Selection principles
and the Minimal Tower problem

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Abstract. We study diagonalizations of covers using various selection principles, where the covers are related to linear quasiorderings (τ-covers). This includes: equivalences and nonequivalences, combinatorial characterizations, critical cardinalities and constructions of special sets of reals. This study leads to a solution of a topological problem which was suggested to the author by Scheepers (and stated in [15]) and is related to the Minimal Tower problem.

We also introduce a variant of the notion of τ-cover, called τ^*-cover, and settle some problems for this variant which are still open in the case of τ-covers. This new variant introduces new (and tighter) topological and combinatorial lower bounds on the Minimal Tower problem.

Keywords: Gerlits-Nagy property γ-sets, γ-cover, ω-cover, τ-cover, tower, selection principles, Borel covers, open covers.

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1 Introduction

1.1 Combinatorial spaces

We consider zero-dimensional sets of real numbers. For convenience, we may consider other spaces with more evident combinatorial structure, such as the *Baire space* $^{\mathbb{N}}\mathbb{N}$ of infinite sequences of natural numbers, and the *Cantor space* $^{\mathbb{N}}\{0,1\}$ of infinite sequences of "bits" (both equipped with the product topology). The Cantor space can be identified with $P(\mathbb{N})$ using characteristic functions. We will often work in the subspace $P_{\infty}(\mathbb{N})$ of $P(\mathbb{N})$, consisting of the infinite sets of natural numbers. These spaces, as well as any separable, zero-dimensional metric space, are homeomorphic to sets of reals, thus our results about sets of reals can be thought of as talking about this more general case.

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[i]This paper constitutes a part of the author's doctoral dissertation at Bar-Ilan University.

1.2   Selection principles

Let $\mathfrak{U}$ and $\mathfrak{V}$ be collections of covers of a space X. The following selection hypotheses have a long history for the case when the collections $\mathfrak{U}$ and $\mathfrak{V}$ are topologically significant.

$\mathsf{S}_1(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{V_n\}_{n\in\mathbb{N}}$ such that for each n $V_n \in \mathcal{U}_n$, and $\{V_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

$\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$, there is a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that each $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$.

$\mathsf{U}_{fin}(\mathfrak{U}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathfrak{U}$ *which do not contain a finite subcover*, there exists a sequence $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ such that for each n $\mathcal{F}_n$ is a finite (possibly empty) subset of $\mathcal{U}_n$, and $\{\cup \mathcal{F}_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

We make the convention that

> *The space X is infinite and all covers we consider are assumed not to have X as an element.*

An ω-cover of X is a cover such that each finite subset of X is contained in some member of the cover. It is a γ-cover if it is infinite, and each element of X belongs to all but finitely many members of the cover. Following [7] and [13], we consider the following types of covers:

$\mathcal{O}$ (respectively, $\mathcal{B}$): The collection of countable open (respectively, Borel) covers of X.

Ω (respectively, $\mathcal{B}_\Omega$): The collection of countable open (respectively, Borel) ω-covers of X.

Γ (respectively, $\mathcal{B}_\Gamma$): The collection of countable open (respectively, Borel) γ-covers of X.

The inclusions among these classes can be summarized as follows:

$$
\begin{array}{ccccc}
\mathcal{B}_\Gamma & \to & \mathcal{B}_\Omega & \to & \mathcal{B} \\
\uparrow & & \uparrow & & \uparrow \\
\Gamma & \to & \Omega & \to & \mathcal{O}
\end{array}
$$

These inclusions and the properties of the selection hypotheses lead to a complicated diagram depicting how the classes defined this way interrelate. However, only a few of these classes are really distinct. Figure 1 contains the distinct ones among these classes, together with their critical cardinalities, which were derived in [7] and in [13]; see definition in Section 3. The only unsettled implications in this diagram are marked with dotted arrows.

