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Isomorphisms between spaces of holomorphic mappings on Banach spaces

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Abstract. Let E and F be Banach spaces. Our objective in this work is to find conditions under which, whenever the topological dual spaces E' and F' are isomorphic, the spaces of holomorphic mappings of bounded type on E and F are isomorphic as well. We also examine the corresponding problem for the spaces of holomorphic mappings of a certain type, for instance nuclear bounded type, compact bounded type or weakly compact bounded type.

Keywords: Banach space , Holomorphic mappings , Isomorphisms

MSC 2000 classification: primary 46G20, secondary 46G25

Introduction

Let E and F be complex Banach spaces. Following an idea of Nicodemi [10], Galindo, García, Maestre and Mujica [6] defined a sequence of extension operators from the space of multinear mappings on E into the space of multilinear mappings on F. When $F = E''$ the Nicodemi extension operators coincide with the extension operators studied by Aron and Berner [1]. In a recent paper [7] the author used the Nicodemi sequences of extension operators to find conditions under which, whenever the duals E' and F' are isomorphic, the spaces of multilinear mappings (resp. homogeneous polynomials) on E and F are isomorphic as well. The case of multilinear mappings or homogeneous polynomials of finite, nuclear, compact or weakly compact type were studied as well. Problems of this type had been previously studied by several other authors. See for instance Díaz-Dineen [4], Cabello-Castillo-García [2], Lassalle-Zalduendo [8] and Carando-Lassalle [3].

In this work the author uses the same approach to find conditions under which, whenever E' and F' are isomorphic, the spaces of holomorphic mappings

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of bounded type on E and F are isomorphic as well. Similar results are obtained for the spaces of holomorphic mappings of nuclear bounded type, of compact bounded type, and of weakly compact bounded type.

Throughout the whole work, E, F and G denote complex Banach spaces. $\mathbb{N}$ denotes the set of all positive integers. $L(^mE;G)$ denotes the Banach space of all continuous m-linear mappings from E^m into G under its natural norm. We write $L(^mE;\mathbb{C}) = L(^mE)$, $L(^1E;G) = L(E;G)$ and $L(E) = E'$, the topological dual of E. Let $\mathcal{P}(^mE;G)$ denote the Banach space of all continuous $m-$ homogeneous polynomials from E into G under its natural norm. Let $\mathcal{P}_N(^mE;G)$ denotes the Banach space of all nuclear polynomials from E^m into G under the nuclear norm. Let $\mathcal{P}_K(^mE;G)$ (resp. $\mathcal{P}_{WK}(^mE;G)$) denote the space of all compact (resp. weakly compact) homogeneous polynomials from E into G.

Let $U \subset E$ be open and non-empty. A mapping $f : U \to G$ is called holomorphic if for each $a \in U$ there exists a power series $\sum_{m=0}^{\infty} P_m(x-a)$, with $P_m \in \mathcal{P}(^mE;G)$ for each $m \in \mathbb{N}$, which converges uniformly to $f(x)$ in a neighborhood of a. For each m, P_m, called the mth homogeneous polynomial in the Taylor series of f at a, is denoted by $P^m f(a)$. The space of all holomorphic mappings from U to G is denoted by $\mathcal{H}(U;G)$. For $\Theta = N, K$ or WK, we let

$$\mathcal{H}_\Theta(U;G) = \{f \in \mathcal{H}(U;G) : P^m f(a) \in \mathcal{P}_\Theta(^mE;G) \text{ for all } m \in \mathbb{N}, a \in U\},$$

$$\mathcal{H}_b(E;G) = \{f \in \mathcal{H}(E;G) : \text{ f is bounded on bounded sets}\},$$

and

$$\mathcal{H}_{\Theta b}(E;G) = \mathcal{H}_\Theta(E;G) \cap \mathcal{H}_b(E;G).$$

For background information on multilinear mappings, homogeneous polynomials and holomorphic mappings, we refer to the books [5] and [9].

1 Isomorphisms between spaces of holomorphic mappings of bounded type

In this work, we use the Nicodemi sequences defined in [6] to prove all the theorems.

We recall the definition of the Nicodemi sequences. The mapping

$$I_m : L(^{m+n}E;G) \to L(^mE; L(^nE;G))$$

defined by $I_m A(x)(y) = A(x,y)$ for all $A \in L(^{m+n}E;G), x \in E^m, y \in E^n$, is an isometric isomorphism. Likewise the mapping

$$A \in L(^mE; L(^nF;G)) \to A^t \in L(^nF; L(^mE;G))$$

defined by $A^t(y)(x) = A(x)(y)$ for all $A \in L(^mE; L(^nF; G)), x \in E^m, y \in F^n$, is an isometric isomorphism.

