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Symmetrical extensions and generalized nearness

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Abstract. A common concept of supertopologies and nearness, called **supernearness**, is considered. Moreover, we will characterize those supernear spaces which can be extended to topological ones in the **symmetrical** case.

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This paper is dedicated to Prof. B. Banaschewski
on the occasion of his 75th birthday

Introduction

Topological extensions are closely related to nearness structures of various kinds.

For example, the *Smirnov compactification* [14] of a proximity space X is a compact Hausdorff space Y which contains X as a dense subspace and for which it is true that a pair of subsets of X is near iff their closures in Y meet. *Lodato* generalized this result to weaker conditions for the proximity and the space Y using "bunches" for the characterization of the extension. Ivanova and Ivanov studied contiguity spaces and bicompact extensions.

Herrlich found a useful generalization of contiguity spaces by introducing nearness spaces, and Bentley [2] showed that those nearness spaces which can be extended to topological ones have a neat *internal* characterization.

Doitchinov introduced the notion of supertopological spaces in order to construct a *unified* theory of topological, proximity and uniform spaces, and he proved a certain relationship of some special classes of supertopologies – called b-supertopologies – with compactly determined extensions.

Now, to study unification and extensions in a more *general* setting, we define the category $\boldsymbol{SN}$ of supernear spaces and related maps and consider its important subcategory of *clump-determined* supernear spaces.

1 Definition. For a set X, a subset $\xi \subseteq \boldsymbol{P}(\boldsymbol{P}X)$ (where $\boldsymbol{P}X$ denotes the set of all subsets of X) is called a *nearness structure* on X, and the pair (X,ξ) is called a *nearness space*, if the following axioms are satisfied

(N1) $\mathcal{N}_2 \ll \mathcal{N}_1 \in \xi$ implies $\mathcal{N}_2 \in \xi$; where $\mathcal{N}_2 \ll \mathcal{N}_1 \in \xi$ ($\mathcal{N}_2$ corefines $\mathcal{N}_1$) iff for each $F_2 \in \mathcal{N}_2$ there exists $F_1 \in \mathcal{N}_1$ such that $F_2 \supseteq F_1$;

(N2) $\bigcap \mathcal{N} \neq \emptyset$ implies $\mathcal{N} \in \xi$;

(N3) $\emptyset \in \xi$ and $\{\emptyset\} \notin \xi$;

(N4) $\mathcal{N}_2 \cup \mathcal{N}_1 \in \xi$ implies $\mathcal{N}_1 \in \xi$ or $\mathcal{N}_2 \in \xi$; where $\mathcal{N}_2 \cup \mathcal{N}_1 := \{ F_1 \cup F_2 \mid F_1 \in \mathcal{N}_1, F_2 \in \mathcal{N}_2 \}$;

(N5) $\{ cl_\xi(F) \mid F \in \mathcal{N} \} \in \xi$ implies $\mathcal{N} \in \xi$; where $cl_\xi(F) := \{ x \in X \mid \{\{x\}, F\} \in \xi \}$.

Elements of ξ are called *near collections*. Given a pair of nearness spaces (X,ξ), (Y,η), a function $f : X \to Y$ is called a *near map* or shortly *n-map* iff

(n) $\mathcal{N} \in \xi$ implies $\{ f[F] \mid F \in \mathcal{N} \} \in \eta$.

We denote by $\boldsymbol{NEAR}$ the corresponding category.

2 Remark. Note that the closure operator cl_ξ as defined above is *always* topological, moreover it is *symmetric* in the following sense:

(s) $x, y \in X$ and $x \in cl_\xi(\{y\})$ imply $y \in cl_\xi(\{x\})$.

As Herrlich shows, it is possible to embed *both* the category $\boldsymbol{R_0 TOP}$ of *symmetrical topological* spaces and continuous maps and the category $\boldsymbol{UNIF}$ of *uniform* spaces and related maps into $\boldsymbol{NEAR}$ as bireflective and bicoreflective subcategories, respectively.

3 Definition. A *supertopology* on a set X is a pair $(\mathcal{M},\Theta)$, where $\mathcal{M}$ is a subset of the power $\boldsymbol{P}X$ and Θ is a map $\Theta : \mathcal{M} \to \boldsymbol{FIL}(X)$ to the set of all filters on X, such that:

(ST1) $\mathcal{M}$ contains $\vartheta X := \{\emptyset\} \cup \{ \{x\} \mid x \in X \}$;

(ST2) $A_2 \subseteq A_1 \in \mathcal{M}$ implies $A_2 \in \mathcal{M}$;

(ST3) $\Theta(\emptyset) = \boldsymbol{P}X$;

(ST4) $A \in \mathcal{M}$ and $U \in \Theta(A)$ imply $U \supseteq A$;

(ST5) $A_2 \subseteq A_1 \in \mathcal{M}$ implies $\Theta(A_1) \subseteq \Theta(A_2)$;

(ST6) $A \in \mathcal{M}$ and $U \in \Theta(A)$ imply there exists a set $V \in \Theta(A)$ such that always $U \in \Theta(B)$ for each $B \in \mathcal{M}$ with $B \subseteq V$.

Then, the triple $(X, \mathcal{M}, \Theta)$ is called a *supertopological space.* For each $A \in \mathcal{M}$, a set $U \in \Theta(A)$ is called a *neighborhood* of A, and $\Theta(A)$ is called the *neighborhood-system* of A with respect to Θ.

For supertopological spaces $(X, \mathcal{M}^X, \Theta_X)$, $(Y, \mathcal{M}^Y, \Theta_Y)$, a function $f\colon X \to Y$ is called *continuous* iff

(i) $\{ f[A] \mid A \in \mathcal{M}^X \} \subseteq \mathcal{M}^Y$;

(ii) $A \in \mathcal{M}^X$ and $V \in \Theta_Y(f[A])$ imply $f^{-1}[V] \in \Theta_X(A)$, where f^{-1} denotes the inverse image under f.

We denote by ***STOP*** the corresponding category.

4 Remark. Doitchinov embedded ***TOP***, the category of *topological* spaces, and ***PROX***, the category of *proximity* spaces into ***STOP*** by restricting $\mathcal{M}$ to ϑX and to $\boldsymbol{P}X$, respectively.

5 Remark. Every supertopology $(\mathcal{M}, \Theta)$ on X induces a *generalized* proximity relation p_Θ from $\mathcal{M}$ to $\boldsymbol{P}X$ by setting

$$A \, p_\Theta \, B \text{ iff } B \in sec\Theta(A),$$

where $sec\Theta(A) := \{ B \subseteq X \mid \forall U \in \Theta(A).\, B \cap U \neq \emptyset \}$.

In terms of the generalized proximity p_Θ the axioms of Definition 3 may be reformulated. The first two concerning $\mathcal{M}$ do not change. In addition we have

(SP1) $A \in \mathcal{M}$ and $B \subseteq X$ imply $A \, \overline{p_\Theta} \, \emptyset$ and $\emptyset \, \overline{p_\Theta} \, B$, which means that the empty set is *not* in relation to A, nor is it in relation to B;

(SP2) $A \, p_\Theta \, (B \cup C)$ iff $A \, p_\Theta \, B$ or $A \, p_\Theta \, C$, for $A \in \mathcal{M}$ and subsets $B, C \subseteq X$;

(SP3) $A \in \mathcal{M}$, $B \subseteq X$ and $A \cap B \neq \emptyset$ imply $A \, p_\Theta \, B$;

(SP4) If $A \, p_\Theta \, B$ and $A \subseteq A' \in \mathcal{M}$ then $A' \, p_\Theta \, B$;

(SP5) $A \, \overline{p_\Theta} \, B$ implies there is a set $V \subseteq X$ such that $A \, \overline{p_\Theta} \, X \setminus V$ and $C \, \overline{p_\Theta} \, B$ for each $C \in \mathcal{M}$ with $C \subseteq V$.

