

## Combinatorics of open covers (IX): Basis properties

**L. Babinkostova**

*Department of Mathematics, Boise State University* liljanab@math.boisestate.edu

**M. Scheepers**<sup>i</sup>

*Department of Mathematics, Boise State University* marion@math.boisestate.edu

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**Abstract.** We introduce the concepts of diagonalization basis property and strong diagonalization basis property. For appropriate spaces having these properties we show that the classical selection properties are equivalent to certain basis properties of the spaces. In particular, these equivalences hold for various metrizable spaces. The Sorgenfrey line, which is not metrizable, has the diagonalization basis property and thus our results also apply in this case. We calculate critical selection cardinals for subspaces of the Sorgenfrey line.

**Keywords:** Selection principle, diagonalization basis property, Lusin set, Sierpiński set, Sorgenfrey line.

**MSC 2000 classification:** Primary 54D70, Secondary 03E17.

The following notation is used throughout the paper: For collections  $\mathcal{A}$  and  $\mathcal{B}$  the symbol:

- $U_{fin}(\mathcal{A}, \mathcal{B})$  denotes the statement that for each sequence  $(U_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  there is a sequence  $(V_n : n \in \mathbb{N})$  of finite sets such that for each  $n$  we have  $V_n \subseteq U_n$ , and  $\{\cup V_n : n \in \mathbb{N}\} \in \mathcal{B}$ .
- $S_{fin}(\mathcal{A}, \mathcal{B})$  denotes the statement that for each sequence  $(U_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  there is a sequence  $(V_n : n \in \mathbb{N})$  of finite sets such that for each  $n$  we have  $V_n \subseteq U_n$ , and  $\cup_{n \in \mathbb{N}} V_n \in \mathcal{B}$ .
- $S_1(\mathcal{A}, \mathcal{B})$  denotes the statement that for each sequence  $(U_n : n \in \mathbb{N})$  of elements of  $\mathcal{A}$  there is a sequence  $(V_n : n \in \mathbb{N})$  such that for each  $n$  we have  $V_n \in U_n$ , and  $\{V_n : n \in \mathbb{N}\} \in \mathcal{B}$ .

$U_{fin}(\mathcal{A}, \mathcal{B})$ ,  $S_{fin}(\mathcal{A}, \mathcal{B})$  and  $S_1(\mathcal{A}, \mathcal{B})$  are selection principles.

From now on  $X$  is a space with no isolated points and  $Y$  is a subspace of  $X$ , possibly equal to  $X$ . The collection of all open covers of  $X$  is denoted  $\mathcal{O}_X$ , while the collection of all covers of  $Y$  by sets open in  $X$  is denoted  $\mathcal{O}_{XY}$ . An open cover  $\mathcal{U}$  of  $X$  is said to be:

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- (1) a *noncompact* cover if: No finite subset of  $\mathcal{U}$  is a cover of  $X$ . The symbol  $\mathcal{O}_X^{nc}$  denotes the collection of non-compact open covers of  $X$ . The symbol  $\mathcal{O}_{XY}^{nc}$  denotes the set of open covers of  $X$  which contain no finite subcovers of  $Y$ <sup>1</sup>.
- (2) a *large* cover if: for each  $x \in X$  the set  $\{U \in \mathcal{U} : x \in U\}$  is infinite. The symbol  $\Lambda_X$  denotes the set of large covers of  $X$ , and  $\Lambda_{XY}$  denotes the collection of large covers of  $Y$  by sets open in  $X$ .
- (3) an  $\omega$ -cover if:  $X$  is not in  $\mathcal{U}$ , and for each finite subset  $F$  of  $X$  there is a  $U \in \mathcal{U}$  such that  $F \subseteq U$ . The symbol  $\Omega_X$  denotes the set of  $\omega$ -covers of  $X$ , and  $\Omega_{XY}$  denotes the collection of  $\omega$ -covers of  $Y$  by sets open in  $X$ .
- (4) a  $\gamma$ -cover if:  $\mathcal{U}$  is infinite and for each  $x \in X$  the set  $\{U \in \mathcal{U} : x \notin U\}$  is finite.  $\Gamma_X$  denotes the collection of  $\gamma$ -covers of  $X$ .  $\Gamma_{XY}$  denotes the collection of  $\gamma$ -covers of  $Y$  by sets open in  $X$ .
- (5) *groupable* if there is a partition  $\mathcal{U} = \cup_{n \in \mathbb{N}} \mathcal{U}_n$  of  $\mathcal{U}$  into finite sets  $\mathcal{U}_n$  such that for  $m \neq n$ ,  $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$ , and for each  $x \in X$ , for all but finitely many  $n$ ,  $x \in \cup \mathcal{U}_n$ . The symbol  $\mathcal{O}_X^{gp}$  denotes the collection of groupable open covers of  $X$ , and  $\Lambda_X^{gp}$  denotes the collection of groupable large covers of  $X$ . The symbols  $\mathcal{O}_{XY}^{gp}$  and  $\Lambda_{XY}^{gp}$  have the obvious definitions.
- (6) *weakly groupable* if there is a partition  $\mathcal{U} = \cup_{n \in \mathbb{N}} \mathcal{U}_n$  of  $\mathcal{U}$  into finite sets  $\mathcal{U}_n$  such that for  $m \neq n$ ,  $\mathcal{U}_m \cap \mathcal{U}_n = \emptyset$ , and for each finite set  $F \subset X$  there is an  $n$  with  $F \subset \cup \mathcal{U}_n$ .  $\mathcal{O}_X^{wgp}$  denotes the collection of weakly groupable covers of  $X$ , and  $\Lambda_X^{wgp}$  denotes the collection of weakly groupable large covers of  $X$ . The symbols  $\mathcal{O}_{XY}^{wgp}$  and  $\Lambda_{XY}^{wgp}$  have the obvious definitions.