Definition 1. ([6]) Given a continuous linear operator $R_1 : L(E; G) \longrightarrow L(F; G)$, let $R_m : L(^mE; G) \longrightarrow L(^mF; G)$ be inductively defined by $R_{m+1}A = I_m^{-1}[R_m \circ (R_1 \circ I_m(A))^t]^t$ for all $A \in L(^{m+1}E; G)$ and $m \in \mathbb{N}$.

Definition 2. ([6]) Given $R_1 \in L(E'; F')$, let $\widetilde{R_1} \in L(L(E; G'), L(F; G'))$ be defined by $\widetilde{R_1}A(y)(z) = R_1(\delta_z \circ A)(y)$ for all $A \in L(E; G')$, $y \in F$ *and* $z \in G$, where $\delta_z : G' \to \mathbb{K}$ is defined by $\delta_z(z') = z'(z)$ for all $z' \in G'$. If (R_m) and $(\widetilde{R_m})$ are the corresponding Nicodemi sequences, then $\widetilde{R_m}A(y)(z) = R_m(\delta_z \circ A)(y)$ for all $A \in L(^mE; G')$, $y \in F^m$ *and* $z \in G$.

Theorem 1. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{H}_b(E)$ and $\mathcal{H}_b(F)$ are isomorphic.*

Proof. Since E' and F' are isomorphic, there exists an isomorphism $R_1 : E' \longrightarrow F'$. Let $R_m : L(^mE) \longrightarrow L(^mF)$ be the Nicodemi sequence beginning with R_1. Let $S_1 = R_1^{-1} : F' \longrightarrow E'$ be the inverse of R_1, and let $S_m : L(^mF) \longrightarrow L(^mE)$ be the Nicodemi sequence beginning with S_1. Let $\widehat{R}_m : \mathcal{P}(^mE) \longrightarrow \mathcal{P}(^mF)$ be defined by $\widehat{R}_m\widehat{A} = \widehat{R_mA}$ for all $A \in L^s(^mE)$ and let $\widehat{S}_m : \mathcal{P}(^mF) \longrightarrow \mathcal{P}(^mE)$ be defined by $\widehat{S}_m\widehat{B} = \widehat{S_mB}$ for all $B \in L^s(^mF)$. By [7, Corollary 18], we have that $\widehat{R}_m$ and $\widehat{S}_m$ are isomorphisms. Let $\widehat{R} : \mathcal{H}_b(E) \longrightarrow \mathcal{H}_b(F)$ be defined by $\widehat{R}f = \sum_{m=0}^{\infty} \widehat{R}_m(P^mf(0))$ for all $f \in \mathcal{H}_b(E)$ and let $\widehat{S} : \mathcal{H}_b(F) \longrightarrow \mathcal{H}_b(E)$ be defined by $\widehat{R}g = \sum_{m=0}^{\infty} \widehat{S}_m(P^mg(0))$ for all $g \in \mathcal{H}_b(F)$. We have that by [6], $\widehat{R}$ and $\widehat{S}$ are well defined and are linear and continuous. We observe that for $f \in \mathcal{H}_b(E)$

$$
\begin{aligned}
(\widehat{S} \circ \widehat{R})(f) &= \widehat{S}(\widehat{R}f) \\
&= \widehat{S}(\sum_{m=0}^{\infty} \widehat{R}_m(P^mf(0))) \\
&= \sum_{m=0}^{\infty} \widehat{S}(\underbrace{\widehat{R}_m(P^mf(0))}_{\in \mathcal{H}_b(E)})) \\
&= \sum_{m=0}^{\infty} (\qquad \underbrace{\sum_{j=0}^{\infty} \widehat{S}_j P_j} \qquad) \\
& \qquad \left\{ \begin{array}{lcl} P_j &=& \widehat{R}_m(P^mf(0)) \quad if \quad j = m \\ P_j &=& 0, \quad if \quad j \neq m. \end{array} \right. \\
&= \sum_{m=0}^{\infty} (\underbrace{\widehat{S_m}\widehat{R}_m}_{identity}(P^mf(0)))
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{m=0}^{\infty}(P^m f(0)) \\
&= f.
\end{aligned}
$$

