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# Isomorphisms between lattices of nearly normal subgroups

**Carmela Musella**
*Dipartimento di Matematica e Applicazioni, Universitá di Napoli Federico II, Compl. Univ. Monte S. Angelo, Via Cintia, I 80126 Italy,* `musella@matna2.dma.unina.it`

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**Abstract.** A subgroup $H$ of a group $G$ is said to be *nearly normal* in $G$ if it has a finite index in its normal closure $H^G$. The set $nn(G)$ of nearly normal subgroups of $G$ is a sublattice of the lattice of all subgroups of $G$. Isomorphisms between lattices of nearly normal subgroups of $FC$-soluble groups are considered in this paper. In particular, properties of images of normal subgroups under such an isomorphism are investigated. Moreover, it is proved that if $G$ is a supersoluble group and $\bar{G}$ is an $FC$-soluble group such that the lattices $nn(G)$ and $nn(\bar{G})$ are isomorphic, then also $\bar{G}$ is supersoluble.

**Keywords:** nearly normal subgroup, lattice isomorphism

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## Introduction

A subgroup $H$ of a group $G$ is said to be *nearly normal* if it has finite index in its normal closure $H^G$. A famous result of B.H. Neumann [4] states that all subgroups of a group $G$ are nearly normal if and only if the commutator subgroup $G'$ is finite. The structure of groups which are rich in some sense of nearly normal subgroups has recently been studied (see for instance [1], [2] and [3]). It is easy to see that the set $nn(G)$ of all nearly normal subgroups of a group $G$ is a sublattice of the lattice $\mathfrak{L}(G)$ of all subgroups of $G$. Over the last fifty years there has been a strong development in the investigation concerning projectivities of groups (recall that a *projectivity* between the groups $G$ and $\bar{G}$ is an isomorphism between the subgroup lattices $\mathfrak{L}(G)$ and $\mathfrak{L}(\bar{G})$), and it seems to be natural to consider isomorphisms between lattices of nearly normal subgroups, taking as a model the theory of projectivities of finite groups. Obviously, an infinite simple group cannot contain proper non-trivial nearly normal subgroups, and hence we will restrict our attention to the case of $FC$-soluble groups (i. e. groups having a series of finite length whose factors are $FC$-groups).

A relevant result of Busetto and Schmidt states that, if $\varphi$ is a projectivity between the groups $G$ and $\bar{G}$, and $N$ is a normal subgroup of $G$, then the preimages of the normal closure and the core of $N^\varphi$ in $\bar{G}$ are normal subgroups of $G$ (see [7], Theorem 6.5.6). In section 2 some theorems of this type for isomorphisms between lattices of nearly normal subgroups will be obtained. In particular, we will prove that, if $\varphi$ is an isomorphism between the lattices of nearly normal subgroups of the $FC$-soluble groups $G$ and $\bar{G}$, and $N$ is a normal subgroup of $G$ such that $G/N$ is polycyclic-by-finite, then the preimages of the normal closure and the core of $N^\varphi$ in $\bar{G}$ are normal subgroups of $G$.

It is well-known that the property of being a supersoluble group can be detected from the behaviour of the subgroup lattice (see [7], Theorem 6.4.11), and in section 3 it will be proved that, if $G$ is a supersoluble group and $\bar{G}$ is an $FC$-soluble group such that the lattices $nn(G)$ and $nn(\bar{G})$ are isomorphic, then also $\bar{G}$ is supersoluble.

Most of our notation is standard and can be found in [5] and [7]. In particular, we will use [7] as a general reference on the theory of projectivities.

## 1. Images of normal subgroups

Our first lemma shows that in any group the intersection and the join of finitely many nearly normal subgroups are likewise nearly normal.