Our beginning point is the following *basis property* introduced by K. Menger in [8]: For each base  $\mathcal{B}$  for the topology of the metric space  $(X, d)$ , there is a sequence  $(B_n : n < \infty)$  such that: For each  $n$ ,  $B_n \in \mathcal{B}$  and  $\lim_{n \rightarrow \infty} \text{diam}(B_n) = 0$  and  $\{B_n : n < \infty\}$  covers  $X$ . Hurewicz proved that for metrizable spaces Menger's basis property is equivalent to a selection property:

**1 Theorem (Hurewicz - [6]).** *A metrizable space  $X$  has property  $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$  if, and only if, it has Menger's basis property with respect to each metric generating the topology.*

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<sup>1</sup>Important Note: Here we require that the open covers be covers of the superspace  $X$ , and has no finite subcovers for the subspace  $Y$ . In all other instances where we use  $_{XY}$  it is only assumed that the family in question consists of sets open in  $X$ , and not that the family also covers  $X$ .

The property  $S_{fin}(\mathcal{O}_X, \mathcal{O}_X)$  is also called the ‘‘Menger property’’. Theorem 1 gives for metrizable spaces an equivalence between a selection property and a basis property.

In this paper we prove theorems which characterize several other selection properties in terms of basis properties. In particular, we generalize Hurewicz’s theorem in two directions: In Theorem 4 we prove the relative version of it, which implies Hurewicz’s theorem, and we prove it in a more general context which in the metric case yields Hurewicz’s theorem. Some of the results in this paper appeared in [1] for metrizable spaces.

## $\mathcal{A}$ -basis and diagonalization basis properties.

**2 Definition.** For  $X$  a space and  $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}(X))$ :  $X$  has the  $\mathcal{A}$ -basis property if there is for each sequence  $(\mathcal{B}_n : n \in \mathbb{N})$  of bases of  $X$  such that for each  $n$   $\mathcal{B}_n \supset \mathcal{B}_{n+1}$  and  $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n = \emptyset$ , a sequence  $(B_n : n \in \mathbb{N})$  such that each  $B_n$  is an element of  $\mathcal{B}_1$ , and for each  $k$ , for all but finitely many  $n$ ,  $B_n \in \mathcal{B}_k$ , and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{A}$ .

With  $\mathcal{O}_X$  the collection of open covers of metric space  $X$ , the  $\mathcal{O}_X$ -basis property implies the Menger basis property. The generalization of metric spaces we will use here is given by:

**3 Definition.** A space  $X$  has the *diagonalization basis property* if there is for each sequence  $(\mathcal{U}_n : n < \infty)$  of open covers a sequence  $(\mathcal{B}_n : n < \infty)$  such that:

- (1)  $\bigcap_{n < \infty} \mathcal{B}_n = \emptyset$ ;
- (2) For each  $n$ ,  $\mathcal{B}_{n+1} \subseteq \mathcal{B}_n$ ;
- (3) For each  $n$ ,  $\mathcal{B}_n$  is a basis for  $X$ ;
- (4) For each  $n$  and each  $B \in \mathcal{B}_n$  there is a  $k > n$  and  $U, V \in \mathcal{U}_k$  with  $B \subseteq U \cup V$ .

In [6] Hurewicz also introduced the selection property  $U_{fin}(\mathcal{O}_X^{nc}, \Gamma_X)$ , which was generalized to  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Gamma_{XY})$  in [5], and was further investigated in [3]. In Theorem 9 of [3] it is shown that for  $X$  a Lindelöf space  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Gamma_{XY})$  if, and only if,  $S_{fin}(\Lambda_X, \Lambda_{XY}^{gp})$ .

Property  $U_{fin}(\mathcal{O}_X^{nc}, \Omega_X)$  is introduced in [10], and was shown in [3] to be equivalent to  $S_{fin}(\Lambda_X, \Lambda_X^{wgp})$ . In [1] it was generalized to  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Omega_{XY})$ .

**4 Theorem.** For a space  $X$  with the diagonalization basis property and a subspace  $Y$  of  $X$ :

- (1)  $S_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$  if, and only if,  $X$  has the  $\mathcal{O}_{XY}$ -basis property.
- (2)  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Omega_{XY})$  if, and only if,  $X$  has the  $\mathcal{O}_{XY}^{wgp}$ -basis property.
- (3)  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Gamma_{XY})$  if, and only if,  $X$  has the  $\mathcal{O}_{XY}^{gp}$ -basis property.

PROOF.  $\Rightarrow$ : Let  $(\mathcal{B}_n : n \in \mathbb{N})$  be a sequence of bases of  $X$  such that for each  $n$  we have  $\mathcal{B}_n \supset \mathcal{B}_{n+1}$ , and  $\bigcap_{n \in \mathbb{N}} \mathcal{B}_n = \emptyset$ .

