $(\omega)$semiopen sets. If some set $A$ satisfies the above relation for some set $U \in \mathcal{J}_n$, we say that $A$ is $(\mathcal{J}_n - \omega)$semiopen. The set of all $(\mathcal{J}_n - \omega)$semiopen sets is denoted by $(\mathcal{J}_n)SO_\omega(X)$. Thus

$$SO_\omega(X) = \bigcup_n(\mathcal{J}_n)SO_\omega(X).$$

**Theorem 7.** $X$ is $(\omega)$hyperconnected iff $SO_\omega(X) - \{\phi\}$ is a filter.

**Proof.** Suppose $X$ is $(\omega)$hyperconnected. Let $A, B \in SO_\omega(X) - \{\phi\}$. Then there exists a $k \in N$ such that for some $U$ and $V$ with $U, V \in \mathcal{J}_n$, we have

$$U \subset A \subset (\omega)clU,$$

$$V \subset B \subset (\omega)clV.$$ 

Since $X$ is $(\omega)$hyperconnected, $U \cap V \neq \phi$ and $(\omega)cl(U \cap V) = X$. Therefore it follows that $A \cap B \neq \phi$ and

$$U \cap V \subset A \cap B \subset (\omega)cl(U \cap V).$$ 

Thus $A \cap B \in SO_\omega(X) - \{\phi\}$. Again if $B \supset A \in SO_\omega(X) - \{\phi\}$, there exists, for some $k$, a $U \in \mathcal{J}_k$ such that

$$U \subset A \subset (\omega)clU \text{ and so } U \subset B \subset (\omega)clU \text{ (since } (\omega)clU = X).$$ 

Hence $B \in SO_\omega(X) - \{\phi\}$. Therefore $SO_\omega(X) - \{\phi\}$ is a filter.

Since every $(\omega)$open set is $(\omega)$semiopen, the converse follows.

It is easy to see that the union of an arbitrary number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Also if $X$ is $(\omega)$hyperconnected, then the intersection of a finite number of $(\mathcal{J}_n - \omega)$semiopen sets is $(\mathcal{J}_n - \omega)$semiopen. Thus if $X$ is $(\omega)$hyperconnected, then the class $(\mathcal{J}_n)SO_\omega(X) = S_n$ forms a topology on $X$ and $S_n \subset S_{n+1}$. Hence $\{S_n\}$ is an $(\omega)$topology on $X$.

From Theorem 7, we get the following result.

**Theorem 8.** If $(X, \{\mathcal{J}_n\})$ is $(\omega)$hyperconnected, then so is $(X, \{S_n\})$.

**Corollary 4.** If $(X, \{\mathcal{J}_n\})$ is maximal $(\omega)$hyperconnected, then $\cup_n \mathcal{J}_n = \cup_n S_n$.

For any set $A \not\in \cup_n \mathcal{J}_n$, let $\mathcal{J}_n(A)$ denote the simple extension (Levine [8]) of $\mathcal{J}_n$. Then $(X, \{\mathcal{J}_n(A)\})$ forms an $(\omega)$topology on $X$ and $\mathcal{J}_n \subset \mathcal{J}_n(A)$ for all $n$. We call $\{\mathcal{J}_n(A)\}$, a simple extension of $\{\mathcal{J}_n\}$. 

Theorem 9. If $(X, \{J_n\})$ is maximal $(\omega)$hyperconnected, then $SO_\omega(X) - \{\phi\}$ is an ultrafilter.