**Lemma 1.** *Let $G$ be a group. Then the set $nn(G)$ of all nearly normal subgroups of $G$ is a sublattice of the lattice $\mathfrak{L}(G)$ of all subgroups of $G$.*

Proof. Let $H$ and $K$ be nearly normal subgroups of $G$, so that the indices $|H^G : H|$ and $|K^G : K|$ are finite, and put $L = \langle H, K\rangle$. If $\{h_1, \dots, h_n\}$ is a left transversal of $H$ in $H^G$ and $\{k_1, \dots, k_m\}$ is a right transversal of $K$ in $K^G$, then we have

$$H^G K^G = \bigcup_{i,j} h_i H K k_j = \bigcup_{i,j} (h_i L h_i^{-1}) h_i k_j,$$

so that $H^G K^G$ is the union of finitely many cosets of conjugates of $L$, and it follows from a result of B.H. Neumann that $L$ has finite index in $H^G K^G$ (see [5] Part 1, Lemma 4.17). Therefore $L$ is nearly normal in $G$. Moreover, the indices $|H^G \cap K^G : H^G \cap K|$ and $|H^G \cap K : H \cap K|$ are finite, so that $H \cap K$ has finite index in $H^G \cap K^G$, and hence it is a nearly normal subgroup of $G$. QED

It is well-known that a soluble group containing only finitely many normal subgroups is finite. We will need the following improvement of this remark, where the hypothesis that the group is soluble is weakened assuming that it is $FC$-soluble.

**Lemma 2.** *Let $G$ be an $FC$-soluble group with finitely many normal subgroups. Then $G$ is finite.*

Proof. Let $J$ be the smallest normal subgroup of finite index of $G$, and let $X$ be any normal subgroup of $J$ such that $J/X$ is an $FC$-group. Since $X$ has finitely many conjugates in $G$, we have that also $J/X_G$ is an $FC$-group. Let $H/K$ be any chief factor of $G$ such that $X_G \leq K < H \leq J$. Then $H/K$ is contained in the $FC$-centre of $G/K$, and hence it is finite. It follows that $J/X_G$ is finite, so that $X_G$ has finite index in $G$ and $X = J$. Therefore the $FC$-soluble group $J$ has no non-trivial homomorphic images which are $FC$-groups, so that $J = \{1\}$ and $G$ is finite. QED

A relevant theorem of Zacher ([9]) states that the image of any subgroup of finite index under a projectivity has likewise finite index. The following lemma provides a similar result for isomorphisms between lattices of nearly normal subgroups of $FC$-soluble groups.

**Lemma 3.** *Let $G$ and $\bar{G}$ be groups, and let $\varphi : nn(G) \to nn(\bar{G})$ be a lattice isomorphism. If $\bar{G}$ is $FC$-soluble and $H$ is a subgroup of finite index of $G$, then $H^\varphi$ has finite index in $\bar{G}$.*

Proof. Let $\bar{N}$ be the normal closure of $H^\varphi$ in $\bar{G}$. Since the lattices $[G/\bar{N}^{\varphi^{-1}}]$ and $nn(\bar{G}/\bar{N})$ are isomorphic, the $FC$-soluble group $\bar{G}/\bar{N}$ has finitely many nearly normal subgroups, and hence it is finite by Lemma 2. On the other hand, $H^\varphi$ is a nearly normal subgroup of $\bar{G}$, so that the index $|\bar{N} : H^\varphi|$ is finite, and $H^\varphi$ has finite index in $\bar{G}$. QED

It should be noted here that, if $G$ is any group and $X$ is a nearly normal subgroup of $G$, then every subgroup of finite index of $X$ is also nearly normal in $G$. On the other hand, if $H$ is a subgroup of finite index of $G$, a nearly normal subgroup of $H$ need not be nearly normal in $G$.