$\mathsf{U}_{fin}(\Gamma,\Gamma)$ $\longrightarrow$ $\mathsf{U}_{fin}(\Gamma,\Omega)$ $\longrightarrow$ $\mathsf{U}_{fin}(\Gamma,\mathcal{O})$
$\mathfrak{b}$ $\mathfrak{d}$ $\mathfrak{d}$

$\mathsf{S}_{fin}(\Gamma,\Omega)$
$\mathfrak{d}$

$\mathsf{S}_1(\Gamma,\Gamma)$ $\longrightarrow$ $\mathsf{S}_1(\Gamma,\Omega)$ $\longrightarrow$ $\mathsf{S}_1(\Gamma,\mathcal{O})$
$\mathfrak{b}$ $\mathfrak{d}$ $\mathfrak{d}$

$\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$ $\longrightarrow$ $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega)$ $\longrightarrow$ $\mathsf{S}_1(\mathcal{B}_\Gamma,\mathcal{B})$
$\mathfrak{b}$ $\mathfrak{d}$ $\mathfrak{d}$

$\mathsf{S}_{fin}(\Omega,\Omega)$
$\mathfrak{d}$

$\mathsf{S}_{fin}(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$
$\mathfrak{d}$

$\mathsf{S}_1(\Omega,\Gamma)$ $\longrightarrow$ $\mathsf{S}_1(\Omega,\Omega)$ $\longrightarrow$ $\mathsf{S}_1(\mathcal{O},\mathcal{O})$
$\mathfrak{p}$ $\mathrm{cov}(\mathcal{M})$ $\mathrm{cov}(\mathcal{M})$

$\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Gamma)$ $\longrightarrow$ $\mathsf{S}_1(\mathcal{B}_\Omega,\mathcal{B}_\Omega)$ $\longrightarrow$ $\mathsf{S}_1(\mathcal{B},\mathcal{B})$
$\mathfrak{p}$ $\mathrm{cov}(\mathcal{M})$ $\mathrm{cov}(\mathcal{M})$
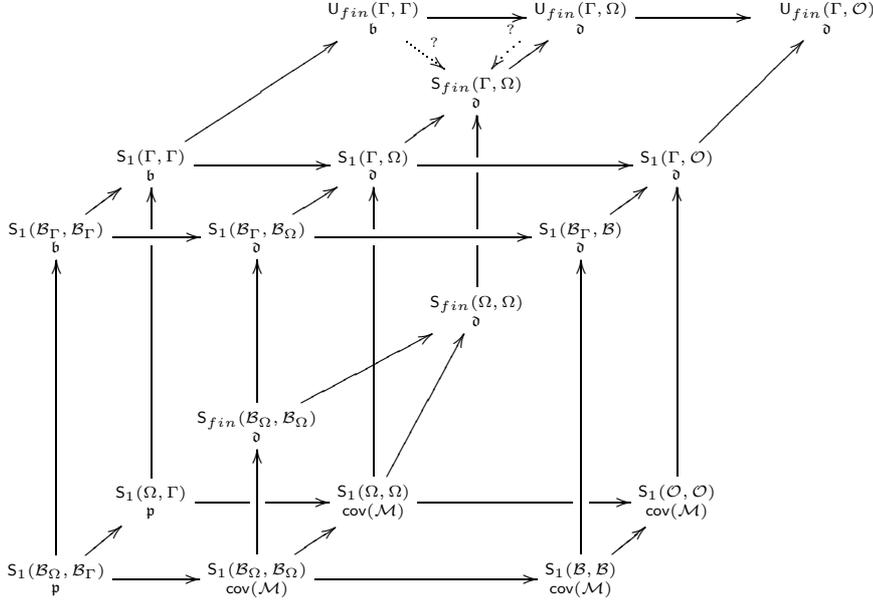
Figure 1. The surviving classes

1.3 τ-covers

A cover of a space X is *large* if each element of X is covered by infinitely many members of the cover. Following [15], we consider the following type of cover. A *large* cover $\mathcal{U}$ of X is a *τ-cover* of X if for each $x,y \in X$ we have either $x \in U$ implies $y \in U$ for all but finitely many members U of the cover $\mathcal{U}$, or $y \in U$ implies $x \in U$ for all but finitely many $U \in \mathcal{U}$.