Proof. Suppose $(X, \{J_n\})$ is maximal $(\omega)$hyperconnected. For $E \subset X$, suppose $E \notin SO_\omega(X, \{J_n\}) - \{\phi\}$. Then $E \notin \bigcup_n J_n$. Let us consider the simple extension $\{J_n(E)\}$ of $\{J_n\}$. Since $(X, \{J_n\})$ is maximal $(\omega)$hyperconnected, $(X, \{J_n(E)\})$ is not $(\omega)$hyperconnected. Therefore for some $n$, there exist two nonempty sets $G, H \in J_n(E)$ such that $G \cap H = \phi$. Let $G = G_1 \cup (G_2 \cap E)$ and $H = H_1 \cup (H_2 \cap E)$ where $G_1, G_2, H_1, H_2 \in J_n$. Then $G_1 \cap H_1 = \phi$. Since $(X, \{J_n\})$ is $(\omega)$hyperconnected, either $G_1 = \phi$ or $H_1 = \phi$. Suppose $G_1 = \phi$. If $H_1 = \phi$, then $G_2 \neq \phi$ and $H_2 \neq \phi$, since $G \neq \phi$ and $H \neq \phi$. Thus by $(\omega)$hyperconnectivity of $(X, \{J_n\})$, $G_2 \cap H_2 \neq \phi$. Again since $G \cap H = \phi$, we have $G_2 \cap H_2 \cap E = \phi$. Hence $G_2 \cap H_2 \subset E^c$, and therefore by Theorem 7, $E^c \in SO_\omega(X, \{J_n\}) - \{\phi\}$. Now consider the case $H_1 \neq \phi$. Since $G \neq \phi$, we have $G_2 \neq \phi$. Therefore $G_2 \cap H_1 \neq \phi$. From the relation $G \cap H = \phi$, it follows that $(G_2 \cap E) \cap H_1 = \phi$. Hence $G_2 \cap H_1 \subset E^c$, and so $E^c \in SO_\omega(X, \{J_n\}) - \{\phi\}$. Thus $SO_\omega(X) - \{\phi\}$ is an ultrafilter.

Using Corollary 4, we get the following result.

**Corollary 5.** If $(X, \{J_n\})$ is maximal $(\omega)$hyperconnected, then the class of all nonempty $(\omega)$open sets is an ultrafilter.

**Definition 10.** $(X, \{J_n\})$ is said to be an $(\omega)$door space if for every subset $E$ of $X$, $E \in J_n$ or $E^c \in J_n$ for some $n$.

We now show that for an $(\omega)$door space $(X, \{J_n\})$, the topological spaces $(X, J_n)$ need not be door (Kelley [6]).

**Example 4.** Let us define an $(\omega)$topological space $(N, \{J_n\})$ as follows:

\[ J_n = \{\phi, N\} \cup \{E \subset \{1, 2, \ldots, n\} \mid 1 \in E\} \text{ for all } n < 10, \text{ and} \]
\[ J_n = \{\phi\} \cup \{E \subset N \mid 1 \in E\} \text{ for all } n \geq 10. \]

Then clearly $(N, \{J_n\})$ is an $(\omega)$door space. But for any $n < 9$, the topological space $(N, J_n)$ is not a door space.

**Example 5.** Taking $X = [0, 1)$, let us define an $(\omega)$topological space $(X, \{J_n\})$ as follows:

$J_n$ is the topology generated by the subbase $\{E \mid E \subset [0, 1 - \frac{1}{n+1}]\}$ and $0 \in E \} \cup \{\phi, \text{ all the sets } c \subset X \text{ containing } 0 \text{ and having 1 as a limit point}. \]

Then it is easy to see that $(X, \{J_n\})$ forms an $(\omega)$door space. However, $(X, J_n)$ is not a door space for any $n$.

**Definition 11.** A property $P$ of an $(\omega)$topological space $(X, \{J_n\})$ is said to be contractive(resp. expansive) if it is possessed by $(\omega)$topological spaces $(X, \{J'_n\})$ whenever it is possessed by $(X, \{J_n\})$, where the $(\omega)$topologies $\{J'_n\}$ are weaker(resp. stronger) than $\{J_n\}$.
It is clear that \((\omega)\)connectedness and \((\omega)\)hyperconnectedness are contractive properties while \((\omega)\)doorness is an expansive property.

**Theorem 10.** \((X, \{J_n\})\) is an \((\omega)\)hyperconnected \((\omega)\)door space iff \(F = (\cup_n J_n) - \{\phi\}\) is an ultrafilter.