**Corollary 1.** *Let $G$ and $\bar{G}$ be groups, and let $\varphi : nn(G) \to nn(\bar{G})$ be a lattice isomorphism. If $\bar{G}$ is $FC$-soluble and $J$ is the finite residual of $G$, then $J^\varphi$ contains the finite residual of $\bar{G}$. In particular, if $G$ is a residually finite group, then $\bar{G}$ is likewise residually finite.*

Proof. Let $\mathfrak{X}$ be the set of all subgroups of finite index of $G$. If $H$ is any element of $\mathfrak{X}$, the image $H^\varphi$ has finite index in $\bar{G}$ by Lemma 3, and so $H^\varphi$ contains the finite residual $\bar{J}$ of $\bar{G}$. On the other hand, $J^\varphi$ is the largest nearly normal subgroup of $\bar{G}$ which is contained in every $H^\varphi$, so that $\bar{J}$ is a subgroup of $J^\varphi$. QED

**Lemma 4.** *Let $G$ and $\bar{G}$ be groups, and let $\varphi : nn(G) \to nn(\bar{G})$ be a lattice isomorphism. If $G$ is cyclic and $\bar{G}$ is $FC$-soluble, then also $\bar{G}$ is cyclic.*

Proof. Let $\bar{N}$ be any normal subgroup of finite index of $\bar{G}$, and put $N = \bar{N}^{\varphi^{-1}}$. Then the finite groups $G/N$ and $\bar{G}/\bar{N}$ are lattice-isomorphic, so that $\bar{G}/\bar{N}$ is cyclic and $\bar{G}' \leq \bar{N}$. As $\bar{G}$ is residually finite by Corollary 1, it follows that $\bar{G}$ is Abelian, and hence even cyclic. QED

Easy examples show that the intersection and the join of infinitely many nearly normal subgroups of a group need not be nearly normal. On the other hand, if $\varphi : nn(G) \to nn(\bar{G})$ is a lattice isomorphism and $(X_i)_{i\in I}$ is a system of nearly normal subgroups of $G$ such that $\cap_{i\in I} X_i$ is nearly normal in $G$ and $\cap_{i\in I} X_i^{\varphi}$ is nearly normal in $\bar{G}$, then $(\cap_{i\in I} X_i)^{\varphi} = \cap_{i\in I} X_i^{\varphi}$. We will frequently use this property in our arguments.

**Theorem 1.** *Let $G$ and $\bar{G}$ be groups, and let $\varphi : nn(G) \to nn(\bar{G})$ be a lattice isomorphism. If $\bar{G}$ is $FC$-soluble and $N$ is a normal subgroup of $G$ such that $G/N$ is a free Abelian group, then $N^{\varphi}$ is a normal subgroup of $\bar{G}$ and the groups $G/N$ and $\bar{G}/N^{\varphi}$ are isomorphic.*

Proof. Let $G/N = \mathrm{Dr}_{i\in I} A_i/N$, where every $A_i/N$ is infinite cyclic, and put $B_i = \langle A_j | j \neq i\rangle$ for each $i \in I$. Then $B_i$ is a normal subgroup of $G$ and $G/B_i \simeq A_i/N$ is infinite cyclic. Let $\bar{H} = H^{\varphi}$ be the normal closure of $B_i^{\varphi}$ in $\bar{G}$, and assume that $B_i$ is a proper subgroup of $H$, so that $H$ has finite index in $G$, and hence $\bar{H}$ has finite index in $\bar{G}$ by Lemma 3. On the other hand, $B_i^{\varphi}$ is a nearly normal subgroup of $\bar{G}$, so that the index $|\bar{H} : B_i^{\varphi}|$ is finite, and $B_i^{\varphi}$ has finite index in $\bar{G}$. This contradiction shows that $B_i^{\varphi}$ is a normal subgroup of $\bar{G}$ for each $i \in I$. Moreover, every factor group $\bar{G}/B_i^{\varphi}$ is cyclic, by Lemma 4, so that

$$\bar{G}' \leq \bigcap_{i\in I} B_i^{\varphi} = (\bigcap_{i\in I} B_i)^{\varphi} = N^{\varphi}.$$

Therefore $N^{\varphi}$ is a normal subgroup of $\bar{G}$, and $\bar{G}/N^{\varphi}$ is an Abelian group. In particular, $G/N$ and $\bar{G}/N^{\varphi}$ have isomorphic subgroup lattices, and hence $G/N \simeq \bar{G}/N^{\varphi}$. QED

Our next lemma is a slight modification of a result by Busetto (see [7], Lemma 6.5.5). Recall that a subgroup $H$ of a group $G$ is said to be *quasinormal* if $HK = KH$ for every subgroup $K$ of $G$ (subgroups with this property are called permutable in [7]).