A *quasiordering* $\preccurlyeq$ on a set X is a reflexive and transitive relation on X. It is *linear* if for all $x,y \in X$ we have $x \preccurlyeq y$ or $y \preccurlyeq x$. A τ-cover $\mathcal{U}$ of a space X induces a linear quasiordering $\preccurlyeq$ on X by:

$$x \preccurlyeq y \Leftrightarrow x \in U \to y \in U \text{ for all but finitely many } U \in \mathcal{U}.$$

If a countable τ-cover is Borel, then the induced $\preccurlyeq = \{\langle x,y \rangle : x \preccurlyeq y\}$ is a Borel subset of $X \times X$. We let T and $\mathcal{B}_T$ denote the collections of countable open and Borel τ-covers of X, respectively. We have the following implications.

$$
\begin{array}{ccccccc}
\mathcal{B}_\Gamma & \to & \mathcal{B}_T & \to & \mathcal{B}_\Omega & \to & \mathcal{B} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow \\
\Gamma & \to & T & \to & \Omega & \to & \mathcal{O}
\end{array}
$$

There is a simple hierarchy between the selection principles: For each $\mathfrak{U}, \mathfrak{V}$ in $\{\mathcal{O}, \Omega, \mathrm{T}, \Gamma\}$ or in $\{\mathcal{B}, \mathcal{B}_\Omega, \mathcal{B}_\mathrm{T}, \mathcal{B}_\Gamma\}$, we have that $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V}) \to \mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V}) \to \mathsf{U}_{fin}(\mathfrak{U}, \mathfrak{V})$. The implication $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V}) \to \mathsf{U}_{fin}(\mathfrak{U}, \mathfrak{V})$ needs a little care when $\mathfrak{V}$ is T or $\mathcal{B}_\mathrm{T}$: It holds due to the following lemma.

1 Lemma. *Assume that* $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$*, where each* $\mathcal{F}_n$ *is finite, is a* τ*-cover of a space* X*. Then either* $\cup \mathcal{F}_n = X$ *for some* n*, or else* $\mathcal{V} = \{\cup \mathcal{F}_n\}_{n \in \mathbb{N}}$ *is also a* τ*-cover of* X*.*

PROOF. Assume that $\cup \mathcal{F}_n \neq X$ for all n. Then, as $\mathcal{U}$ is an ω-cover of X, so is $\mathcal{V}$. In particular, $\mathcal{V}$ is a large cover of X. Now fix any $x, y \in X$ such that $x \in U \to y \in U$ for all but finitely many $U \in \mathcal{U}$, and let $F = \{n : (\exists U \in \mathcal{F}_n)\, x \in U \text{ and } y \notin U\}$. Then F is finite and contains the set of n's such that $x \in \cup \mathcal{F}_n$ and $y \notin \cup \mathcal{F}_n$. $\boxed{QED}$

1.4 Equivalences

The notion of τ-covers introduces seven new pairs—namely, $(\mathrm{T}, \mathcal{O})$, (T, Ω), (T, T), (T, Γ), $(\mathcal{O}, \mathrm{T})$, (Ω, T), and (Γ, T)—to which any of the selection operators S_1, S_{fin}, and U_{fin} can be applied. This makes a total of 21 new selection hypotheses. Fortunately, some of them are easily eliminated, using the arguments of [11] and [7]. We will repeat the reasoning briefly for our case. The details can be found in the cited references.

First, the properties $\mathsf{S}_1(\mathcal{O}, \mathrm{T})$ and $\mathsf{S}_{fin}(\mathcal{O}, \mathrm{T})$ imply $\mathsf{S}_{fin}(\mathcal{O}, \Omega)$, and thus hold only in trivial cases (see Section 6 of [17]). Next, $\mathsf{S}_{fin}(\mathrm{T}, \mathcal{O})$ is equivalent to $\mathsf{U}_{fin}(\mathrm{T}, \mathcal{O})$, since if the finite unions cover, then the original sets cover as well. Now, since finite unions can be used to turn any countable cover which does not contain a finite subcover into a γ-cover [7], we have the following equivalences[1]:

- $\mathsf{U}_{fin}(\mathrm{T}, \Gamma) = \mathsf{U}_{fin}(\Gamma, \Gamma)$,

- $\mathsf{U}_{fin}(\mathcal{O}, \mathrm{T}) = \mathsf{U}_{fin}(\Omega, \mathrm{T}) = \mathsf{U}_{fin}(\mathrm{T}, \mathrm{T}) = \mathsf{U}_{fin}(\Gamma, \mathrm{T})$,

- $\mathsf{U}_{fin}(\mathrm{T}, \Omega) = \mathsf{U}_{fin}(\Gamma, \Omega)$; and

- $\mathsf{U}_{fin}(\mathrm{T}, \mathcal{O}) = \mathsf{U}_{fin}(\Gamma, \mathcal{O})$.

In Corollary 6 we get that $\mathsf{S}_1(\mathrm{T}, \Gamma) = \mathsf{S}_{fin}(\mathrm{T}, \Gamma)$. We are thus left with eleven new properties, whose positions with respect to the other properties are described in Figure 2. In this Figure, as well as in the one to come, there still exist quite many unsettled possible implications.

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[1] We identify each property with the collection of sets satisfying this property. Thus, for properties P and Q, we may write $X \in P$, $P \subseteq Q$, etc.

$$U_{fin}(\Gamma,\Gamma) \xrightarrow{\ \ } U_{fin}(\Gamma,T) \longrightarrow U_{fin}(\Gamma,\Omega) \longrightarrow U_{fin}(\Gamma,\mathcal{O})$$
$$\ \ \mathfrak{b} \qquad\qquad \mathfrak{x} \qquad\qquad\qquad \mathfrak{d} \qquad\qquad\qquad \mathfrak{d}$$

$$S_{fin}(\Gamma,T) \longrightarrow S_{fin}(\Gamma,\Omega)$$
$$? \qquad\qquad\qquad \mathfrak{d}$$

$$S_1(\Gamma,\Gamma) \longrightarrow S_1(\Gamma,T) \longrightarrow S_1(\Gamma,\Omega) \longrightarrow S_1(\Gamma,\mathcal{O})$$
$$\mathfrak{b} \qquad\qquad ? \qquad\qquad\qquad \mathfrak{d} \qquad\qquad\qquad \mathfrak{d}$$

$$S_{fin}(T,T) \longrightarrow S_{fin}(T,\Omega)$$
$$? \qquad\qquad\qquad \mathfrak{d}$$

$$S_1(T,\Gamma) \longrightarrow S_1(T,T) \longrightarrow S_1(T,\Omega) \longrightarrow S_1(T,\mathcal{O})$$
$$\mathfrak{t} \qquad\qquad ? \qquad\qquad\qquad ? \qquad\qquad\qquad ?$$

$$S_{fin}(\Omega,T) \longrightarrow S_{fin}(\Omega,\Omega)$$
$$\mathfrak{p} \qquad\qquad\qquad \mathfrak{d}$$