**Proof.** Suppose \((X, \{J_n\})\) is an \((\omega)\)hyperconnected \((\omega)\)door space. Then for \(A, B \in F\), \(A \cap B \in F\). Now let \(B \supseteq A \in F\). If \(B = X\), then \(B \in F\). If \(B \neq X\), then \(B^c \notin F\), since otherwise \(A \cap B^c \neq \phi\). Therefore \(B \in F\). Hence \(F\) is a filter and so an ultrafilter.

The converse part is obvious. \(\Box\)

**Theorem 11.** If \((X, \{J_n\})\) is \((\omega)\)hyperconnected and \((\omega)\)door, then \((X, \{J_n\})\) is maximal \((\omega)\)hyperconnected and minimal \((\omega)\)door.

**Proof.** Let \(\{J'_n\}\) be an \((\omega)\)topology on \(X\) stronger than \(\{J_n\}\) such that \((X, \{J'_n\})\) is \((\omega)\)hyperconnected. If possible, suppose \(G\) is a nonempty set with \(G \in J'_m\) for some \(m\) and \(G \notin \cup_n J_n\). Since \((X, \{J_n\})\) is \((\omega)\)door, \(X - G \in J_l\) for some \(l\). Hence \(X - G \in \cup_n J_n\). This contradicts the fact that \((X, \{J'_n\})\) is \((\omega)\)hyperconnected. Thus \(G \in \cup_n J_n\). Therefore \(\cup_n J'_n = \cup_n J_n\).

Again let \((X, \{J'_n\})\) be an \((\omega)\)door space such that \(\cup_n J'_n \subset \cup_n J_n\). Suppose, if possible, \(G\) is a nonempty set with \(G \in \cup_n J_n\) and \(G \notin \cup_n J'_n\). But then \(X - G \in \cup_n J_n\). So \(X - G \in \cup_n J_n\) which contradicts the \((\omega)\)hyperconnectedness of \((X, \{J_n\})\). Therefore \(\cup_n J'_n = \cup_n J_n\). \(\Box\)

**Definition 12.** A set \(E \subset X\) is said to be \((\omega)\)dense if \((\omega)\)cl\(E = X\).

**Definition 13.** \(X\) is said to be submaximal if every \((\omega)\)dense subset of \(X\) is \((\omega)\)open.

**Theorem 12.** \((X, \{J_n\})\) is maximal \((\omega)\)hyperconnected iff it is submaximal and \((\omega)\)hyperconnected.

**Proof.** Suppose \((X, \{J_n\})\) is maximal \((\omega)\)hyperconnected. Let \(E \subset X\) be \((\omega)\)dense. By Corollary 5, \((\cup_n J_n) - \{\phi\}\) is an ultrafilter. Therefore \(E\) must be \((\omega)\)open. For, if \(E\) is not \((\omega)\)open, then \(E^c\) must be \((\omega)\)open, since \((\cup_n J_n) - \{\phi\}\) is an ultrafilter. Therefore \(E\) is \((\omega)\)closed and hence \((\omega)\)cl\(E = X\). Again since \(E\) is \((\omega)\)dense, \((\omega)\)cl\(E = X\). Therefore \(E = X\). Thus \(X\) is submaximal.

Conversely, suppose \((X, \{J_n\})\) is submaximal and \((\omega)\)hyperconnected. Let \((X, \{J'_n\})\) be \((\omega)\)hyperconnected with \(\cup_n J'_n \supset \cup_n J_n\). If \(G \in \cup_n J'_n\) be a nonempty set, then, since \((X, \{J'_n\})\) is \((\omega)\)hyperconnected, \((\omega)\)cl\(G\) (the \((\omega)\)closure of \(G\) in \((X, \{J'_n\})\)) coincides with \(X\). This implies that \((\omega)\)cl\(G\) (the \((\omega)\)closure of \(G\) in \((X, \{J_n\})) = X\) (since \((\omega)\)cl\(G\) (w.r.t. \((X, \{J_n\})\)) \(\supset \omega)\)cl\(G\) (w.r.t. \((X, \{J'_n\})\))). and so it follows that \(G\) is \((\omega)\)dense in \((X, \{J_n\})\). Hence \(G \in \cup_n J_n\). Thus \(\cup_n J'_n = \cup_n J_n\). \(\Box\)
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