**Lemma 5.** *Let $G$ and $\bar{G}$ be groups, $X$ a subgroup of $G$ and $\bar{X}$ a normal subgroup of $\bar{G}$, and let $\varphi : [G/X] \to [\bar{G}/\bar{X}]$ be a lattice isomorphism. If $N$ is a normal subgroup of finite index of $G$ containing $X$ such that $N^{\varphi}$ has finite index in $\bar{G}$, and $\bar{H}$ is the normal closure of $N^{\varphi}$ in $\bar{G}$, then the preimage $\bar{H}^{\varphi^{-1}}$ is a normal subgroup of $G$.*

Proof. Assume by contradiction that the lemma is false, and choose a counterexample for which the index of $N$ in $H = \bar{H}^{\varphi^{-1}}$ is smallest possible. If $L$ is any normal subgroup of $G$ such that $N \leq L < H$, then we have $(L^{\varphi})^{\bar{G}} = \bar{H}$ and the minimal choice of the index $|H : N|$ yields that $L = N$. Thus $N$ is the core of $H$ in $G$. Clearly, $N^{\varphi}$ is a permodular element of the lattice $[\bar{G}/\bar{X}]$, so that $N^{\varphi}$ is a permodular subgroup of $\bar{G}$, and the proof of Lemma 6.5.5 of [7] can be used to show that $N^\varphi$ is quasinormal in $\bar{G}$. As $N^\varphi$ is not normal in $\bar{G}$, there exists an element $y$ of $\bar{G}$ such that $(N^\varphi)^y \neq N^\varphi$, and $\langle y\rangle/\langle y\rangle \cap N^\varphi$ is a finite group of prime-power order (see [7], Remark 6.5.2). Put $\bar{T} = \langle N^\varphi, y\rangle$ and $T = \bar{T}^{\varphi^{-1}}$. Then the lattice $\mathfrak{L}(T/N)$ is isomorphic to $[\bar{T}/N^\varphi]$, and so to $\mathfrak{L}(\langle y\rangle/\langle y\rangle \cap N^\varphi)$. It follows that $T/N$ is a finite cyclic group, and its order is a power of a prime number $p$. Let $x$ be an element of $G$ such that $T = \langle N, x\rangle$. As in the proof of Lemma 6.5.5 of [7], it can now be proved that $R/N = \Omega(T/N)$ is a quasinormal subgroup of order $p$ of $G/N$, which is contained in $H/N$. Since $N$ is the core of $H$ in $G$, the subgroup $R$ is not normal in $G$, and so there exists $z \in G$ such that $R^z \neq R$ and the coset $zN$ has order $p^m$ for some positive integer $m$ (see [7], Lemma 5.2.9). Let $G_0 = \langle N, x, z\rangle$, and consider the normal closure $H_0^\varphi = (N^\varphi)^{G_0^\varphi}$. Clearly $\varphi$ induces a lattice isomorphism between $[G_0/X]$ and $[G_0^\varphi/\bar{X}]$, and $|H_0 : N| \leq |H : N|$. As in the proof of Lemma 6.5.5 of [7] it can be shown that $H_0$ is not normal in $G_0$, so that $|H_0 : N| = |H : N|$, and hence replacing $G$ by $G_0$ we may assume without loss of generality that $G = \langle N, x, z\rangle$. Let $A/N$ be a cyclic $q$-subgroup of $G/N$ for some prime $q \neq p$, and suppose that $N^\varphi$ is not normal in $A^\varphi$. Then there exists an element $b$ of $A^\varphi$ such that $(N^\varphi)^b \neq N^\varphi$ and $\langle b\rangle/\langle b\rangle \cap N^\varphi$ is a finite group of prime power order. As above, we have that $R_1/N = \Omega(A/N)$ is a quasinormal subgroup of order $q$ of $G/N$ which is contained in $H/N$. The proof can now be completed using the same arguments of Lemma 6.5.5 of [7]. QED