$$S_1(\Omega,\Gamma) \longrightarrow S_1(\Omega,T) \longrightarrow S_1(\Omega,\Omega) \longrightarrow S_1(\mathcal{O},\mathcal{O})$$
$$\mathfrak{p} \qquad\qquad \mathfrak{p} \qquad\qquad \mathrm{cov}(\mathcal{M}) \qquad \mathrm{cov}(\mathcal{M})$$
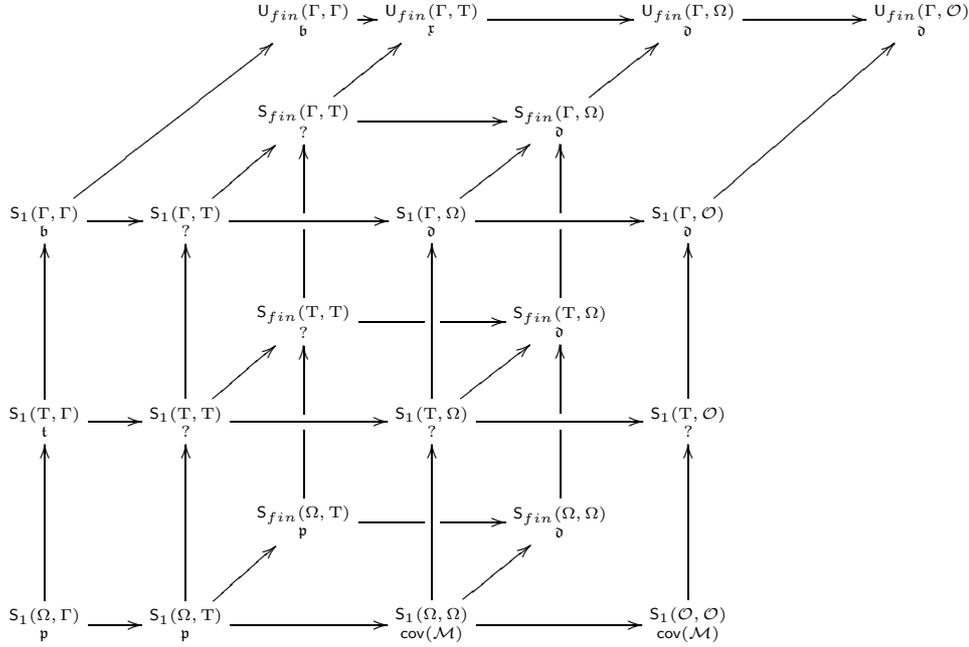
Figure 2. The surviving classes for the open case

1.5 Equivalences for Borel covers

For the Borel case we have the diagram corresponding to Figure 2, but in this case, more equivalences are known [13]: $S_1(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma) = U_{fin}(\mathcal{B}_\Gamma,\mathcal{B}_\Gamma)$, $S_1(\mathcal{B}_\Gamma,\mathcal{B}_\Omega) = S_{fin}(\mathcal{B}_\Gamma,\mathcal{B}_\Omega) = U_{fin}(\mathcal{B}_\Gamma,\mathcal{B}_\Omega)$, and $S_1(\mathcal{B}_\Gamma,\mathcal{B}) = U_{fin}(\mathcal{B}_\Gamma,\mathcal{B})$. In addition, each selection principle for Borel covers implies the corresponding selection principle for open covers.

This paper is divided into two parts. Part 1 consists of Sections 2–4, and Part 2 consists of the remaining sections. In Section 2 we study subcover-type properties and their applications to the study of the new selection principles. In Section 3 we characterize some of the properties in terms of combinatorial properties of Borel images. In Section 4 we find the critical cardinalities of most of the new properties, and apply the results to solve a topological version of the minimal tower problem, which was suggested to us by Scheepers and stated in [15].

It seems that some new mathematical tools are required to solve some of the remaining open problems, as the special properties of τ-covers usually do not

allow application of standard methods developed during the study of classical
selection principles. For this very reason, we believe that these are the important
problems which must be addressed in the future. However, we suggest in the
second part of this paper two relaxations of the notion of τ-cover, which are
easier to work with and may turn out useful in the study of the original problems.
We demonstrate this by proving results which are still open for the case of usual
τ-covers.

Part 1: τ-covers

2 Subcovers with stronger properties

2 Definition. Let X be a set of reals, and $\mathfrak{U}, \mathfrak{V}$ collections of covers of X.
We say that X satisfies $\binom{\mathfrak{U}}{\mathfrak{V}}$ (read: $\mathfrak{U}$ *choose* $\mathfrak{V}$) if for each cover $\mathcal{U} \in \mathfrak{U}$ there
exists a subcover $\mathcal{V} \subseteq \mathcal{U}$ such that $\mathcal{V} \in \mathfrak{V}$.

Observe that for any pair $\mathfrak{U}, \mathfrak{V}$ of collections of countable covers we have
that the property $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ implies $\binom{\mathfrak{U}}{\mathfrak{V}}$. Gerlits and Nagy [6] proved that for
$\mathfrak{U} = \Omega$ and $\mathfrak{V} = \Gamma$, the converse also holds, in fact, $\mathsf{S}_1(\Omega, \Gamma) = \binom{\Omega}{\Gamma}$. But in
general the property $\binom{\mathfrak{U}}{\mathfrak{V}}$ can be strictly weaker than $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$.