If we relax (SP5) to the following condition

(SP5') $A \, p_\Theta \, B$ and $B \subseteq cl_{p_\Theta}(C)$ imply $A \, p_\Theta \, C$, where $cl_{p_\Theta}(C) := \{ x \in X \mid \{x\} \, p_\Theta \, C \}$,

then we obtain a *superproximity*, which reduces to a proximity in the sense of Leader, provided we set $\mathcal{M} = \boldsymbol{P}X$ and require *additivity*, i.e., (Add) $(B \cup C)\, p_\Theta\, A$ iff $B\, p_\Theta\, A$ or $C\, p_\Theta\, A$ with $B \cup C \in \mathcal{M}$.

Hence it is possible to describe Leader proximity spaces or topological spaces as special *superproximity* spaces.

Now, to study unifications and extensions in a more general setting, we define the category ***SN*** of supernear spaces and related maps and analyze its relationship to "symmetrical" extensions.

6 Definition. For a set X, a subset $\mathcal{B}^X$ is called a *prebornology*, or shortly ***B****-structure* or ***B****-set*, on X, and its elements are called *bounded sets*, if the following axioms are satisfied:

(B1) $B' \subseteq B \in \mathcal{B}^X$ implies $B' \in \mathcal{B}^X$;

(B2) $\emptyset \in \mathcal{B}^X$;

(B3) $x \in X$ implies $\{x\} \in \mathcal{B}^X$.

Given a pair of ***B***-structures $\mathcal{B}^X$, $\mathcal{B}^Y$ on sets X and Y, respectively, a map $f : X \to Y$ is called *bounded* iff

(b) $\{\, f[B] \mid B \in \mathcal{B}^X \,\} \subseteq \mathcal{B}^Y$.

7 Remark. The category ***BOUND***, whose objects are pairs $(X, \mathcal{B}^X)$, where X is a set and $\mathcal{B}^X$ is a ***B***-structure, and whose morphisms are bounded maps is *topological*, *cartesian closed* and has *universal one-point extensions*, hence ***BOUND*** is a *topological universe*!

8 Definition. For a ***B***-set $\mathcal{B}^X$ on X, a function $N : \mathcal{B}^X \to \boldsymbol{P}(\boldsymbol{P}(\boldsymbol{P}X))$ is called a *supernear operator* or a *supernearness* on $\mathcal{B}^X$, and the pair $(\mathcal{B}^X, N)$ is called a *supernear space* (*supernearness space*), iff

(SN1) $B \in \mathcal{B}^X$ and $\mathcal{H}_2 \ll \mathcal{H}_1 \in N(B)$ imply $\mathcal{H}_2 \in N(B)$;

(SN2) $N(\emptyset) = \{\emptyset\}$ and $\mathcal{B}^X \notin N(B)$ for each $B \in \mathcal{B}^X$;

(SN3) $B' \subseteq B \in \mathcal{B}^X$ implies $N(B') \subseteq N(B)$;

(SN4) $x \in X$ implies $\{\,\{x\}\} \in N(\{x\})$;

(SN5) $B \in \mathcal{B}^X$ and $\mathcal{H}_1 \cup \mathcal{H}_2 \in N(B)$ imply $\mathcal{H}_1 \in N(B)$ or $\mathcal{H}_2 \in N(B)$;

(SN6) $B \in \mathcal{B}^X$ and $\{\, cl_N(F) \mid F \in \mathcal{H} \,\} \in N(B)$ for some $\mathcal{H} \subseteq \boldsymbol{P}(\boldsymbol{P}X)$ imply $\mathcal{H} \in N(B)$, where $cl_N(F) := \{\, x \in X \mid \{\{x\}, F\} \in N(\{x\}) \,\}$.

Elements of $N(B)$ are called ***B****-near collections*. Given a pair of supernear spaces $(\mathcal{B}^X, N_X)$, $(\mathcal{B}^Y, N_Y)$, a bounded map $f : \mathcal{B}^X \to \mathcal{B}^Y$ is called a *supernear map* or shortly *sn-map*, iff (sn) $B \in \mathcal{B}^X$ and $\mathcal{H} \in N_X(B)$ imply $\{ f[F] \mid F \in \mathcal{H} \} \in N_Y(f[B])$.

A map will also be referred to as a *supernear* map by saying it preserves $\boldsymbol{B}$-near collections in the above sense. We denote by $\boldsymbol{SN}$ the corresponding category.

1 Example. (i) Given a superproximity p on $\mathcal{B}^X$, we obtain a supernear operator if for $B \in \mathcal{B}^X$ we set $N_p(B) := \{ \mathcal{H} \mid \mathcal{H} \subseteq p(B) \}$, where $p(B) := \{ F \subseteq X \mid B\, p\, F \}$;

(ii) Given a nearness space (X, ξ), we obtain a supernear operator on $\mathcal{B}^X := \boldsymbol{P}X$ by setting

$$N_\xi(B) := \begin{cases} \{\emptyset\} & \text{if } B = \emptyset; \\ \{ \mathcal{H} \mid \{B\} \cup \mathcal{H} \in \xi \} & \text{otherwise.} \end{cases}$$

(iii) $\boldsymbol{EXT}$ denotes the category whose objects are triples $(e, \mathcal{B}^X, Y)$ – called *extensions* – where $X = (X, cl_X)$, $Y = (Y, cl_Y)$ are topological spaces (given by closure operators), $\mathcal{B}^X$ is a $\boldsymbol{B}$-set on X and $e : X \to Y$ is a function satisfying the following conditions:

(E1) $A \in \boldsymbol{P}X$ implies $cl_X(A) = e^{-1}[cl_Y(e[A])]$;

(E2) $cl_Y(e[X]) = Y$, which means that the image of X under e is *dense* in Y.

Morphisms in $\boldsymbol{EXT}$ have the form $(f, g) : (e, \mathcal{B}^X, Y) \to (e', \mathcal{B}^{X'}, Y')$, where $f : X \to X'$, $g : Y \to Y'$ are *continuous* maps such that f is also *bounded*, and the following diagram *commutes*:

$$\begin{array}{ccc} X & \xrightarrow{\;e\;} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ X' & \xrightarrow[\;e'\;]{} & Y' \end{array}$$

One can *composed* $\boldsymbol{EXT}$-morphisms $(f, g) : (e, \mathcal{B}^X, Y) \to (e', \mathcal{B}^{X'}, Y')$ and $(f', g') : (e', \mathcal{B}^{X'}, Y') \to (e'', \mathcal{B}^{X''}, Y'')$ according to the rule $(f', g') \circ (f, g) := (f' \circ f, g' \circ g) : (e, \mathcal{B}^X, Y) \to (e'', \mathcal{B}^{X''}, Y'')$, where "$\circ$" denotes the *composition* of maps.