In a similar way, we can get that $(\widehat{R}\circ\widehat{S})g = g$ for all $g \in \mathcal{H}_b(F)$. Therefore $\mathcal{H}_b(E)$ and $\mathcal{H}_b(F)$ are isomorphic. $\boxed{QED}$

Theorem 2. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{H}_b(E;G')$ and $\mathcal{H}_b(F;G')$ are isomorphic.*

Proof. We use the notations from the proof of Theorem 1. Let $\widetilde{R_m} : L(^mE;G') \longrightarrow L(^mF;G')$ and $\widetilde{S_m} : L(^mF;G') \longrightarrow L(^mE;G')$ be the corresponding Nicodemi sequence for vector-valued multilinear mappings. Let $\widehat{\widetilde{R}}_m : \mathcal{P}(^mE;G') \longrightarrow \mathcal{P}(^mF;G')$ be defined by $\widehat{\widetilde{R}}_m\widehat{A} = \widehat{\widetilde{R}_m A}$ for all $A \in L^s(^mE;G')$, and let $\widehat{\widetilde{S}}_m : \mathcal{P}(^mF;G') \longrightarrow \mathcal{P}(^mE;G')$ be defined by $\widehat{\widetilde{S}}_m\widehat{B} = \widehat{\widetilde{S}_m B}$ for all $B \in L^s(^mF;G')$. By [7, Corollary 20], we have that $\widehat{\widetilde{R}}_m$ and $\widehat{\widetilde{S}}_m$ are isomorphisms. Let $\widehat{R} : \mathcal{H}_b(E;G') \longrightarrow \mathcal{H}_b(F;G')$ be defined by $\widehat{R}f = \sum_{m=0}^{\infty}\widehat{\widetilde{R}}_m(P^m f(0))$ for all $f \in \mathcal{H}_b(E;G')$ and let $\widehat{S} : \mathcal{H}_b(F;G') \longrightarrow \mathcal{H}_b(E;G')$ be defined by $\widehat{R}g = \sum_{m=0}^{\infty}\widehat{\widetilde{S}}_m(P^m g(0))$ for all $g \in \mathcal{H}_b(F;G')$. We have that by [6], $\widehat{R}$ and $\widehat{S}$ are well defined and are linear and continuous. We observe that for $f \in \mathcal{H}_b(E;G')$

$$
\begin{aligned}
(\widehat{S}\circ\widehat{R})(f) &= \widehat{S}(\widehat{R}f) \\
&= \widehat{S}\Big(\sum_{m=0}^{\infty}\widehat{\widetilde{R}}_m(P^m f(0))\Big) \\
&= \sum_{m=0}^{\infty}\widehat{S}(\underbrace{\widehat{\widetilde{R}}_m(P^m f(0))}_{\in\mathcal{H}_b(E;G')}) \\
&= \sum_{m=0}^{\infty}\Big(\underbrace{\sum_{j=0}^{\infty}\widehat{\widetilde{S}}_j P_j}\Big) \\
&\qquad \left\{\begin{array}{lll} P_j &=& \widehat{\widetilde{R}}_m(P^m f(0)) \quad if \quad j = m \\ P_j &=& 0, \quad if \quad j \neq m. \end{array}\right. \\
&= \sum_{m=0}^{\infty}(\underbrace{\widehat{\widetilde{S}}_m\widehat{\widetilde{R}}_m}_{identity}(P^m f(0))) \\
&= \sum_{m=0}^{\infty}(P^m f(0))
\end{aligned}
$$

$$= \quad f.$$

In a similar way, we can get that $(\widehat{R}\circ\widehat{S})g = g$ for all $g \in \mathcal{H}_b(F;G')$. Therefore $\mathcal{H}_b(E;G')$ and $\mathcal{H}_b(F;G')$ are isomorphic. $\boxed{QED}$

2 Isomorphisms between spaces of holomorphic mappings of nuclear bounded type.

Theorem 3. *If E' and F' are isomorphic, then $\mathcal{H}_{Nb}(E)$ and $\mathcal{H}_{Nb}(F)$ are isomorphic.*