A useful property of this notion is the following.

3 Lemma (Cancellation Laws). *For collections of covers $\mathfrak{U}, \mathfrak{V}, \mathfrak{W}$,*

(1) $\binom{\mathfrak{U}}{\mathfrak{V}} \cap \binom{\mathfrak{V}}{\mathfrak{W}} \subseteq \binom{\mathfrak{U}}{\mathfrak{W}}$,

(2) $\binom{\mathfrak{U}}{\mathfrak{V}} \cap \mathsf{S}_{fin}(\mathfrak{V}, \mathfrak{W}) \subseteq \mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{W})$,

(3) $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V}) \cap \binom{\mathfrak{V}}{\mathfrak{W}} \subseteq \mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{W})$; *and*

(4) $\binom{\mathfrak{U}}{\mathfrak{V}} \cap \mathsf{S}_1(\mathfrak{V}, \mathfrak{W}) \subseteq \mathsf{S}_1(\mathfrak{U}, \mathfrak{W})$,

(5) If $\mathfrak{W}$ is closed under taking supersets, then $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V}) \cap \binom{\mathfrak{V}}{\mathfrak{W}} \subseteq \mathsf{S}_1(\mathfrak{U}, \mathfrak{W})$.

Moreover, if $\mathfrak{U} \supseteq \mathfrak{V} \supseteq \mathfrak{W}$, then equality holds in (1)–(5).

PROOF. (1) is immediate. To prove (2), we can apply $\mathsf{S}_{fin}(\mathfrak{V}, \mathfrak{W})$ to $\mathfrak{V}$-
subcovers of the given covers. (4) is similar to (2).

(3) Assume that $\mathcal{U}_n \in \mathfrak{U}$, $n \in \mathbb{N}$, are given. Apply $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ to choose finite
subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\mathcal{V} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n \in \mathfrak{V}$. By $\binom{\mathfrak{V}}{\mathfrak{W}}$, there exists a
subset $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{W} \in \mathfrak{W}$. Then for each n $\mathcal{W} \cap \mathcal{F}_n$ is a finite (possibly
empty) subset of $\mathcal{U}_n$, and $\bigcup_{n \in \mathbb{N}} (\mathcal{W} \cap \mathcal{F}_n) = \mathcal{W} \in \mathfrak{W}$. To prove (5), observe that
the resulting cover $\mathcal{V}$ contains an element of $\mathfrak{W}$, and as $\mathfrak{W}$ is closed under taking
supersets, $\mathcal{V} \in \mathfrak{W}$ as well.

It is clear that reverse inclusion (and therefore equality) hold in (1)–(5) when $\mathfrak{U} \supseteq \mathfrak{V} \supseteq \mathfrak{W}$. $\boxed{QED}$

4 Corollary. *Assume that* $\mathfrak{U} \supseteq \mathfrak{V}$. *Then the following equivalences hold:*

(1) $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V}) = \binom{\mathfrak{U}}{\mathfrak{V}} \cap \mathsf{S}_{fin}(\mathfrak{V}, \mathfrak{U})$.

(2) *If* $\mathfrak{V}$ *is closed under taking supersets, then* $\mathsf{S}_1(\mathfrak{U}, \mathfrak{V}) = \binom{\mathfrak{U}}{\mathfrak{V}} \cap \mathsf{S}_1(\mathfrak{V}, \mathfrak{U})$.