1. Apply  $S_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$  to the sequence  $(\mathcal{B}_n : n \in \mathbb{N})$  of open covers of  $X$ : For each  $n$  choose a finite set  $\mathcal{F}_n \subset \mathcal{B}_n$  such that  $\bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathcal{O}_{XY}$ . Then we have for each  $k$  that for all but finitely many  $n$ ,  $\mathcal{F}_n \subset \mathcal{B}_k$ .

2. We may assume that for infinitely many  $n$   $\mathcal{B}_n \in \mathcal{O}_{XY}^{nc}$ . We get from  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Omega_{XY})$  a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  of finite sets such that for each  $n$  we have  $\mathcal{V}_n \subset \mathcal{B}_n$ , and  $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$  is in  $\mathcal{O}_{XY}^{wgp}$ . Choose  $n_1 < n_2 < \dots < n_k < \dots$  so that for  $i \neq j$  we have  $\mathcal{V}_{n_i} \cap \mathcal{V}_{n_j} = \emptyset$ . Let  $(B_n : n \in \mathbb{N})$  be a bijective enumeration of  $\bigcup_{j \in \mathbb{N}} \mathcal{V}_{n_j}$ . Then this sequence witnesses the  $\mathcal{O}_{XY}^{wgp}$ -basis property for  $X$ .

3. We may assume that for infinitely many  $n$   $\mathcal{B}_n \in \mathcal{O}_{XY}^{nc}$ . We get from  $U_{fin}(\mathcal{O}_{XY}^{nc}, \Gamma_{XY})$  a sequence  $(\mathcal{V}_n : n \in \mathbb{N})$  of finite sets such that for each  $n$  we have  $\mathcal{V}_n \subset \mathcal{B}_n$ , and  $\bigcup_{n \in \mathbb{N}} \mathcal{V}_n$  is in  $\mathcal{O}_{XY}^{gp}$ . Choose  $n_1 < n_2 < \dots < n_k < \dots$  so that for  $i \neq j$  we have  $\mathcal{V}_{n_i} \cap \mathcal{V}_{n_j} = \emptyset$ . Let  $(B_n : n \in \mathbb{N})$  be a bijective enumeration of  $\bigcup_{j \in \mathbb{N}} \mathcal{V}_{n_j}$ . Then this sequence witnesses the  $\mathcal{O}_{XY}^{gp}$ -basis property for  $X$ .

$\Leftarrow$ : Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of large covers of  $X$ . For each  $n$  put  $\mathcal{H}_n = \{U_1 \cap \dots \cap U_n : (\forall i \leq n)(U_i \in \mathcal{U}_i) \text{ and } |\{U_i : i \leq n\}| = n\}$ . Let  $(\mathcal{B}_n : n < \infty)$  be a sequence of bases as in the Basis Diagonalization Property for  $(\mathcal{H}_n : n \in \mathbb{N})$ . Apply the  $\mathcal{A}$ -basis property to  $(\mathcal{B}_n : n \in \mathbb{N})$  and choose a sequence  $(B_n : n \in \mathbb{N})$  such that for each  $k$ , for almost every  $n$ ,  $B_n \in \mathcal{B}_k$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{A}$ .

1.  $\mathcal{A} = \mathcal{O}_{XY}$ : For each  $n$  choose a  $k_n > n$  and  $U_n, V_n \in \mathcal{H}_{k_n}$  such that  $B_n \subset U_n \cup V_n$ . For each  $k$  put  $\mathcal{S}_k = \{U_n : k_n = k\} \cup \{V_n : k_n = k\}$ . Then for each  $k$ ,  $\mathcal{S}_k \subset \mathcal{H}_k$  is finite. Since  $\{B_n : n \in \mathbb{N}\}$  is a cover of  $Y$ , so is  $\bigcup_{k \in \mathbb{N}} \mathcal{S}_k$ . This confirms  $S_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$ .

2.  $\mathcal{A} = \mathcal{O}_{XY}$ : We may assume that we have  $m_1 < m_2 < \dots < m_k < \dots$  such that for each finite  $F \subset Y$  there is a  $k$  with  $F \subset \bigcup_{m_k \leq j < m_{k+1}} B_j$ .

For each  $k$ , choose for each  $j$  with  $m_k \leq j < m_{k+1}$  a  $U_j, V_j \in \mathcal{U}_k$  with  $B_j \subset U_j \cup V_j$  and put  $\mathcal{V}_k = \{U_j : m_k \leq j < m_{k+1}\} \cup \{V_j : m_k \leq j < m_{k+1}\}$ . Then for each  $k$  we have a finite  $\mathcal{V}_k \subset \mathcal{U}_k$  and for each finite  $F \subset Y$  there is a  $k$

with  $F \subset \cup \mathcal{V}_k$ . This implies  $\cup_{fin}(\mathcal{O}_X^{nc}, \Omega_{XY})$ .

3.  $\mathcal{A} = \mathcal{O}_{XY}$ : We may assume that we have  $m_1 < m_2 < \dots < m_k < \dots$  such that for each  $x \in Y$ , for all but finitely many  $k$ ,  $x \in \cup_{m_k \leq j < m_{k+1}} B_j$ .