**Lemma 6.** *Let $G$ and $\bar{G}$ be FC-soluble groups, $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. If $N$ is a normal subgroup of finite index of $G$, and $H^\varphi$ and $K^\varphi$ are the normal closure and the core of $N^\varphi$ in $\bar{G}$, respectively, then $H$ and $K$ are normal subgroups of $G$.*

Proof. The subgroup $N^\varphi$ has finite index in $\bar{G}$ by Lemma 3, so that the indices $|G : K|$ and $|\bar{G} : K^\varphi|$ are finite, and $\varphi$ induces an isomorphism between the lattices $[G/K]$ and $[\bar{G}/K^\varphi]$. Thus it follows from Lemma 5 that $H$ is a normal subgroup of $G$. This property applied to the isomorphism $\varphi^{-1}$ yields now that the image $(K^G)^\varphi$ is a normal subgroup of $\bar{G}$. On the other hand, $K^G$ is obviously contained in $N$, so that $(K^G)^\varphi \leq K^\varphi$ and hence $(K^G)^\varphi = K^\varphi$. Therefore $K = K^G$ is a normal subgroup of $G$. QED

**Lemma 7.** *Let $G$ and $\bar{G}$ be FC-soluble groups, $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. If $N$ is a normal subgroup of $G$ such that the factor group $G/N$ is residually finite, and $K^\varphi$ is the core of $N^\varphi$ in $\bar{G}$, then $K$ is a normal subgroup of $G$.*

Proof. Let $(N_i)_{i\in I}$ be a system of normal subgroups of finite index of $G$ such that $\bigcap_{i\in I} N_i = N$, and for each $i \in I$ let $K_i^\varphi$ be the core of $N_i^\varphi$ in $\bar{G}$. As $N^\varphi \leq N_i^\varphi$ for all $i$, we have that $K^\varphi$ is contained in the normal subgroup

$\bigcap_{i\in I} K_i^\varphi$ of $\bar{G}$. Moreover, it follows from Lemma 6 that every $K_i$ is a normal subgroup of $G$, so that $\bigcap_{i\in I} K_i$ is also normal in $G$, and $(\bigcap_{i\in I} K_i)^\varphi = \bigcap_{i\in I} K_i^\varphi$. Clearly the subgroup $\bigcap_{i\in I} K_i$ is contained in $N$, so that

$$\bigcap_{i\in I} K_i^\varphi = (\bigcap_{i\in I} K_i)^\varphi \leq N^\varphi,$$

and hence $K^\varphi = (\bigcap_{i\in I} K_i)^\varphi$. Therefore $K = \bigcap_{i\in I} K_i$ is a normal subgroup of $G$. QED

We can now prove the main result of this section.

**Theorem 2.** *Let $G$ and $\bar{G}$ be $FC$-soluble groups, $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. If $N$ is a normal subgroup of $G$ such that the factor group $G/N$ is polycyclic-by-finite, and $H^\varphi$ and $K^\varphi$ are the normal closure and the core of $N^\varphi$ in $\bar{G}$, respectively, then $H$ and $K$ are normal subgroups of $G$.*

Proof. As the factor group $G/N$ is residually finite, it follows from Lemma 7 that $K$ is a normal subgroup of $G$. Consider now the finite residual $\bar{J}/H^\varphi$ of $\bar{G}/H^\varphi$, and let $X$ be any subgroup of finite index of $G$ containing $H$. Then $X^\varphi$ has finite index in $\bar{G}$ by Lemma 3, so that $X^\varphi$ contains $\bar{J}$, and hence the preimage $J = \bar{J}^{\varphi^{-1}}$ of $\bar{J}$ is a subgroup of $X$. It is well-known that both subgroups $J$ and $H$ are intersections of subgroups of finite index of $G$ (see [8, p. 18]), so that the above argument shows that $H = J$. Thus $H^\varphi = \bar{J}$, and hence $\bar{G}/H^\varphi$ is a residually finite group. Application of Lemma 7 to the isomorphism $\varphi^{-1}$ yields that the image $(H_G)^\varphi$ is a normal subgroup of $\bar{G}$. Clearly $N$ is contained in $H_G$, so that $N^\varphi \leq (H_G)^\varphi$ and $(H_G)^\varphi = H^\varphi$. Therefore $H = H_G$ is a normal subgroup of $G$. QED

## 2. Supersoluble groups

The first result of this section deals with $FC$-soluble groups having the same nearly normal structure of a soluble residually finite group.