Now, for each $B \in \mathcal{B}^X$ we put

$$N^{(1)}_{\boldsymbol{EXT}}(B) := \begin{cases} \{\emptyset\} & \text{if } B = \emptyset; \\ \{ \mathcal{H} \mid e[B] \cap \bigcap \{ cl_Y(e[F]) \mid F \in \mathcal{H} \} \neq \emptyset \} & \text{otherwise.} \end{cases}$$

$$N^{(2)}_{\boldsymbol{EXT}}(B) := \{ \mathcal{H} \mid e[B] \in sec\{ cl_Y(e[F]) \mid F \in \mathcal{H} \} \}.$$

(iv) ***SEXT*** denotes the full subcategory of ***EXT*** whose objects are the *symmetric* extensions, that means its objects in addition satisfy the axiom

(SYM) $x \in X$ and $y \in cl_Y(\{e(x)\})$ imply $e(x) \in cl_Y(\{y\})$.
Now, for each $B \in \mathcal{B}^X$ we put

$$N^{(1)}_{\boldsymbol{SEXT}}(B) := \begin{cases} \{\emptyset\} & \text{if } B = \emptyset; \\ \{\, \mathcal{H} \mid \bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H} \cup \{B\}\,\} \neq \emptyset & \text{otherwise.} \end{cases}$$
$$N^{(2)}_{\boldsymbol{SEXT}}(B) := \{\, \mathcal{H} \mid cl_Y(e[B]) \in sec\{\, cl_Y(e[F]) \mid F \in \mathcal{H}\,\}\,\}.$$

9 Remark. Observe that axiom (E1) in the *definition* of an *extension* is automatically satisfied, if $e : X \rightarrow Y$ is a topological *embedding* with respect to the topologies determined by the closure operators cl_X and cl_Y. Note also that in *general* no *symmetry* or *separation* axiom is needed! Additionally, ordinary *prebornologies* on X are allowed, which need not coincide with the power $\boldsymbol{P}X$. More *specifically*, we recall that an extension is called T_1 (T_1-*extension*) iff

(T_1) $x \in X$ and $y \in cl_Y(\{e(x)\})$ imply $y = e(x)$.

This axiom *strengthens* the symmetry axiom. Moreover, if e is injective, then we have a topological *embedding* of X into Y.

Finally, we mention that an extension is called *strict* iff

(STR) $\{\, cl_Y(e[A]) \mid A \subseteq X\,\}$ is a *base* for the *closed* subsets of Y.

On the other hand, we recall that each symmetric topology "*cl*" on a given set defines a *compatible* Lodato proximity (=symmetrical Leader proximity) on the set by setting
$$B\,p\,A \quad \text{iff} \quad cl(B) \cap cl(A) \neq \emptyset.$$

In addition, we obtain a *compatible* nearness structure by setting

$$\mathcal{A} \in \xi \quad \text{iff} \quad \bigcap\{\, cl(F) \mid F \in \mathcal{A}\,\} \neq \emptyset.$$

Analogously – with respect to the above definitions – each *symmetric* extension gives rise to a *functorial* relationship between ***SEXT*** and ***SN*** (see also Example 1(iv)).

10 Theorem. *The category **SUPPROX** whose objects are the superproximity spaces is isomorphic to a full subcategory of **SN**.*

PROOF. With respect to Example 1(i) every superproximity p on $\mathcal{B}^X$ defines in this *natural* way a supernear operator N on $\mathcal{B}^X$, which is *endoformic*, i.e.,

(e) $B \in \mathcal{B}^X$ implies $\bigcup\{\, \mathcal{H} \mid \mathcal{H} \in N(B)\,\} \in N(B)$, where $\bigcup\{\, \mathcal{H} \mid \mathcal{H} \in N(B)\,\} := \{\, F \subseteq X \mid \exists \mathcal{H} \in N(B)\, F \in \mathcal{H}\,\}$.

Moreover, N is *grill-determined*, which means

(g) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply there exists a grill $\mathcal{G} \in N(B)$ such that $\mathcal{H} \subseteq \mathcal{G}$.

We recall that a *grill* $\mathcal{G}$ is a non-empty subset of the power $\boldsymbol{P}X$ such that

(g_1) $\emptyset \notin \mathcal{G}$;

(g_2) $stack\,\mathcal{G} \subseteq \mathcal{G}$;

(g_3) $G_1 \cup G_2 \in \mathcal{G}$ implies $G_1 \in \mathcal{G}$ or $G_2 \in \mathcal{G}$.

By setting $B\, q_M\, A$ iff $\{A\} \in M(B)$ for a given endoformic and grilldetermined supernear operator M, we get a *bijection* between the set of all superproximities and of all so-defined supernear operators on $\mathcal{B}^X$. With respect to the corresponding morphisms this yields an *isomorphism* between the above-mentioned categories. QED

11 Theorem. *The category* ***NEAR*** *is isomorphic to a full subcategory of* ***SN****.*

Proof. With respect to Example 1(ii) every nearness ξ on X defines a supernear operator N on $\boldsymbol{P}X$ that in addition is

(sa) *strongly additive*, which means $B_1 \cup B_2 \in \mathcal{B}^X$ implies $N(B_1 \cup B_2) \subseteq N(B_1) \cup N(B_2)$;

(ss) *strongly symmetric*, which means $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply $\{B\} \cup \mathcal{H} \in \bigcap\{\, N(F) \mid F \in (\mathcal{H} \cap \mathcal{B}^X) \cup \{B\}\,\}$, and

(ci) *closure-isotonic*, which means $cl_N(B) \in \mathcal{B}^X$ implies $N(cl_N(B)) \subseteq N(B)$.

By setting $\mathcal{A} \in \eta_M$ iff $\mathcal{A} \in \bigcap\{\, M(A) \,|\, A \in \mathcal{A}\,\}$ for a given supernear operator M on $\boldsymbol{P}X$ which satisfies the axioms (sa), (ss) and (ci), we get a *bijection* between the set of all nearness structures and of all so-defined supernear operators on $\boldsymbol{P}X$. With respect to the corresponding morphisms this yields an *isomorphism* between the above-mentioned categories. QED

12 Remark. We pointed out that $N^{(1)}_{\boldsymbol{SEXT}}$, as defined in Example 1(iv), is grill-determined and also satisfies the axioms (sa), (ss) and (ci), but is is not necessarily specified on $\boldsymbol{P}X$, nor is it endoformic.

Moreover we note that N_p, as defined in Example 1(i), in general satisfies *none* of the above-mentioned axioms in brackets. However, if the relation p is additive (see Remark 5) or symmetric, then N_p is *additive* respectively *symmetric* as well, which means

(a) $B_1 \cup B_2 \in \mathcal{B}^X$ and $\mathcal{H} \in N(B_1 \cup B_2)$ imply $\{F\} \in N(B_1) \cup N(B_2)$ for each $F \in \mathcal{H}$;

(s) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply $\{B\} \in \bigcap\{\, N(F) \mid F \in \mathcal{H} \cap \mathcal{B}^X \,\}$.

Note that a supernear operator that is strongly additive, respectively strongly symmetric, is additive, respectively symmetric, as well. Additionally we mention that each *symmetric* and *endoformic* supernear operator on $\boldsymbol{P}X$ is *automatically closure-isotonic*! Moreover, we can state that in *general* N_ξ, as defined in Example 1(ii), is not necessarily *grill-determined* nor is it *endoformic*. Furthermore we note that $N^{(2)}_{\boldsymbol{SEXT}}$, as defined in Example 1(iv), satisfies the axioms (a), (s), (ci), (e) and (g).

Finally, $N^{(1)}_{\boldsymbol{EXT}}$, as defined in Example 1(iii), is *grill-determined* and also *pointed*, i.e.,

(p) $B \in \mathcal{B}^X \setminus \{\emptyset\}$ implies $N(B) = \bigcup\{\, N(\{x\}) \mid x \in B \,\}$.

But in general $N^{(1)}_{\boldsymbol{EXT}}$ satisfies *none* of the above mentioned axioms (s), (ci), and (e). However, each pointed supernear operator is *strongly additive*!