Put  $\mathcal{R}_1 = \{B_j : j < m_1\}$  and for all  $k$ ,  $\mathcal{R}_{k+1} = \{B_j : m_k \leq j < m_{k+1}\}$ . Then for each  $j$  define

$$\xi(j) = \max\{m : (\forall B \in \mathcal{R}_j)(B \in \mathcal{B}_m)\}.$$

Then choose  $i_1 = 1 < i_2 < \dots < i_t < \dots$  so that  $\xi(i_1) < \xi(i_2) < \dots$ .

For each  $j$ , for each  $B \in \mathcal{R}_{i_j}$  there is an  $n > \xi(i_j) (> j)$  and sets  $C, D \in \mathcal{H}_n$  with  $B \subseteq C \cup D$ . By the construction of the sets  $\mathcal{H}_n$  we can choose for each  $j$ , for each  $B \in \mathcal{R}_{i_j}$ , sets  $C(B), D(B) \in \mathcal{U}_j$  with  $B \subseteq C(B) \cup D(B)$ . For such  $j$  put  $\mathcal{V}_j = \{C(B) : B \in \mathcal{R}_{i_j}\} \cup \{D(B) : B \in \mathcal{R}_{i_j}\}$ . Then each  $\mathcal{V}_j$  is a finite subset of  $\mathcal{U}_j$ , and for each  $x \in Y$ , for all but finitely many  $j$ ,  $x \in \cup \mathcal{V}_j$ .  $\square$

## The strong $\mathcal{A}$ -basis and strong diagonalization basis properties.

**5 Definition.** For  $X$  a space and  $\mathcal{A} \subseteq \mathcal{P}(\mathcal{P}(X))$ :  $X$  has the strong  $\mathcal{A}$ -basis property if there is for each sequence  $(\mathcal{B}_n : n \in \mathbb{N})$  of bases of  $X$  such that for each  $n$   $\mathcal{B}_n \supset \mathcal{B}_{n+1}$  and  $\cap_{n \in \mathbb{N}} \mathcal{B}_n = \emptyset$ , a sequence  $(B_n : n \in \mathbb{N})$  such that for each  $n$ ,  $B_n \in \mathcal{B}_n$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{A}$ .

**6 Definition.** A space  $X$  has the *strong diagonalization basis property* if there is for each sequence  $(\mathcal{U}_n : n < \infty)$  of open covers a sequence  $(\mathcal{W}_n : n < \infty)$  of finite sets such that:

- (1) For each  $n$ ,  $\mathcal{W}_n$  refines  $\mathcal{U}_n$ ;
- (2)  $\mathcal{B} = \cup_{n < \infty} \mathcal{W}_n$  is a basis for  $X$ .

**7 Theorem.** Let  $X$  be a  $\tau_3$ -space with no isolated points, and which has the strong diagonalization basis property. Let  $Y$  be a subspace of  $X$ . For  $\mathcal{A} \in \{\mathcal{O}_{XY}, \mathcal{O}_{XY}^{wgp}, \mathcal{O}_{XY}^{gp}\}$

$S_1(\Lambda_X, \mathcal{A})$  holds if, and only if,  $X$  has the strong  $\mathcal{A}$ -basis property.

PROOF. Since a basis for  $X$  is a large cover of  $X$ , it is evident that for each  $\mathcal{A}$  the  $\mathcal{A}$ -basis property of  $X$  is implied by  $S_1(\Lambda_X, \mathcal{A})$ . We must prove the converse implications. For all three parts the beginning of the proof that the basis property implies the selection property is the same. Let  $(\mathcal{U}_n : n \in \mathbb{N})$  be a sequence of large open covers of  $X$ . For each  $n$  define  $\mathcal{V}_n$  to be the set

$\{V \subset X : V \text{ nonempty open and } |\{U \in \mathcal{U}_n : V \subseteq U\}| > n\}$ . Then each  $\mathcal{V}_n$  is still a large cover of  $X$ . For each  $n$  define

$$\mathcal{H}_n = \{V_1 \cap \cdots \cap V_n : V_i \in \mathcal{V}_i, 1 \leq i \leq \frac{n \cdot (n+1)}{2}\} \setminus \{\emptyset\}.$$

For each element of  $\mathcal{H}_n$  choose a representation as an intersection of  $n$  sets as above, and let this representation be fixed for the duration of this proof.

By the strong diagonalization basis property choose a sequence  $(\mathcal{W}_n : n \in \mathbb{N})$  of finite sets such that for each  $n$ ,  $\mathcal{W}_n$  refines  $\mathcal{H}_n$ , and  $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{W}_n$  is a basis for  $X$ . For each  $n$  define  $\mathcal{B}_n = \mathcal{B} \setminus \cup_{j \leq n} \mathcal{W}_j$ . Since  $X$  is  $T_3$  and has no isolated points, each  $\mathcal{B}_n$  is a basis, and for each  $n$  we have  $\mathcal{B}_{n+1} \subset \mathcal{B}_n$ , and  $\cap_{n \in \mathbb{N}} \mathcal{B}_n = \emptyset$ . Now apply the fact that  $X$  has the strong  $\mathcal{A}$ -basis property, and choose a sequence  $(B_n : n \in \mathbb{N})$  from  $\mathcal{B}_n$  such that  $\{B_n : n < \infty\} \in \mathcal{A}$ .