**Lemma 8.** *Let $G$ be a soluble residually finite group with derived length $n$, $\bar{G}$ an $FC$-soluble group and $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. Then $\bar{G}$ is soluble with derived length at most $3n - 1$.*

Proof. Let $\bar{N}$ be any normal subgroup of finite index of $\bar{G}$, and let $K$ be the core of $N = \bar{N}^\varphi$ in $G$. Since $N$ has finite index in $G$ by Lemma 3, also the indices $|G : K|$ and $|\bar{G} : K^\varphi|$ are finite. Moreover, $K^\varphi$ is a normal subgroup of $\bar{G}$ by Lemma 6, and the lattices $\mathfrak{L}(G/K)$ and $\mathfrak{L}(\bar{G}/K^\varphi)$ are isomorphic, so that $\bar{G}/K^\varphi$ is soluble with derived length at most $3n - 1$ (see [7], Corollary 6.6.4), and hence $\bar{G}^{(3n-1)} \leq K^\varphi \leq \bar{N}$. As $\bar{G}$ is residually finite by Corollary 1, it follows that $\bar{G}$ is soluble with derived length at most $3n - 1$. QED

**Lemma 9.** *Let $G$ and $\bar{G}$ be groups, and let $\varphi : nn(G) \to nn(\bar{G})$ be a lattice isomorphism. If $X$ is a subgroup of $G$ and $Y$ is a normal subgroup of $X$ such that the indices $|G : Y|$ and $|\bar{G} : Y^\varphi|$ are finite, then $Y^\varphi$ is a modular subgroup of $X^\varphi$.*

Proof. Let $\bar{K}$ be the core of $Y^\varphi$ in $\bar{G}$ and put $K = \bar{K}^{\varphi^{-1}}$. Then $\bar{K}$ has finite index in $\bar{G}$, and $\varphi$ induces an isomorphism between the interval $[X/K]$ of the lattice $nn(G)$ and the subgroup lattice $\mathfrak{L}(X^\varphi/\bar{K})$. Since $Y$ is a modular element of $[X/K]$, we have that $Y^\varphi/\bar{K}$ is a modular element of $\mathfrak{L}(X^\varphi/\bar{K})$, and hence $Y^\varphi$ is a modular subgroup of $X^\varphi$. QED

**Proposition 1.** *Let $G$ be a finitely generated abelian-by-finite group, $\bar{G}$ an $FC$-soluble group and $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. Then $\bar{G}$ is a finitely generated abelian-by-finite group.*