1 Resumé. Since Leader proximity spaces as well as nearness spaces *essentially* can be described by corresponding "special" supernear spaces (see also Remark 5 and Theorem 11, respectively), our new concept of "supernearness spaces" is a *common generalization* of topological spaces, proximity spaces and uniform spaces (see also Herrlich's paper [8]). Moreover, supertopological spaces are subsumed by supernearness spaces as well.

1 Special extensions and related supernear operators

Well known "topological extensions" in the literature are the Smirnov-compactification of an Efremovic proximity space, or the T_1-extension related to Lodato proximity spaces, or, more generally, the "Herrlich-Bentley"-extension of the so-called "bunch-determined" nearness spaces.

All the above-mentioned constructions on a nearness structure can be viewed as special cases of a more general theory of symmetric extensions and their related supernear operators.

13 Definition. An extension $E := (e, \mathcal{B}^X, Y)$ is called a *power-extension* iff

(pow) $\mathcal{B}^X = \boldsymbol{P}X$.

14 Lemma. *For a symmetric power extension E let N^E be defined by setting for each $B \in \boldsymbol{P}X$*

$$N^E(B) := \begin{cases} \{\emptyset\} & \text{if } B = \emptyset \\ \{\,\mathcal{H} \mid \bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H} \cup \{B\}\,\} \neq \emptyset\,\} & \text{otherwise} \end{cases}$$

and

$$\xi^E := \{\,\mathcal{A} \mid \bigcap\{\, cl_Y(e[A]) \mid A \in \mathcal{A}\,\} \neq \emptyset\}.$$

Then the operators N^E and N_{ξ^E} coincide (see also Example 1(ii)).

PROOF. Straightforward. QED

15 Remark. With respect to Theorem 11 the induced nearness ξ^E essentially coincides with the special supernear operator N^E. Hence it is also possible to describe the “Herrlich-Bentley extension process” by corresponding supernear operators on $\boldsymbol{P}X$.

16 Lemma. *For a symmetric power extension E let N^E be defined by setting for each $B \in \boldsymbol{P}X$*

$$N^E(B) := \{\,\mathcal{H} \mid cl_Y(e[B]) \in sec\{\, cl_Y(e[F]) \mid F \in \mathcal{H}\,\}\,\}$$

and let δ^E be given by

$$B\,\delta^E\,A :\Longleftrightarrow cl_Y(e[B]) \cap cl_Y(e[A]) \neq \emptyset.$$

Then the operators N^E and N_{δ^E} coincide (see also Example 1(i)).

PROOF. Straightforward. QED

17 Remark. With respect to Remark 5 and Theorem 10, the induced Lodato proximity δ^E (where $\mathcal{B}^X$ is restricted to $\boldsymbol{P}X$) essentially coincides with the special supernear operator N^E. Hence it is also possible to describe the “Lodato-extension process” by means of supernear operators on $\boldsymbol{P}X$.

18 Definition. A power-extension $E = (e, \boldsymbol{P}X, Y)$ is called *compactly determined* (see also Doitchinov’s paper [5]), iff

(1) $e : X \rightarrow Y$ is injective;

(2) for any $y \in Y$ there exists a set $A \subseteq X$ such that $y \in cl_Y(e[A])$ and $cl_Y(e[A])$ is compact.

If, moreover, (Y, cl_Y) is a T_2-space, we call E a compactly determined *Hausdorff-extension.*

19 Remark. Doitchinov showed in his paper (see also [5]) that the compactly generated Hausdorff-extensions are *closely connected* with a class of supertopologies on X, which he called *b-supertopologies*. With respect to Definition 3 and Remark 5, b-supertopologies can be described by special superproximities. But these last mentioned structures are also a special case of corresponding supernear operators. Hence, Doitchinov's "extension process" are closely related to some special kind of supernear operators as well.

20 Definition. For an extension $E = (e, \mathcal{B}^X, Y)$ a supernear operator $N : \mathcal{B}^X \to \boldsymbol{P}^3 X$ is called *E-compatible* iff

(EC) $cl_N = cl_X$.

2 Example. (i) For an arbitrary extension E, the supernear operators $N^{(1)}_{\boldsymbol{EXT}}$ and $N^{(2)}_{\boldsymbol{EXT}}$, as defined in Example 1(iii), are E-compatible.

(ii) For any symmetric extension E, the supernear operators $N^{(1)}_{\boldsymbol{SEXT}}$ and $N^{(2)}_{\boldsymbol{SEXT}}$, as defined in Example 1(iv), are E-compatible.

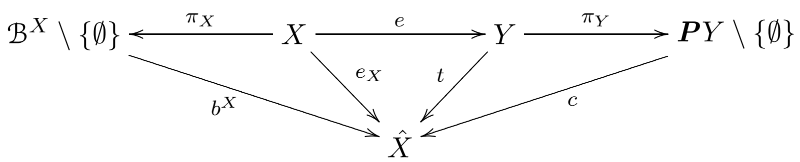
21 Definition. Let $(\mathcal{B}^X, N)$ be a supernear space. A grill $\mathcal{G} \subseteq \boldsymbol{P}X$ is called *N-compressed* iff

(NC) $cl_N(F) \in \mathcal{G}$ implies $F \in \mathcal{G}$.

22 Theorem. *For an extension $E := (e, \mathcal{B}^X, Y)$, let N be an E-compatible supernear operator. If $\hat{X}$ denotes the set of all N-compressed grills on X and*

$\pi_X(x) := \{x\}$	for each	$x \in X$;
$\pi_Y(y) := \{y\}$	for each	$y \in Y$;
$b^X(B) := \{\, T \subseteq X \mid B \cap cl_N(T) \neq \emptyset \,\}$	for each	$B \in \mathcal{B}^X \setminus \{\emptyset\}$;
$e_X(x) := \{\, T \subseteq X \mid x \in cl_N(T) \,\}$	for each	$x \in X$;
$t(y) := \{\, T \subseteq X \mid y \in cl_Y(e[T]) \,\}$	for each	$y \in Y$;
$c(D) := \{\, T \subseteq X \mid D \cap cl_Y(e[T]) \neq \emptyset \,\}$	for each	$D \in \boldsymbol{P}Y \setminus \{\emptyset\}$

then the following diagram commutes



Proof. Straightforward. QED

23 Remark. We also note that for each superproximity space $(\mathcal{B}^X, p)$ and for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$ the grill

$$\mathcal{L}^p(B) := \{ T \subseteq X \mid B \cap cl_{N_p}(T) \neq \emptyset \}$$

is N_p-compressed. Furthermore, $p(B)$ is N_p-compressed (see also Example 1(i)). In addition, for each nearness space (X, ξ) every ξ-bunch $\mathcal{G}$ is N_ξ-compressed.

24 Definition. Let $(\mathcal{B}^X, N)$ be a supernear space. For $B \in \mathcal{B}^X \setminus \{\emptyset\}$ a grill $\mathcal{G} \subseteq \boldsymbol{P}X$ is called

(i) *B-absorbed* iff $B \in \mathcal{G}$;

(ii) a *B-neargrill* iff $\mathcal{G} \in N(B)$;

(iii) a *B-clump* iff it is an N-compressed B-absorbed B-neargrill.

3 Example. (i) Let (X, ξ) be a nearness space. Then each ξ-bunch is an X-clump with respect to N_ξ (see also Remark 23).

(ii) Let (X, δ) be a Lodato proximity space and let σ be a bunch over X. Then σ is an X-clump with respect to N_δ (see also Example 1(i)).

(iii) In connection with Remark 23 we have that $\mathcal{L}^p(B)$ and $p(B)$, respectively, are also B-clumps with respect to N_p.

(iv) In general, $b^X(B)$ is an N-compressed grill that is B-absorbed (see also Theorem 22).