Observe that if  $B_n$  is a member of  $\mathcal{W}_m$  then we have  $n < m$ . Thus, for each  $m$  we have  $\{n : B_n \in \mathcal{W}_m\} \subseteq \{1, \dots, m-1\}$ . For each  $n$  choose  $\phi(n) > n$  with  $B_n \in \mathcal{W}_{\phi(n)}$ . For each  $k$ , put  $\mathcal{S}_k = \{n : \phi(n) = k\}$ . Thus, for each  $k$ ,  $|\mathcal{S}_k| \leq k$ .

For each  $n$  consider  $B_n$ . Since it is an element of  $\mathcal{W}_{\phi(n)}$  choose an element  $V_n$  of  $\mathcal{V}_{\phi(n)}$  with  $B_n \subseteq V_n$ . Then choose an element  $U_n \in \mathcal{U}_n \setminus \{U_1, \dots, U_{n-1}\}$  with  $V_n \subseteq U_n$  (this is possible since each  $V_n$  is contained in more than  $n$  elements of  $\mathcal{U}_n$ ). This describes how to choose for each  $k$  a  $U_k \in \mathcal{U}_k$ .

- (1)  $\mathcal{A} = \mathcal{O}_{XY}$ : Since  $\mathcal{B}$  refines  $\{U_k : k \in \mathbb{N}\}$  it follows that  $\{U_k : k \in \mathbb{N}\}$  is a cover of  $Y$  and we have verified  $S_1(\Lambda_X, \mathcal{O}_{XY})$ .
- (2)  $\mathcal{A} = \mathcal{O}_{XY}^{wgp}$ : Choose a sequence  $z_1 < z_2 < \cdots < z_n < \cdots$  such that for each finite  $F \subset Y$  there is an  $n$  with  $F \subset \cup_{z_n \leq i < z_{n+1}} B_i$ . Thus for each finite  $F \subset Y$  there is an  $n$  with  $F \subset \cup_{z_n \leq i < z_{n+1}} U_i$ . Thus, we verified  $S_1(\Lambda_X, \mathcal{O}_{XY}^{wgp})$ .
- (3)  $\mathcal{A} = \mathcal{O}_{XY}^{gp}$ : Choose a sequence  $z_1 < z_2 < \cdots < z_n < \cdots$  such that for each  $y \in Y$ , for all but finitely many  $n$ , we have  $y \in \cup_{z_n \leq i < z_{n+1}} B_i$ . Thus for each  $y \in Y$ , for all but finitely many  $n$ , we have  $y \in \cup_{z_n \leq i < z_{n+1}} U_i$ , and we derived  $S_1(\Lambda_X, \mathcal{O}_{XY}^{gp})$ .

□

## Examples

### Metrizable spaces.

The following lemma appearing in a proof of a result in [4] is an important tool in establishing the relationship between basis properties and selection properties.

**8 Lemma.** *If  $(X, d)$  is a metric space with no isolated points then it has the diagonalization basis property.*

PROOF. Let  $(\mathcal{U}_n : n < \infty)$  be a sequence of open covers of  $X$ . For each  $n$  put  $\mathcal{H}_n = \{U \subset X : U \text{ open and } (\exists V \in \mathcal{U}_n)(U \subseteq V)\} \setminus \{\emptyset\}$ . Define

$$\mathcal{B} = \{U \cup V : (\exists n)(U, V \in \mathcal{H}_n \text{ and } \text{diam}_d(U \cup V) > \frac{1}{n})\}$$

Then  $\mathcal{B}$  is a basis for  $X$  (this uses that  $X$  has no isolated points). For  $n \in \mathbb{N}$  put  $\mathcal{B}_n = \{B \in \mathcal{B} : \text{diam}(B) < \frac{1}{n}\}$ . For  $B \in \mathcal{B}_n$  choose a  $k$  and  $U, V \in \mathcal{H}_k$  such that  $\frac{1}{k} < \text{diam}_d(U \cup V)$  and  $B = U \cup V$ . Then evidently  $k > n$ . Moreover, from the construction of  $\mathcal{H}_k$ , we find sets  $C$  and  $D$  in  $\mathcal{U}_k$  with  $U \subseteq C$  and  $V \subseteq D$ .  $\square$

Thus Theorem 4 holds for metrizable spaces. Indeed, for metrizable spaces one gets the following results:

**9 Theorem.** *Let  $X$  be a metric space with no isolated points and let  $Y$  be a subspace of  $X$ . The following are equivalent:*

- (1)  $X$  has property  $\mathcal{S}_{fin}(\mathcal{O}_X, \mathcal{O}_{XY})$ .
- (2)  $X$  has the  $\mathcal{O}_{XY}$ -basis property.
- (3) For each basis  $\mathcal{B}$  of  $X$  there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that  $\lim \text{diam}(B_n) = 0$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}$ .

In the case when  $X = Y$ , the equivalence  $1 \Leftrightarrow 3$  is Hurewicz's classical characterization of Menger's basis property.

**10 Theorem.** *Let  $X$  be a metric space with no isolated points and let  $Y$  be a subspace of  $X$ . The following are equivalent:*

- (1)  $X$  has property  $\mathcal{U}_{fin}(\mathcal{O}_X^{nc}, \Gamma_{XY})$ .
- (2)  $X$  has the  $\mathcal{O}_{XY}^{gp}$ -basis property.
- (3) For each basis  $\mathcal{B}$  of  $X$  there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that  $\lim \text{diam}(B_n) = 0$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}^{gp}$ .