PROOF. We prove (1). Clearly $\mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V})$ implies $\binom{\mathfrak{U}}{\mathfrak{V}}$ and $\mathsf{S}_{fin}(\mathfrak{V}, \mathfrak{U})$. On the other hand, by applying the Cancellation Laws (2) and then (3) we have that

$$\binom{\mathfrak{U}}{\mathfrak{V}} \cap \mathsf{S}_{fin}(\mathfrak{V}, \mathfrak{U}) \subseteq \mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{U}) \cap \binom{\mathfrak{U}}{\mathfrak{V}} \subseteq \mathsf{S}_{fin}(\mathfrak{U}, \mathfrak{V}).$$

$\boxed{QED}$

2.1 When every τ-cover contains a γ-cover

5 Theorem. *The following equivalences hold:*

(1) $\mathsf{S}_1(\mathrm{T}, \Gamma) = \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Gamma, \mathrm{T})$,

(2) $\mathsf{S}_1(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Gamma}) = \binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}} \cap \mathsf{S}_{fin}(\mathcal{B}_{\Gamma}, \mathcal{B}_{\mathrm{T}})$,

PROOF. (1) By the Cancellation Laws 3, $\binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Gamma, \mathrm{T}) \subseteq \mathsf{S}_{fin}(\Gamma, \Gamma)$. In [7] it was proved that $\mathsf{S}_{fin}(\Gamma, \Gamma) = \mathsf{S}_1(\Gamma, \Gamma)$. Thus, $\binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Gamma, \mathrm{T}) \subseteq \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_1(\Gamma, \Gamma)$, which by the Cancellation Laws is a subset of $\mathsf{S}_1(\mathrm{T}, \Gamma)$. The other direction is immediate.

(2) is similar. $\boxed{QED}$

6 Corollary. *The following equivalences hold:*

(1) $\mathsf{S}_1(\mathrm{T}, \Gamma) = \mathsf{S}_{fin}(\mathrm{T}, \Gamma)$;

(2) $\mathsf{S}_1(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Gamma}) = \mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Gamma})$.

Using similar arguments, we have the following.

7 Theorem. *The following equivalences hold:*

(1) $\mathsf{S}_1(\Omega, \Gamma) = \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Omega, \mathrm{T})$;

(2) $\mathsf{S}_1(\mathcal{B}_{\Omega}, \mathcal{B}_{\Gamma}) = \binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}} \cap \mathsf{S}_{fin}(\mathcal{B}_{\Omega}, \mathcal{B}_{\mathrm{T}})$.

2.2   When every ω-cover contains a τ-cover

8 Theorem. *The following inclusions hold:*

(1) $\binom{\Omega}{T} \subseteq \mathsf{S}_{fin}(\Gamma, T)$.

(2) $\binom{\mathcal{B}_\Omega}{\mathcal{B}_T} \subseteq \mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_T)$.

PROOF. We will prove (1) (the proof of (2) is identical). Assume that X satisfies $\binom{\Omega}{T}$. If X is countable then it satisfies all of the properties mentioned in this paper. Otherwise let x_n, $n \in \mathbb{N}$, be distinct elements in X. Assume that $\mathcal{U}_n = \{U_m^n\}_{m\in\mathbb{N}}$, $n \in \mathbb{N}$, are open γ-covers of X. Define $\tilde{\mathcal{U}}_n = \{U_m^n \setminus \{x_n\}\}_{m\in\mathbb{N}}$. Then $\mathcal{U} = \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{U}}_n$ is an open ω-cover of X, and thus contains a τ-cover $\mathcal{V}$ of X. Let $\preccurlyeq$ be the induced quasiordering.

9 Lemma. *If $\langle X, \preccurlyeq \rangle$ has a least element, then $\mathcal{V}$ contains a γ-cover of X.*

PROOF. Write $\mathcal{V} = \{V_n\}_{n\in\mathbb{N}}$. Let x_0 be a least element in $\langle X, \preccurlyeq \rangle$. Consider the subsequence $\{V_{n_k}\}_{k\in\mathbb{N}}$ consisting of the elements V_n such that $x_0 \in V_n$. Since τ-covers are large, this sequence is infinite. For all $x \in X$ we have $x_0 \preccurlyeq x$, thus $x_0 \in V_n \to x \in V_n$ for all but finitely many n. Since $x_0 \in V_{n_k}$ for all k, we have that for all but finitely many k, $x \in V_{n_k}$.   QED

There are two cases to consider.