Proof. The group $G$ contains a finitely generated torsion-free abelian normal subgroup $A$ such that $G/A$ is finite. Then $\bar{A} = A^\varphi$ has finite index in $\bar{G}$ by Lemma 3, so that also the indices $|\bar{G} : \bar{A}_{\bar{G}}|$ and $|G : (\bar{A}_{\bar{G}})^{\varphi^{-1}}|$ are finite. Moreover, the subgroup $(\bar{A}_{\bar{G}})^{\varphi^{-1}}$ is normal in $G$ by Lemma 6, and hence replacing $A$ by $(\bar{A}_{\bar{G}})^{\varphi^{-1}}$ it can be assumed without loss of generality that $\bar{A}$ is a normal subgroup of $\bar{G}$. Put $A = A_1 \times \cdots \times A_t$, where every $A_i$ is infinite cyclic, and $B_i = \langle A_j | j \neq i\rangle$ for all $i$. Then $A/B_i$ is infinite cyclic, and $B_i$ is intersection of maximal subgroups of $A$. Let $\mathfrak{M}$ be the set of all maximal subgroups of $A$ containing $B_i$ and let $M$ be any element of $\mathfrak{M}$. Then $\bar{M} = M^\varphi$ has finite index in $\bar{G}$ by Lemma 3, and hence it is a maximal subgroup of $\bar{A}$. Consider now the unique subgroup $X$ of $G$ such that $B_i \leq X \leq M$ and $|A : M| = |M : X| = q$ (where $q$ is a prime number). Since $X$ has finite index in $G$, also the index $|\bar{G} : X^\varphi|$ is finite, and so $\bar{X} = X^\varphi$ is a modular subgroup of $\bar{A}$, by Lemma 9. Moreover, the sublattice $[\bar{A}/\bar{X}]$ of $\mathfrak{L}(\bar{A})$ is a chain, and hence $\bar{A}/\bar{X}_{\bar{A}}$ is a finite $p$-group for some prime $p$ (see [7], Lemma 5.1.3). It follows that the maximal subgroup $\bar{M}$ of $\bar{A}$ is normal, and so the commutator $\bar{A}'$ of $\bar{A}$ is contained in $\bar{M}$. Thus

$$(\bar{A}')^{\varphi^{-1}} \leq \bigcap_{M \in \mathfrak{M}} M = B_i,$$

and hence

$$(\bar{A}')^{\varphi^{-1}} \leq \bigcap_{i=1}^{t} B_i = \{1\}$$

so that $\bar{A}' = \{1\}$ and $\bar{A}$ is abelian. Therefore the group $\bar{G}$ is abelian-by-finite. Finally, as $G$ satisfies the maximal condition on subgroups,we obtain that $\bar{G}$ satisfies in particular the maximal condition on normal subgroups, and hence it is finitely generated (see for instance [5] Part 1, Theorem 5.31). QED

We will also need the following easy remark on nearly normal subgroups of residually finite groups.

**Lemma 10.** *Let $G$ be a residually finite group, and let $X$ be a nearly normal subgroup of $G$. If $X$ contains only finitely many nearly normal subgroups of $G$, then $X$ is finite.*

PROOF. Every subgroup of finite index of $X$ is nearly normal in $G$, so that $X$ contains only finitely many subgroups of finite index, and hence it is finite. QED

If $\mathfrak{X}$ is a class of groups, a group $G$ is called *just-non-$\mathfrak{X}$* if it is not an $\mathfrak{X}$-group but all its proper homomorphic images belong to $\mathfrak{X}$. The structure of soluble just-non-supersoluble groups has been described by Robinson and Wilson [6], and we will use their results in the proof of our last theorem. It shows that $FC$-soluble groups having the same nearly normal structure of a supersoluble group are likewise supersoluble.

**Theorem 3.** *Let $G$ be a supersoluble group, $\bar{G}$ an $FC$-soluble group and $\varphi : nn(G) \to nn(\bar{G})$ a lattice isomorphism. Then $\bar{G}$ is supersoluble.*