Proof. We use the notations from the proof of Theorem 1. We have by [7, Theorem 27] $\widehat{R}_m(\mathcal{P}_N(^mE)) \subset \mathcal{P}_N(^mF)$ and $\widehat{S}_m(\mathcal{P}_N(^mF)) \subset \mathcal{P}_N(^mE)$ and by the proof of [7, Theorem 31] $\widehat{R}_m$ and $\widehat{S}_m$ are isomorphisms. Let $\widehat{R} : \mathcal{H}_{Nb}(E) \longrightarrow \mathcal{H}_{Nb}(F)$ be defined by $\widehat{R}f = \sum_{m=0}^{\infty} \widehat{R}_m(P^m f(0))$ for all $f \in \mathcal{H}_{Nb}(E)$ and let $\widehat{S} : \mathcal{H}_{Nb}(F) \longrightarrow \mathcal{H}_{Nb}(E)$ be defined by $\widehat{R}g = \sum_{m=0}^{\infty} \widehat{S}_m(P^m g(0))$ for all $g \in \mathcal{H}_{Nb}(F)$. We have that $\widehat{R}$ and $\widehat{S}$ are well defined and by [6] are linear and continuous. We observe that for $f \in \mathcal{H}_{Nb}(E)$

$$\begin{aligned}
(\widehat{S}\circ\widehat{R})(f) &= \widehat{S}(\widehat{R}f) \\
&= \widehat{S}(\sum_{m=0}^{\infty} \widehat{R}_m(P^m f(0))) \\
&= \sum_{m=0}^{\infty} \widehat{S}(\underbrace{\widehat{R}_m(P^m f(0))}_{\in \mathcal{H}_{Nb}(E)})) \\
&= \sum_{m=0}^{\infty} (\qquad \underbrace{\sum_{j=0}^{\infty} \widehat{S}_j P_j}_{} \qquad) \\
& \qquad \left\{ \begin{array}{lcl} P_j &=& \widehat{R}_m(P^m f(0)) \quad if \quad j = m \\ P_j &=& 0, \quad if \quad j \neq m. \end{array} \right. \\
&= \sum_{m=0}^{\infty} (\underbrace{\widehat{S_m}\widehat{R}_m}_{identity}(P^m f(0))) \\
&= \sum_{m=0}^{\infty} (P^m f(0)) \\
&= f.
\end{aligned}$$

In a similar way, we can get that $(\widehat{R}\circ\widehat{S})g = g$ for all $g \in \mathcal{H}_{Nb}(F)$. Therefore $\mathcal{H}_{Nb}(E)$ and $\mathcal{H}_{Nb}(F)$ are isomorphic. $\boxed{QED}$

Theorem 4. *If E' and F' are isomorphic, then $\mathcal{H}_{Nb}(E;G')$ and $\mathcal{H}_{Nb}(F;G')$ are isomorphic.*

Proof. We use the notations from the proof of Theorem 1 and Theorem 2. By [7, Theorem 28] we have that $\widehat{\widehat{R}}_m(\mathcal{P}_N({}^mE;G')) \subset \mathcal{P}_N({}^mF;G')$ and $\widehat{\widehat{S}}_m(\mathcal{P}_N({}^mF;G')) \subset \mathcal{P}_N({}^mE;G')$, and by the proof of [7, Theorem 32] we have that $\widehat{\widehat{R}}_m$ and $\widehat{\widehat{S}}_m$ are isomorphisms. Let $\widehat{R} : \mathcal{H}_{Nb}(E;G') \longrightarrow \mathcal{H}_{Nb}(F;G')$ be defined by $\widehat{R}f = \sum_{m=0}^{\infty} \widehat{\widehat{R}}_m(P^m f(0))$ for all $f \in \mathcal{H}_{Nb}(E;G')$ and let $\widehat{S} : \mathcal{H}_{Nb}(F;G') \longrightarrow \mathcal{H}_{Nb}(E;G')$ be defined by $\widehat{R}g = \sum_{m=0}^{\infty} \widehat{\widehat{S}}_m(P^m g(0))$ for all $g \in \mathcal{H}_{Nb}(F;G')$. We have that $\widehat{R}$ and $\widehat{S}$ are well defined and by [6] are linear and continuous. We observe that for $f \in \mathcal{H}_{Nb}(E;G')$

$$
\begin{aligned}
(\widehat{S}\circ\widehat{R})(f) &= \widehat{S}(\widehat{R}f)\\
&= \widehat{S}\Big(\sum_{m=0}^{\infty}\widehat{\widehat{R}}_m(P^m f(0))\Big)\\
&= \sum_{m=0}^{\infty}\widehat{S}(\underbrace{\widehat{\widehat{R}}_m(P^m f(0))}_{\in\mathcal{H}_{Nb}(E;G')})\\
&= \sum_{m=0}^{\infty}\Big(\underbrace{\sum_{j=0}^{\infty}\widehat{\widehat{S}}_j P_j}_{\left\{\begin{array}{lcl} P_j &=& \widehat{\widehat{R}}_m(P^m f(0)) \quad if \quad j=m\\ P_j &=& 0, \quad if \quad j\neq m.\end{array}\right.}\Big)\\
&= \sum_{m=0}^{\infty}(\underbrace{\widehat{\widehat{S}}_m\widehat{\widehat{R}}_m}_{identity}(P^m f(0)))\\
&= \sum_{m=0}^{\infty}(P^m f(0))\\
&= f.
\end{aligned}
$$