This gives a characterization in terms of basis properties of the covering property introduced by Hurewicz in [6].

**11 Theorem.** *Let  $X$  be a metric space with no isolated points and let  $Y$  be a subspace of  $X$ . The following are equivalent:*

- (1)  $X$  has property  $\mathcal{U}_{fin}(\mathcal{O}_X^{nc}, \Omega_{XY})$ .
- (2)  $X$  has the  $\mathcal{O}_{XY}^{wgp}$ -basis property.

- (3) For each basis  $\mathcal{B}$  of  $X$  there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that  $\lim \text{diam}(B_n) = 0$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}^{wgp}$ .

For metrizable spaces with property  $\mathcal{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$  stronger versions of these basis properties hold.

**12 Lemma.** Let  $(X, d)$  be a metric space with property  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$ . Then  $X$  has the strong diagonalization basis property.

PROOF. Let  $(\mathcal{U}_n : n < \infty)$  be a sequence of open covers of  $X$ . For each  $n$  define:  $\mathcal{V}_n = \{U \text{ open} : (\exists V \in \mathcal{U}_n)(U \subseteq V \text{ and } \text{diam}_d(U) < \frac{1}{n})\}$ . Then each  $\mathcal{V}_n$  is a large cover of  $X$ . Since  $X$  has property  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$ , choose for each  $n$  a finite set  $\mathcal{W}_n \subset \mathcal{V}_n$  such that  $\mathcal{B} = \cup_{n \in \mathbb{N}} \mathcal{W}_n$  is a large cover of  $X$ .  $\square$  QED

Thus, Theorem 7 holds for metrizable spaces which have property  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$ . Specifically, one has the following theorems:

**13 Theorem.** Let  $X$  be a metric space with no isolated points and with property  $\mathcal{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ . Let  $Y$  be a subspace of  $X$ . The following are equivalent:

- (1)  $X$  has property  $\mathcal{S}_1(\Lambda_X, \mathcal{O}_{XY})$ .
- (2)  $X$  has the strong  $\mathcal{O}_{XY}$ -basis property.
- (3) For each basis  $\mathcal{B}$  of  $X$  and for each sequence  $(\epsilon_n : n \in \mathbb{N})$  of positive reals, there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that for each  $n$ ,  $\text{diam}(B_n) < \epsilon_n$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}$ .

And also, one has the following two theorems:

**14 Theorem.** Let  $X$  be a metric space with no isolated points and with property  $\mathcal{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ . Let  $Y$  be a subspace of  $X$ . The following are equivalent:

- (1)  $X$  has property  $\mathcal{S}_1(\Lambda_X, \mathcal{O}_{XY}^{wgp})$ .
- (2)  $X$  has the strong  $\mathcal{O}_{XY}^{wgp}$ -basis property.
- (3) For each basis  $\mathcal{B}$  of  $X$  and for each sequence  $(\epsilon_n : n \in \mathbb{N})$  of positive reals, there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that for each  $n$ ,  $\text{diam}(B_n) < \epsilon_n$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}^{wgp}$ .

**15 Theorem.** Let  $X$  be a metric space with no isolated points and with property  $\mathcal{S}_{fin}(\mathcal{O}_X, \mathcal{O}_X)$ . Let  $Y$  be a subspace of  $X$ . The following are equivalent:

- (1)  $X$  has property  $\mathcal{S}_1(\Lambda_X, \mathcal{O}_{XY}^{gp})$ .
- (2)  $X$  has the strong  $\mathcal{O}_{XY}^{gp}$ -basis property.
- (3) For each basis  $\mathcal{B}$  of  $X$  and for each sequence  $(\epsilon_n : n \in \mathbb{N})$  of positive reals, there is a sequence  $(B_n : n \in \mathbb{N})$  in  $\mathcal{B}$  such that for each  $n$ ,  $\text{diam}(B_n) < \epsilon_n$  and  $\{B_n : n \in \mathbb{N}\} \in \mathcal{O}_{XY}^{gp}$ .

### The Sorgenfrey line.

Consider the topology on the real line generated by the basis  $\{[a, b) : a < b \in \mathbb{R}\}$ . This is the well-known Sorgenfrey topology on the real line. Let  $\mathbb{R}$  now denote the space of real numbers with the usual Euclidean topology and let  $\mathbb{S}$  denote the space of real numbers with the Sorgenfrey topology.  $\mathbb{S}$  is also known as the Sorgenfrey line.

We now discuss selection properties of some subspaces of  $\mathbb{S}$ . Observe that if a subset of the real line satisfies some selection principle as subspace of  $\mathbb{S}$ , then it has that property also as subspace of  $\mathbb{R}$ . Also, every subspace of  $\mathbb{S}$  is a first countable  $\mathbb{T}_3$  Lindelöf space and thus paracompact and  $\mathbb{T}_4$ .