(v) $t(y)$ as well as $c(D)$ are N-compressed grills.

(vi) $e_X(x)$ is an $\{x\}$-clump; moreover, it is a maximal element in the set $N(\{x\}) \setminus \{\emptyset\}$, ordered by the natural inclusion "$\subseteq$".

Proof. We only show some parts of (vi). It is easy to see that $e_X(x)$ is an $\{x\}$-absorbed N-compressed grill. In proving that is an $\{x\}$-neargrill as well, we have that $\{ cl_N(T) | T \in e_X(x) \}$ corefines $\{\{x\}\} \in N(\{x\})$ (cf. (SN4)). Hence by (SN1) $\{ cl_N(T) | T \in e_X(x) \} \in N(\{x\})$, which by (SN6) implies $e_X(x) \in N(\{x\})$. It remains to establish the maximality of $e_X(x)$. To this end, let $\mathcal{H} \in N(\{x\})$ be nonempty and suppose $e_X(x) \subseteq \mathcal{H}$. By assumption we have $\{x\} \in \mathcal{H}$. For $F \in \mathcal{H}$ we then get $\{\{x\}, F\} \ll \mathcal{H}$. By (SN1) this implies $\{\{x\}, F\} \in N(\{x\})$, and hence $x \in cl_N(F)$. This means $F \in e_X(x)$. Therefore $e_X(x) = \mathcal{H}$, which proves the maximality of $e_X(x)$. QED

2 Symmetric extensions and related supernear operators

25 Theorem. *We obtain a functor* $F : \boldsymbol{SEXT} \to \boldsymbol{SN}$ *by setting*

(a) $F(E) := (\mathcal{B}^X, N^{(1)}_{\boldsymbol{SEXT}})$ *for a* $\boldsymbol{SEXT}$*-object* $E := (e, \mathcal{B}^X, Y)$*;*

(b) $F(f,g) := f$ *for a* $\boldsymbol{SEXT}$*-morphism* $(f,g) : E := (e, \mathcal{B}^X, Y) \to E' := (e', \mathcal{B}^{X'}, Y')$.

Proof. In view of Example 1(iv) and Remark 12 we already know that $F(E)$ is an object of $\boldsymbol{SN}$ that is strongly additive, strongly symmetric and closure-isotonic. Moreover, in connection with Example 2(ii) we will show that $N^{(1)}_{\boldsymbol{SEXT}}$ is also E-compatible.

We have to verify the equality of the closure operators cl_X and $cl_{N^{(1)}_{\boldsymbol{SEXT}}}$. So consider $A \in \boldsymbol{P}X$ and $x \in cl_X(A)$. Then, by (E1), $e(x) \in cl_Y(e[A]) \cap cl_Y(\{e(x)\})$, hence $\{\{x\}, A\} \in N^{(1)}_{\boldsymbol{SEXT}}(\{x\})$. Thus $x \in cl_{N^{(1)}_{\boldsymbol{SEXT}}}(A)$. Conversely, let $x \in cl_{N^{(1)}_{\boldsymbol{SEXT}}}(A)$. Then $\{\{x\}, A\} \in N^{(1)}_{\boldsymbol{SEXT}}(\{x\})$, which implies $y \in cl_Y(e[A]) \cap cl_Y(\{e(x)\})$ for some $y \in Y$. As a consequence of (SYM) we get $e(x) \in cl_Y(\{y\})$, hence $e(x) \in cl_Y(cl_Y(e[A])) \subseteq cl_Y(e[A])$ follows. In view of (E1) we have $x \in e^{-1}[cl_Y(e[A])] = cl_X(A)$, which was to be shown.

Now, let $(f,g) : E := (e, \mathcal{B}^X, Y) \to E' := (e', \mathcal{B}^{X'}, Y')$ be a $\boldsymbol{SEXT}$-morphism. It has to be shown that f preseves the near-collections from $F(E) = (\mathcal{B}^X, N^{(1)}_{\boldsymbol{SEXT}})$ to $F(E') = (\mathcal{B}^{X'}, N'^{(1)}_{\boldsymbol{SEXT}})$. Without loss of generality, let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N^{(1)}_{\boldsymbol{SEXT}}(B)$. By definition, $\bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H} \cup \{B\} \,\} \neq \emptyset$, hence $y \in \bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H} \,\}$ for some $y \in cl_Y(e[B])$. Or goal is to verify that $\bigcap\{\, cl_{Y'}(e'[f[F]]) \mid F \in \mathcal{H} \cup \{B\} \,\} \neq \emptyset$. By hypothesis, we have $g(y) \in g[cl_Y(e[B])]$ and therefore $g(y) \in cl_{Y'}(g[e[B]]) = cl_{Y'}(e'[f[B]])$, since (f,g) is a $\boldsymbol{SEXT}$-morphism. Now consider some $F \in \mathcal{H}$. Because $y \in cl_Y(e[F])$, we have $g(y) \in cl_{Y'}(e'[f[F]])$, which results in $\{\, f[F] \mid F \in \mathcal{H} \,\} \in N^{(1)}_{\boldsymbol{SEXT}}(f[B])$. QED

3 Supernear operators and related symmetric extensions

In the previous section we have defined a functor from $\boldsymbol{SEXT}$ to $\boldsymbol{SN}$. Now we are going to introduce a related functor in the *opposite* direction.

26 Lemma. *Let* $(\mathcal{B}^X, N)$ *be a supernear space. We put*

$$\hat{X} := \{\, \mathcal{C} \subseteq \boldsymbol{P}X \mid \exists B \in \mathcal{B}^X \setminus \{\emptyset\}\ \mathcal{C} \text{ is a } B\text{-clump} \,\}$$

and for each $\hat{A} \subseteq \hat{X}$ *we set*

$$cl_{\hat{X}}(\hat{A}) := \{\, \mathcal{C} \in \hat{X} \mid \bigcap \hat{A} \subseteq \mathcal{C} \,\}\ ,$$

where $\bigcap \hat{A} := \{ F \subseteq X \mid \forall \mathcal{C} \in \hat{A}\, F \in \mathcal{C} \}$ *(so that, by convention,* $\bigcap \hat{A} = \boldsymbol{P}X$ *if* $\hat{A} = \emptyset$*). Then* $\mathrm{cl}_{\hat{X}}$ *is a topological closure operator on* $\hat{X}$*.*