In a similar way, we can get that $(\widehat{R}\circ\widehat{S})g = g$ for all $g \in \mathcal{H}_{Nb}(F;G')$. Therefore $\mathcal{H}_{Nb}(E;G')$ and $\mathcal{H}_{Nb}(F;G')$ are isomorphic. $\boxed{QED}$

3 Isomorphisms between spaces of holomorphic mappings of compact or weakly compact bounded type.

Theorem 5. *If E and F are symmetrically Arens - regular, and E' and F' are isomorphic, then $\mathcal{H}_{\Theta b}(E;G')$ and $\mathcal{H}_{\Theta b}(F;G')$ are isomorphic, where $\Theta = K$ or WK.*

Proof. We use the notations from the proof of Theorem 1 and Theorem 2. By [7, Theorem 34], we have that, for $\Theta = K$ or WK, $\widehat{\widehat{R}}_m(\mathcal{P}_\Theta(^mE;G')) \subset \mathcal{P}_\Theta(^mF;G')$, and $\widehat{\widehat{S}}_m(\mathcal{P}_\Theta(^mF;G')) \subset \mathcal{P}_\Theta(^mE;G')$. By the proof of [7, Theorem 36], we have that $\widehat{\widehat{R}}_m$ and $\widehat{\widehat{S}}_m$ are isomorphisms. Let $\widehat{R} : \mathcal{H}_{\Theta b}(E;G') \longrightarrow \mathcal{H}_{\Theta b}(F;G')$ be defined by $\widehat{R}f = \sum_{m=0}^{\infty} \widehat{\widehat{R}}_m(P^m f(0))$ for all $f \in \mathcal{H}_{\Theta b}(E;G')$ and let $\widehat{S} : \mathcal{H}_{\Theta b}(F;G') \longrightarrow \mathcal{H}_{\Theta b}(E;G')$ be defined by $\widehat{R}g = \sum_{m=0}^{\infty} \widehat{\widehat{S}}_m(P^m g(0))$ for all $g \in \mathcal{H}_{\Theta b}(F;G')$. We have that $\widehat{R}$ and $\widehat{S}$ are well defined and by [6] are linear and continuous. We observe that for $f \in \mathcal{H}_{\Theta b}(E;G')$

$$\begin{array}{rcl}
(\widehat{S}\circ\widehat{R})(f) & = & \widehat{S}(\widehat{R}f) \\
& = & \widehat{S}(\sum_{m=0}^{\infty} \widehat{\widehat{R}}_m(P^m f(0))) \\
& = & \sum_{m=0}^{\infty} \widehat{S}(\underbrace{\widehat{\widehat{R}}_m(P^m f(0))}_{\in \mathcal{H}_{\Theta b}(E;G')}) \\
& = & \sum_{m=0}^{\infty} (\qquad \underbrace{\sum_{j=0}^{\infty} \widehat{\widehat{S}}_j P_j}_{\left\{\begin{array}{lcl} P_j & = & \widehat{\widehat{R}}_m(P^m f(0)) \quad if \quad j=m \\ P_j & = & 0, \quad if \quad j \neq m. \end{array}\right.} \qquad) \\
& = & \sum_{m=0}^{\infty} (\underbrace{\widehat{\widehat{S}}_m \widehat{\widehat{R}}_m}_{identity}(P^m f(0))) \\
& = & \sum_{m=0}^{\infty} (P^m f(0)) \\
& = & f.
\end{array}$$

In a similar way, we can get that $(\widehat{R}\circ\widehat{S})g = g$ for all $g \in \mathcal{H}_{\Theta b}(F;G')$. Therefore $\mathcal{H}_{\Theta b}(E;G')$ and $\mathcal{H}_{\Theta b}(F;G')$ isomorphic, where $\Theta = K$ or WK. $\boxed{QED}$

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