**16 Proposition.** *If a subspace  $X$  of  $\mathbb{S}$  has property  $\mathcal{S}_{fin}(\Omega_X, \Omega_X)$  then  $X$  is countable.*

PROOF. Recall that  $\mathcal{S}_{fin}(\Omega_X, \Omega_X)$  holds if and only if  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$  holds in all finite powers of  $X$ . Now: If  $X \subset \mathbb{S}$  is uncountable then  $X \times X$  contains an uncountable closed, discrete, subspace and thus is not Lindelöf. Since a space with property  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$  is a Lindelöf space, for no uncountable subspace  $X$  of  $\mathbb{S}$  could  $X \times X$  have the property  $\mathcal{S}_{fin}(\mathcal{O}, \mathcal{O})$ .  $\square$

**17 Lemma.**  *$\mathbb{S}$  does not have the property  $\mathcal{S}_{fin}(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}})$ .*

PROOF. Define for each finite sequence  $n_1, \dots, n_k$  of elements of  $\omega$  a point  $x_{n_1, \dots, n_k}$  in  $[0, 1)$  as follows:

- For each  $n < \omega$ ,  $x_n = \frac{2^n - 1}{2^n}$ ;
- $x_{n_1, \dots, n_k, 0} = x_{n_1, \dots, n_k}$  and
- $x_{n_1, \dots, n_k, j+1} = x_{n_1, \dots, n_k, j} + \left(\frac{1}{2}\right)^{n_1 + \dots + n_k + (j+1) + 1}$ .

Then, for each  $m \in \mathbb{Z}$  and  $n_1, \dots, n_k < \omega$  put

$$S(m, n_1, \dots, n_k) = [m + x_{n_1, \dots, n_k}, m + x_{n_1, \dots, n_{k-1}, n_k + 1}).$$

For each  $k < \omega$  define:

$$\mathcal{U}_k = \{S(m, n_1, \dots, n_k) : n_1, \dots, n_k < \omega \text{ and } m \in \mathbb{Z}\}.$$

Then each  $\mathcal{U}_k$  is an open cover of  $\mathbb{S}$ .

Consider any sequence  $(\mathcal{F}_n : n < \omega)$  of finite sets where for each  $n$  we have  $\mathcal{F}_n \subset \mathcal{U}_n$ . To show that  $\cup_{n < \omega} \mathcal{F}_n$  is not an open cover of  $\mathbb{S}$ , argue as follows:

Since  $\mathcal{F}_0$  is finite, choose an integer  $m_0 > 0$  so large that  $\cup \mathcal{F}_0 \cap [m_0, m_0 + 1) = \emptyset$ . Put  $C_1 = [m_0, m_0 + 1)$ .

Since  $\mathcal{F}_1$  is finite, choose an  $n_1 > 0$  so large that  $\cup \mathcal{F}_1 \cap [m_0 + x_{n_1}, m_0 + x_{n_1 + 1}) = \emptyset$ .

Put  $C_2 = [m_0 + x_{n_1}, m_0 + x_{n_1+1}]$ .

Since  $\mathcal{F}_2$  is finite, choose an  $n_2 > 0$  so large that  $\cup \mathcal{F}_2 \cap [m_0 + x_{n_1, n_2}, m_0 + x_{n_1, n_2+1}] = \emptyset$  and put  $C_3 = [m_0 + x_{n_1, n_2}, m_0 + x_{n_1, n_2+1}]$ , and so on.

With  $m_0, n_1, \dots, n_k > 0$  chosen choose by the finiteness of  $\mathcal{F}_{k+1}$  an  $n_{k+1} > 0$  so large that  $\cup \mathcal{F}_{k+1} \cap [m_0 + x_{n_1, \dots, n_{k+1}}, m_0 + x_{n_1, \dots, n_{k+1}+1}] = \emptyset$ . Define  $C_{k+2} = [m_0 + x_{n_1, \dots, n_{k+1}}, m_0 + x_{n_1, \dots, n_{k+1}+1}]$ .

Then we have:

$$C_1 \supset [m_0, m_0 + 1] \supset C_2 \supset \dots \supset [m_0 + x_{n_1, \dots, n_k}, m_0 + x_{n_1, \dots, n_{k+1}}] \supset C_{k+2} \dots$$

and each  $C_k$  is compact and for each  $k$ ,  $\cup \mathcal{F}_k \cap C_{k+2} = \emptyset$ . By the Cantor intersection theorem, choose an  $x \in \cap_{k < \infty} C_k$ . Then  $x$  is an element of  $\mathbb{S}$ , but not covered by any  $\mathcal{F}_k$ .  $\square$

**18 Lemma.**  $\mathbb{S}$  has the diagonalization basis property.

PROOF. The proof is identical to that of Lemma 8, but now we use the metric  $d$  generating the usual Euclidean topology to define the bases.  $\square$

It follows that the Sorgenfrey line, though not metrizable, has the following properties:

**19 Corollary.** Let  $Y$  be a subspace of  $\mathbb{S}$ .

- (1)  $S_{fin}(\mathcal{O}_{\mathbb{S}}, \mathcal{O}_{\mathbb{S}Y})$  if, and only if,  $\mathbb{S}$  has the  $\mathcal{O}_{\mathbb{S}Y}$ -basis property.
- (2)  $U_{fin}(\mathcal{O}_{\mathbb{S}Y}^{nc}, \Omega_{\mathbb{S}Y})$  if, and only if,  $\mathbb{S}$  has the  $\mathcal{O}_{\mathbb{S}Y}^{wgp}$ -basis property.
- (3)  $U_{fin}(\mathcal{O}_{\mathbb{S}Y}^{nc}, \Gamma_{\mathbb{S}Y})$  if, and only if,  $X$  has the  $\mathcal{O}_{\mathbb{S}Y}^{gp}$ -basis property.