PROOF. The group $G$ is soluble and residually finite, so that $\bar{G}$ is likewise soluble by Lemma 8. Let $\bar{N}$ be any normal subgroup of finite index of $\bar{G}$, and let $K$ be the core of $N = \bar{N}^{\varphi^{-1}}$ in $G$. Then $K^{\varphi}$ is a normal subgroup of $\bar{G}$ by Lemma 6, and the finite groups $G/K$ and $\bar{G}/K^{\varphi}$ have isomorphic subgroup lattices, so that $\bar{G}/K^{\varphi}$ is supersoluble (see [7], Corollary 5.3.8). Therefore $\bar{G}/\bar{N}$ is supersoluble. Assume now by contradiction that the group $\bar{G}$ is not supersoluble, so that it is not even polycyclic by a result of Baer (see [8], p. 54). As $\bar{G}$ obviously satisfies the maximal condition on normal subgroups, it contains a normal subgroup $\bar{H}$ which is maximal with respect to the condition that the factor group $\bar{G}/\bar{H}$ is not polycyclic. Clearly every proper homomorphic image of $\bar{G}/\bar{H}$ is polycyclic, and so supersoluble, so that $\bar{G}/\bar{H}$ is a just-non-supersoluble group. In particular, the group $\bar{G}/\bar{H}$ is residually finite and does not contain polycyclic non-trivial nearly normal subgroups. The preimage $H = \bar{H}^{\varphi^{-1}}$ is a nearly normal subgroup of $G$, so that the index $|H^G : H|$ is finite and $(H^G)^{\varphi}/\bar{H}$ contains only finitely many nearly normal subgroups of $\bar{G}/\bar{H}$. Then $(H^G)^{\varphi}/\bar{H}$ is finite by Lemma 10, so that $(H^G)^{\varphi} = \bar{H}$ and $H = H^G$ is a normal subgroup of $G$. Moreover, $\varphi$ induces an isomorphism between the lattices $nn(G/H)$ and $nn(\bar{G}/\bar{H})$, and hence replacing $G$ by $G/H$ it can be assumed without loss of generality that the group $\bar{G}$ is just-non-supersoluble. Therefore the Fitting subgroup $\bar{A}$ of $\bar{G}$ is isomorphic to $Q_{\pi}$ for some finite set $\pi$ of prime numbers (here $Q_{\pi}$ denotes the additive group of rational numbers whose denominators are $\pi$-numbers), and there exists a finitely generated abelian subgroup $\bar{X}$ of $\bar{G}$ such that $\bar{U} = \bar{A}\bar{X}$ has finite index in $\bar{G}$ and $\bar{A} \cap \bar{X} = \{1\}$ (see [6]). As $G$ is supersoluble, its Fitting subgroup $F$ has finite index, so that also $\bar{F} = F^{\varphi}$ has finite index in $\bar{G}$, and hence $\bar{L} = \bar{U} \cap \bar{F}_{\bar{G}}$ is a subgroup of finite index of $\bar{G}$. Consider now the subgroups $\bar{B} = \bar{A} \cap \bar{L}$ and $\bar{Y} = \bar{X} \cap \bar{L}$, and put $\bar{V} = \bar{B}\bar{Y}$ and $\bar{W}_p = \bar{B}^p\bar{Y}$ for each prime $p \notin \pi$. Since $\bar{B}$ is isomorphic to a subgroup of $Q_{\pi}$, we have

$$\bigcap_{p\notin\pi} \bar{B}^p = \{1\},$$

and so also

$$\bigcap_{p\notin\pi} \bar{W}_p = \bar{Y}.$$

For each prime $p \notin \pi$ the subgroup $\bar{W}_p$ has finite index in $\bar{G}$. It follows that $W_p = \bar{W}_p^{\varphi^{-1}}$ has finite index in $G$, and hence it is a maximal subgroup of the nilpotent group $V = \bar{V}^{\varphi^{-1}}$. Thus $W_p$ is normal in $V$, so that also

$$W = \bigcap_{p\notin\pi} W_p$$

is a normal subgroup of $V$, and $V/W$ is an abelian group. Since

$$(W_G)^{\varphi} \leq \bigcap_{p\notin\pi} \bar{W}_p = \bar{Y},$$

the image $(W_G)^{\varphi}$ is a polycyclic nearly normal subgroup of $\bar{G}$, so that $(W_G)^{\varphi} = \{1\}$ and hence $W_G = \{1\}$. For each element $x$ of $G$, the group $V^x/W^x$ is abelian, so that $(V^x)'$ lies in $W^x$. Since

$$\bigcap_{x\in G} W^x = W_G = \{1\},$$

it follows that also

$$V_G = \bigcap_{x\in G} V^x$$

is an abelian group. On the other hand, $V$ has finite index in $G$, so that $G/V_G$ is finite and $G$ is a finitely generated abelian-by-finite group. Proposition 1 yields now that $\bar{G}$ is likewise a finitely generated abelian-by-finite group, and this contradiction completes the proof of the theorem. QED

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