Proof. Since according to (g_1) $\emptyset \notin \mathcal{C}$ for each $\mathcal{C} \in \hat{X}$, we first note that $\mathcal{C} \notin cl_{\hat{X}}(\emptyset)$. Let $\hat{A}$ be a subset of $\hat{X}$ and consider $\mathcal{C} \in \hat{A}$. Then $F \in \bigcap \hat{A}$ implies $F \in \mathcal{C}$, hence $\hat{A} \subseteq cl_{\hat{X}}(\hat{A})$. Now consider $\hat{A}_1 \subseteq \hat{A}_2 \subseteq \hat{X}$. Then $\bigcap \hat{A}_2 \subseteq \bigcap \hat{A}_1$, which implies $cl_{\hat{X}}(\hat{A}_1) \subseteq cl_{\hat{X}}(\hat{A}_2)$. Now let $\hat{A}_1 \subseteq \hat{A}_2$ be arbitrary subsets of $\hat{X}$ and consider an element $\mathcal{C} \in \hat{X}$ such that $\mathcal{C} \notin cl_{\hat{X}}(\hat{A}_1) \cup cl_{\hat{X}}(\hat{A}_2)$. Then we have $\bigcap \hat{A}_1 \nsubseteq \mathcal{C}$ and $\bigcap \hat{A}_2 \nsubseteq \mathcal{C}$. Choose $F_1 \in \bigcap \hat{A}_1$ with $F_1 \notin \mathcal{C}$ and $F_2 \in \bigcap \hat{A}_2$ with $F_2 \notin \mathcal{C}$. By (g_3) we get $F_1 \cup F_2 \notin \mathcal{C}$. On the other hand, by (g_2) we have $F_1 \cup F_2 \in \big(\bigcap \hat{A}_1\big) \cup \big(\bigcap \hat{A}_2\big) = \bigcap \big(\hat{A}_1 \cup \hat{A}_2\big)$, hence $\mathcal{C} \notin cl_{\hat{X}}(\hat{A}_1 \cup \hat{A}_2)$. At last, let $\mathcal{C}$ be an element of $cl_{\hat{X}}(cl_{\hat{X}}(\hat{A}))$ and suppose $\mathcal{C} \notin cl_{\hat{X}}(\hat{A})$. Choose $F \in \bigcap \hat{A}$ with $F \notin \mathcal{C}$. By assumption, we have $\bigcap cl_{\hat{X}}(\hat{A}) \subseteq \mathcal{C}$, hence $F \notin \bigcap cl_{\hat{X}}(\hat{A})$. Choose $\mathcal{D} \in cl_{\hat{X}}(\hat{A})$ satisfying $F \notin \mathcal{D}$. Then $\bigcap \hat{A} \subseteq \mathcal{D}$. Hence $F \in \mathcal{D}$, which leads us to a contradiction. QED

27 Theorem. *For supernear spaces* $(\mathcal{B}^X, N_1)$ *and* $(\mathcal{B}^Y, N_2)$ *let* $f : X \to Y$ *be an sn-map. Define a function* $\hat{f} : \hat{X} \to \hat{Y}$ *by setting for each* $\mathcal{C} \in \hat{X}$*:*

$$\hat{f}(\mathcal{C}) := \{ D \subseteq Y \mid f^{-1}[cl_{N_2}(D)] \in \mathcal{C} \}.$$

Then the following statements are valid:

(1) $\hat{f}$ *is a continuous map from* $(X, \mathrm{cl}_{\hat{X}})$ *to* $(Y, \mathrm{cl}_{\hat{Y}})$*.*

(2) The composites $\hat{f} \circ e_X$ *and* $e_Y \circ f$ *coincide, where* $e_X : X \to \hat{X}$ *denotes the function which assigns the* $\{x\}$*-clump* $e_X(x)$ *to each* $x \in X$ *(see also Theorem 22 or Example 3(vi)).*

(3) $f\mathcal{C} \subseteq \hat{f}(\mathcal{C})$ *for each* $\mathcal{C} \in \hat{X}$*, where* $f\mathcal{C} := \{ f[F] \mid f \in \mathcal{C} \}$*.*

(4) $\bigcap e_X[B] := \bigcap \{ e_X(x) \mid x \in B \} = \{ F \subseteq X \mid B \subseteq cl_{N_1}(F) \}$ *for every* $B \subseteq X$*.*

Proof. First, let $\mathcal{C}$ be a B-clump with respect to N_1. We must show that $\hat{f}(\mathcal{C})$ is an $f[B]$-clump with respect to N_2. It is easy to show that $\hat{f}(\mathcal{C})$ is an N_2-compressed grill. In order to establish that $\hat{f}(\mathcal{C})$ is an $f[B]$-neargrill, we observe that $\mathcal{C} \in N(B)$ by hypothesis. We will verify that

$$\{ cl_{N_2}(D) \mid D \in \hat{f}(\mathcal{C}) \} \ll f\mathcal{C} \in N_2(f[B]).$$

(Note that f is an sn-map.) For any $D \in \hat{f}(\mathcal{C})$ we have $f^{-1}[cl_{N_2}(D)] \in \mathcal{C}$ and hence $cl_{N_2}(D) \supseteq f[f^{-1}[cl_{N_2}(D)]] \in f\mathcal{C}$. Since $\mathcal{C}$ is B-absorbed, we get $B \in \mathcal{C}$ and $B \subseteq f^{-1}[f[B]] \subseteq f^{-1}[cl_{N_2}(f[B])]$, and therefore $f^{-1}[cl_{N_2}(f[B])] \in \mathcal{C}$, which shows that $\hat{f}(\mathcal{C})$ is $f[B]$-absorbed. Consequently, $\hat{f}(\mathcal{C}) \in \hat{Y}$.

(1) Let $\hat{A} \subseteq \hat{X}$, $\mathcal{C} \in cl_{\hat{X}}(\hat{A})$ and suppose $\hat{f}(\mathcal{C}) \notin cl_{\hat{Y}}(\hat{f}[\hat{A}])$. Then $\bigcap \hat{f}[\hat{A}] \nsubseteq \hat{f}(\mathcal{C})$, hence $F \notin \hat{f}(\mathcal{C})$ for some $F \in \bigcap \hat{f}[\hat{A}]$, which means $f^{-1}[cl_{N_2}(F)] \notin \mathcal{C}$. Since $\bigcap \hat{A} \subseteq \mathcal{C}$, we have $f^{-1}[cl_{N_2}(F)] \notin \mathcal{D}$ for some $\mathcal{D} \in \hat{A}$. Therefore $F \notin \hat{f}(\mathcal{D})$, which leads us to a contradiction, because $F \in \bigcap \hat{f}[\hat{A}]$.

(2) Let x be an element of X. We will prove the validity of $\hat{f}(e_X(x)) = e_Y(f(x))$. To this end, let $F \in e_Y(f(x))$. Then $f(x) \in cl_{N_2}(F)$, hence $x \in f^{-1}[cl_{N_2}(F)]$, and consequently $f^{-1}[cl_{N_2}(F)] \in e_X(x)$. Thus $F \in \hat{f}(e_X(x))$, proving the inclusion $e_Y(f(x)) \subseteq \hat{f}(e_X(x))$. Since $e_Y(f(x))$ is maximal with respect to $(N_2(\{f(x)\}) \setminus \{\emptyset\}, \subseteq)$ – see Example 3(vi) and also note that $\{\, cl_{N_2}(D) \mid D \in \hat{f}(e_X(x)) \,\} \ll f\, e_X(x)$, since by hypothesis f is an sn-map – we obtain the desired equality.

(3) Let $\mathcal{C}$ be an element of $\hat{X}$ and $D := f[F]$ for some $F \in \mathcal{C}$. Then, according to (g_2), $F \subseteq f^{-1}[D] \subseteq f^{-1}[cl_{N_2}[D]] \in \mathcal{C}$, which yields $D \in \hat{f}(\mathcal{C})$.

(4) Straightforward.

QED

1 Remark. In view of Theorem 11 and Remark 12 we summarize that the supernear operators N_ξ and $N^{(1)}_{\boldsymbol{SEXT}}$ both satisfy the axioms (sa), (ss) and (ci).

28 Definition. A supernear operator on a B-set, and also the corresponding space, is called *strong*, if the above-mentioned axioms for the operator are satisfied. Moreover, we denote by ***SSN*** the corresponding full subcategory of ***SN***.

29 Theorem. *We obain a functor $G : \boldsymbol{SSN} \to \boldsymbol{SEXT}$ by setting*

(a) $G(\mathcal{B}^X, N) := (e_X, \mathcal{B}^X, \hat{X})$ for any strong supernear space $(\mathcal{B}^X, N)$ with $X = (X, cl_N)$ and $\hat{X} = (\hat{X}, cl_{\hat{X}})$;

(b) $G(f) := (f, \hat{f})$ for any sn-map $f : (\mathcal{B}^X, N) \to (\mathcal{B}^Y, N')$.