Observe that a space with the strong basis diagonalization property must be second countable. Since no dense subspace of the Sorgenfrey line is second countable, no dense subspace of the Sorgenfrey line has the strong basis diagonalization property. But using Proposition 16 and the methods of Section 4 of [7] one can prove

**20 Proposition.** For  $X$  a subspace of the Sorgenfrey line:

- (1)  $\text{non}(S_{fin}(\Omega_X, \Omega_X)) = \aleph_1$
- (2)  $\text{non}(S_{fin}(\mathcal{O}_X, \mathcal{O}_X)) = \mathfrak{d}$
- (3)  $\text{non}(U_{fin}(\mathcal{O}_X^{nc}, \Gamma_X)) = \mathfrak{b}$
- (4)  $\text{non}(S_1(\mathcal{O}_X, \mathcal{O}_X)) = \text{cov}(\mathcal{M})$
- (5)  $\text{non}(S_1(\Gamma_X, \Gamma_X)) = \mathfrak{b}$
- (6)  $\text{non}(S_1(\Gamma_X, \Omega_X)) = \mathfrak{d}$

Each of  $\mathfrak{b}$ ,  $\mathfrak{d}$  and  $\text{cov}(\mathcal{M})$  is uncountable, and for each it is consistent that the cardinal number is larger than  $\aleph_1$ . Though the selection properties considered here implies for topological spaces the Lindelöf property, we see now that they do not imply second countability: For example, consider a model of Set Theory in which  $\mathfrak{d} > \aleph_1$ , and consider a dense subspace  $X$  of  $\mathbb{S}$  of cardinality  $\aleph_1$ . Then in the relative topology  $X$  is not second countable, yet has the Menger property.

Absence of the strong basis diagonalization property also does not preclude subspaces from satisfying strong selection properties. For consider special subsets of the real line for these two topological spaces  $\mathbb{R}$  and  $\mathbb{S}$ . Observe that open sets in the Sorgenfrey topology are Borel sets in the usual topology. Let  $\mathcal{B}$  denote the collection of countable covers consisting of Borel sets. In [12], Theorem 1.6, the following was proved:

**21 Theorem ([12], Theorem 1.6).** *For a set  $X$  of real numbers the following are equivalent:*

- (1)  $X$  has property  $S_1(\mathcal{B}, \mathcal{B})$ .
- (2) For each meager set  $M \subset \mathbb{R}$ ,  $X \cap M$  has property  $S_1(\mathcal{B}, \mathcal{B})$ .

A subset  $L$  of the real line is a Lusin set if it is uncountable and for each meager set  $M$  of real numbers,  $L \cap M$  is countable.

**22 Proposition.** *Let  $L$  be a Lusin set of real numbers. Then in the Sorgenfrey topology  $L$  satisfies  $S_1(\mathcal{O}_L, \mathcal{O}_L)$ .*

PROOF. If  $L$  is a Lusin set in  $\mathbb{R}$ , then for each meager set  $M$  we have  $L \cap M$  is countable. But countable sets satisfy  $S_1(\mathcal{B}, \mathcal{B})$ , and so by Theorem 21 Lusin sets satisfy  $S_1(\mathcal{B}, \mathcal{B})$ . But sets open in the Sorgenfrey topology, are Borel sets in the usual topology. Moreover, subspaces of the Sorgenfrey line are Lindelof in the Sorgenfrey topology. Thus in the Sorgenfrey topology each Lusin set  $L$  satisfies  $S_1(\mathcal{O}_L, \mathcal{O}_L)$ .  $\square$

A subset of the real line is a Sierpinski set if it is uncountable but its intersection with each Lebesgue measure zero set is countable. Let  $\mathcal{B}_\Gamma$  denote the collection of  $\gamma$ -covers consisting of Borel sets. In [12], Theorem 1.15, the following was proved:

**23 Theorem ([12], Theorem 1.15).** *For a set  $X$  of real numbers the following are equivalent:*

- (1)  $X$  has property  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ .
- (2) For each Lebesgue measure zero set  $N \subset \mathbb{R}$ ,  $X \cap N$  has property  $S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ .

**24 Proposition.** *Let  $S$  be a Sierpinski set of real numbers. Then in the Sorgenfrey topology  $S$  satisfies  $S_1(\Gamma_S, \Gamma_S)$ .*

PROOF. If  $S$  is a Sierpinski set in  $\mathbb{R}$ , then for each Lebesgue measure zero set  $N$  we have  $S \cap N$  is countable. But countable sets satisfy  $\mathfrak{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ , and so by Theorem 23 Sierpinski sets satisfy  $\mathfrak{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ . But sets open in the Sorgenfrey topology, are Borel sets in the usual topology. Moreover, subspaces of the Sorgenfrey line are Lindelof in the Sorgenfrey topology. Thus in the Sorgenfrey topology each Sierpinski set  $S$  satisfies  $\mathfrak{S}_1(\Gamma_S, \Gamma_S)$ .  $\square$

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