Proof. In view of (SN6) it is straightforward to verify that cl_N is a topological closure operator on X. By Lemma 26, we also have the topological closure operator $cl_{\hat{X}}$ on $\hat{X}$. Therefore we obtain topological spaces with B-structure $\mathcal{B}^X$, and $e_X : X \to \hat{X}$ is a continuous map according to Theorem 27.

To establish (E1), let A be a subset of X and suppose $x \in cl_N(A)$. Then, by Theorem 27(4), the inclusion $\bigcap e_X[A] \subseteq e_X(x)$ follows. This means that $e_X(x) \in cl_{\hat{X}}(e_X[A])$, hence $x \in e_X^{-1}[cl_{\hat{X}}(e_X[A])]$. Conversely, let x be an element of $e_X^{-1}[cl_{\hat{X}}(e_X[A])]$. Then by definition we have $e_X(x) \in cl_{\hat{X}}(e_X[A])$, and consequently $\bigcap e_X[A] \subseteq e_X(x)$. By Theorem 27(4) we obtain $A \in e_X(x)$, which means $x \in cl_N(A)$.

To establish (E2), let $\mathcal{C} \in \hat{X}$ and suppose $\mathcal{C} \notin cl_{\hat{X}}(e_X[X])$. By definition we get $\bigcap e_X[X] \nsubseteq \mathcal{C}$, so that there exists a set $F \in \bigcap e_X[X]$ with $F \notin \mathcal{C}$. By Theorem 27(4) the inclusion $X \subseteq cl_N(F)$ holds. Since $\mathcal{C} \neq \emptyset$ and in view of axiom (g_2), we get $cl_N(F) \in \mathcal{C}$, hence $F \in \mathcal{C}$, because $\mathcal{C}$ is N-compressed. But this is a contradiction, which shows $\mathcal{C} \in cl_{\hat{X}}(e_X[X])$.

To establish (SYM), let x be an element of X such that $\mathcal{C} \in cl_{\hat{X}}(\{e_X(x)\})$. We must show $e_X(x) \in cl_{\hat{X}}(\{\mathcal{C}\})$. By hypothesis we have $e_X(x) \subseteq \mathcal{C}$ and moreover $\mathcal{C} \in N(B)$ for some $B \in \mathcal{B}^X \setminus \{\emptyset\}$. Since $\{x\} \in \mathcal{C}$ and since N is strongly symmetric, we get $\{B\} \cup \mathcal{C} \in N(\{x\})$ with $\mathcal{C} \ll \{B\} \cup \mathcal{C}$. According to (SN1) we then get $\mathcal{C} \in N(\{x\})$, and since $e_X(x)$ is maximal with respect to $(N(\{x\}) \setminus \{\emptyset\}, \subseteq)$ (see also Example 3(vi)), $\mathcal{C}$ coincides with $e_X(x)$.

By hypothesis $f : (\mathcal{B}^X, N) \twoheadrightarrow (\mathcal{B}^Y, N')$ is an sn-map so that f is continuous and bounded from $(\mathcal{B}^X, cl_N)$to $(\mathcal{B}^Y, cl_{N'})$. It remains to prove that the following diagram commutes:

$$\begin{array}{ccc} X & \xrightarrow{e_X} & \hat{X} \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \hat{f}} \\ Y & \xrightarrow[e_Y]{} & \hat{Y} \end{array}$$

To this end let x be an element of X. We must show $(\hat{f} \circ e_X)(x) = (e_Y \circ f)(x)$.

"$\subseteq$": $D \in (\hat{f} \circ e_X)(x)$ implies $D \in \hat{f}(e_X(x))$, which means $f^{-1}[cl_{N'}(D)] \in e_X(x)$, hence $x \in cl_N(f^{-1}[cl_{N'}(D)])$. Especially, since f is continuous, we have $f(x) \in cl_{N'}(f[f^{-1}[cl_{N'}(D)]])$. But now $cl_{N'}(cl_{N'}(D)) \subseteq cl_{N'}(D)$ implies $D \in e_Y(f(x))$.

"$\supseteq$": $D \in e_Y(f(x))$ implies $f(x) \in cl_{N'}(D)$, hence $x \in f^{-1}[cl_{N'}(D)])$ and consequently $x \in cl_N(f^{-1}[cl_{N'}(D)])$. This implies $f^{-1}[cl_{N'}(D)] \in e_X(x)$, which means $D \in \hat{f}(e_X(x))$. Finally, this establishes that the composition of sn-maps is preserved by G. QED

2 Remark. We denote by ***STREXT*** the full subcategory of ***EXT*** whose objects are those triples $(e, \mathcal{B}^X, Y)$ for which $\{\, cl_Y(e[A]) \mid A \subseteq X \,\}$ is a base for the closed subsets of Y. (Banaschewski has called these extensions *strict*, see also Remark 9.)

30 Theorem. *Let G : **SSN** $\twoheadrightarrow$ **SEXT** be the functor of Theorem 29. Then the image of G is contained in **STREXT**.*

Proof. For a strong supernear space $(\mathcal{B}^X, N)$ set $G(\mathcal{B}^X, N) := (e_X, \mathcal{B}^X, \hat{X})$. To show that $(e_X, \mathcal{B}^X, \hat{X})$ is strict, consider $\mathcal{C} \in \hat{X}$ and let $\hat{A}$ be closed in $\hat{X}$ with $\mathcal{C} \notin \hat{A}$. Then $\mathcal{C} \notin cl_{\hat{X}}(\hat{A})$ and so $\bigcap \hat{A} \nsubseteq \mathcal{C}$. There exists $F \in \bigcap \hat{A}$ such that $F \notin \mathcal{C}$. Now for each $D \in \hat{A}$ we have $F \in \mathcal{D}$, which implies $\bigcap e_X[F] \subseteq D$, and therefore we conclude $\mathcal{D} \in cl_{\hat{X}}(e_X[F])$. Since $F \notin \mathcal{C}$ we have $\bigcap e_X[F] \nsubseteq \mathcal{C}$ and so $\mathcal{C} \notin cl_{\hat{X}}(e_X[F])$. QED

4 Clump-determined strong supernear spaces

31 Theorem. *Let $F : \boldsymbol{SEXT} \to \boldsymbol{SN}$ and $G : \boldsymbol{SSN} \to \boldsymbol{SEXT}$ be the functors given in Theorems 25 and 29. For each object $(\mathcal{B}^X, N)$ of $\boldsymbol{SSN}$ let $t(\mathcal{B}^X, N)$ denote the identity map $t(\mathcal{B}^X, N) = id_X : F(G(\mathcal{B}^X, N)) \to (\mathcal{B}^X, N)$. Then $t : F \circ G \to 1_{\boldsymbol{SSN}}$ is a natural transformation from $F \circ G$ to the identity functor $1_{\boldsymbol{SSN}}$, i.e., $id_X : F(G(\mathcal{B}^X, N)) \to (\mathcal{B}^X, N)$ is an sn-map for each object $(\mathcal{B}^X, N)$, and the following diagram commutes for each sn-map $f : (\mathcal{B}^X, N_1) \to (\mathcal{B}^Y, N_2)$:*

$$\begin{array}{ccc} F(G(\mathcal{B}^X, N_1)) & \xrightarrow{id_X} & (\mathcal{B}^X, N_1) \\ {\scriptstyle F(G(f))}\Big\downarrow & & \Big\downarrow{\scriptstyle f} \\ F(G(\mathcal{B}^Y, N_2)) & \xrightarrow[id_Y]{} & (\mathcal{B}^Y, N_2) \end{array}$$

Proof. The commutativity of the diagram is obvious, because $F(G(f)) = f$. It remains to prove that $id_X : F(G(\mathcal{B}^X, N)) \to (\mathcal{B}^X, N)$ is an sn-map for each object $(\mathcal{B}^X, N)$ of $\boldsymbol{SSN}$.

To fix the notation, let N be such that $F(G(\mathcal{B}^X, N_1)) = F(e_X, \mathcal{B}^X, \hat{X}) = (\mathcal{B}^X, N)$. It suffices to show that for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$ we have $N(B) \subseteq N_1(B)$. To this end, assume $\mathcal{H} \in N(B)$. Then $\bigcap\{\, cl_{\hat{X}}(e_X[F]) | F \in \mathcal{H} \cup \{B\}\,\} \neq \emptyset$. Choose $\mathcal{C} \in cl_{\hat{X}}(e_X[B])$ such that $\mathcal{C} \in \bigcap\{\, cl_{\hat{X}}(e_X[F]) \mid F \in \mathcal{H}\,\}$, hence $\bigcap e_X[B] \subseteq \mathcal{C}$. In view of Theorem 27(4) we get $B \in \mathcal{C}$ and $\mathcal{C} \in N_1(B')$ for some $B' \in \mathcal{B}^X \setminus \{\emptyset\}$ (note in particular that $\mathcal{C}$ is a B'-neargrill for some bounded set B'). Since N_1 is strongly symmetric, we get $\{B'\} \cup \mathcal{C} \in N_1(B)$ and $\mathcal{C} \ll \{B'\} \cup \mathcal{C}$, hence $\mathcal{C} \in N_1(B)$, according to (SN1).

We will now show $\mathcal{H} \subseteq \mathcal{C}$. Any element F of $\mathcal{H}$ satisfies $\mathcal{C} \in cl_{\hat{X}}(e_X[F])$, hence $\bigcap e_X[F] \subseteq \mathcal{C}$. Since by Theorem 27(4) $F \in \bigcap e_X[F]$, we have $F \in \mathcal{C}$, which concludes the proof.

QED

To capture the whole "extension process", we only need to characterize those strong supernear spaces, for which t is a natural equivalence.

32 Definition. A strong supernear space $(\mathcal{B}^X, N)$, and also the supernear operator N, are called *clump-determined*, if $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and $\mathcal{H} \in N(B)$ imply the existence of a B-clump $\mathcal{C}$ such that $\mathcal{H} \subseteq \mathcal{C}$.

33 Example. In view of Example 1(iii) we point out that $N^{(1)}_{\boldsymbol{SEXT}}$ automatically is clump-determined. This can easily be seen as follows.

Proof. Let $B \in \mathcal{B}^X \setminus \{\emptyset\}$ and consider an element $\mathcal{H}$ of $N^{(1)}_{\boldsymbol{SEXT}}(B)$. Then by definition $\bigcap\{\, cl_Y(e[F] \mid F \in \mathcal{H} \cup \{B\}\,\} \neq \emptyset$. Choose $y \in cl_Y(e[B])$ such that $y \in \bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H}\,\}$. In view of Example 3(v) we have $t(y)$ is an $N^{(1)}_{\boldsymbol{SEXT}}$-compressed grill (see also Remark 12), and by hypothesis it is B-absorbed. Moreover, by definition, $t(y) \in N^{(1)}_{\boldsymbol{SEXT}}(B)$, hence $t(y)$ is a B-clump. Finally, for $F \in \mathcal{H}$ we have $y \in cl_Y(e[F])$, hence $F \in t(y)$, which concludes the proof. $\boxed{QED}$

34 Lemma. *Let* $\boldsymbol{CSSN}$ *denote the full subcategory of* $\boldsymbol{SSN}$*, whose objects are the clump-determined strong supernear spaces. Let* F *and* G *be the functors as defined above, and for each object* $(\mathcal{B}^X, N)$ *of* $\boldsymbol{CSSN}$ *let* $t(\mathcal{B}^X, N)$ *be given in the same way as in Theorem 31. If* G *and* F *are restricted, respectively, corestricted to* $\boldsymbol{CSSN}$*, then* $t : F \circ G \to 1_{\boldsymbol{CSSN}}$ *is a natural equivalence from* $F \circ G$ *to the identity functor* $1_{\boldsymbol{CSSN}}$

Proof. In view of Theorem 31 and Example 33, it remains to show that for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$ we have $N_1(B) \subseteq N(B)$. For a non-empty bounded set $B \in \mathcal{B}^X$ consider $\mathcal{H} \in N_1(B)$. Since N_1 in particular is clump-determined, we can choose a B-clump $\mathcal{C}$ such that $\mathcal{H} \subseteq \mathcal{C}$. In order to show $\mathcal{H} \in N(B)$, we need to verify $\mathcal{C} \in \bigcap\{\, cl_{\hat{X}}(e_X[F]) \mid F \in \mathcal{H} \cup \{B\}\,\}$. Therefore it suffices to prove the following claims

(1) $\mathcal{C} \in cl_{\hat{X}}(e_X[B])$, and

(2) $F \in \mathcal{H}$ implies $\mathcal{C} \in cl_{\hat{X}}(e_X[F])$.

(1): By definition of $cl_{\hat{X}}$, it suffices to establish $\bigcap e_X[B] \subseteq \mathcal{C}$. So let D be an element of $\bigcap e_X[B]$, which means $B \subseteq cl_{N_1}(D)$. Since $\mathcal{C}$ is B-absorbed, we get $cl_{N_1}(D) \in \mathcal{C}$. But $\mathcal{C}$ is also N_1-compressed, consequently we get $D \in \mathcal{C}$.

(2): Let F be an element of $\mathcal{H}$ and let D be an element of $\bigcap e_X[F]$, which means $F \subseteq cl_{N_1}(D)$. Since $F \in \mathcal{C}$ by hypothesis, we get $cl_{N_1}(D) \in \mathcal{C}$, and analogously as above we infer $D \in \mathcal{C}$, which concludes the proof. $\boxed{QED}$

Now we are able to formulate the main theorem of this paper, which is a consequence of the preceding Lemmata and Theorems, respectively.

35 Theorem. *Let* $(\mathcal{B}^X, N)$ *be a strong supernear space. Then the following are equivalent:*

(i) $(\mathcal{B}^X, N)$ *is clump-determined;*

(ii) there exists a $\boldsymbol{SEXT}$*-object* $(e, \mathcal{B}^X, Y)$ *that is strict in the sense that for each* $B \in \mathcal{B}^X \setminus \{\emptyset\}$ *the elements* $\mathcal{H} \in N(B)$ *are characterized by* $\bigcap\{\, cl_Y(e[F]) \mid F \in \mathcal{H} \cup \{B\}\,\} \neq \emptyset$*;*

(iii) there exists a topological space (Y, cl_Y) *and a map* $f : X \to Y$ *that satisfies*

- $cl_N(A) = f^{-1}[cl_Y(f[A])$ *for each* $A \subseteq X$ *;*
- $f[X]$ *is dense in* Y*;*

– *$\{ cl_Y(e[A]) \mid A \subseteq X \}$ forms a base for the closed subsets of Y; and*
– *for each $B \in \mathcal{B}^X \setminus \{\emptyset\}$ the elements $\mathcal{H} \in N(B)$ are characterized by $\bigcap\{ cl_Y(e[F]) \mid F \in \mathcal{H} \cup \{B\} \} \neq \emptyset$.*

36 Remark. It seems to be of interest to study the question, which special supernear operators are obtained by given compact (Hausdorff) extensions.

Moreover, can it be shown that the functors constructed above are adjoints, and, if not, which additional conditions are needed to guarantee this?

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