Case 1. For some n x_n is a least element in $\langle X, \preccurlyeq \rangle$. Then $\mathcal{V}$ contains a γ-cover $\tilde{\mathcal{V}}$ of X. In this case, for all n x_n belongs to all but finitely many members of $\tilde{\mathcal{V}}$, thus $\tilde{\mathcal{V}} \cap \tilde{\mathcal{U}}_n$ is finite for each n, and $\mathcal{W} = \{U : (\exists n)U \setminus \{x_n\} \in \tilde{\mathcal{V}}\}$ is a γ-cover of X.

Case 2. For each n there exists $x \neq x_n$ with $x \preccurlyeq x_n$. For each n, $\mathcal{U}_n$ is a γ-cover of X, thus x belongs to all but finitely many members of $\mathcal{V} \cap \tilde{\mathcal{U}}_n$. Since x_n does not belong to any of the members in $\mathcal{V} \cap \tilde{\mathcal{U}}_n$, $\mathcal{V} \cap \tilde{\mathcal{U}}_n$ must be finite. Thus, $\mathcal{W} = \{U : (\exists n)U \setminus \{x_n\} \in \tilde{\mathcal{V}}\}$ is a τ-cover of X.   QED

If $\binom{\Omega}{T} \subseteq \mathsf{S}_{fin}(T, \Omega)$, then by Corollary 4 $\binom{\Omega}{T} = \mathsf{S}_{fin}(\Omega, T)$.

10 Problem. Is $\binom{\Omega}{T} = \mathsf{S}_{fin}(\Omega, T)$?

3   Combinatorics of Borel images

In this section we characterize several properties in terms of Borel images in the spaces $^{\mathbb{N}}\mathbb{N}$ and $P_\infty(\mathbb{N})$, using the combinatorial structure of these spaces.

3.1   The combinatorial structures

A quasiorder $\leq^*$ is defined on the Baire space $^{\mathbb{N}}\mathbb{N}$ by eventual dominance:

$$f \leq^* g \quad \text{if } f(n) \leq g(n) \text{ for all but finitely many } n.$$

A subset Y of $^{\mathbb{N}}\mathbb{N}$ is called *unbounded* if it is unbounded with respect to $\leq^*$. Y is *dominating* if it is cofinal in $^{\mathbb{N}}\mathbb{N}$ with respect to $\leq^*$, that is, for each $f \in {}^{\mathbb{N}}\mathbb{N}$ there exists $g \in Y$ such that $f \leq^* g$. $\mathfrak{b}$ is the minimal size of an unbounded subset of $^{\mathbb{N}}\mathbb{N}$, and $\mathfrak{d}$ is the minimal size of a dominating subset of $^{\mathbb{N}}\mathbb{N}$.

Define a quasiorder $\subseteq^*$ on $P_\infty(\mathbb{N})$ by $a \subseteq^* b$ if $a \setminus b$ is finite. An infinite set $a \subseteq \mathbb{N}$ is a *pseudo-intersection* of a family $Y \subseteq P_\infty(\mathbb{N})$ if for each $b \in Y$, $a \subseteq^* b$. A family $Y \subseteq P_\infty(\mathbb{N})$ is a *tower* if it is linearly quasiordered by $\subseteq^*$, and it has no pseudointersection. $\mathfrak{t}$ is the minimal size of a tower.

A family $Y \subseteq P_\infty(\mathbb{N})$ is *centered* if the intersection of each (nonempty) finite subfamily of Y is infinite. Note that every tower in $P_\infty(\mathbb{N})$ is centered. A centered family $Y \subseteq P_\infty(\mathbb{N})$ is a *power* if it does not have a pseudointersection. $\mathfrak{p}$ is the minimal size of a power.

3.2   The property $\binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}}$

For a set of reals X and a topological space Z, we say that Y is a Borel image of X in Z if there exists a Borel function $f : X \to Z$ such that $f[X] = Y$. The following classes of sets were introduced in [8]:

  P: The set of $X \subseteq \mathbb{R}$ such that no Borel image of X in $P_\infty(\mathbb{N})$ is a power,

  B: The set of $X \subseteq \mathbb{R}$ such that every Borel image of X in $^{\mathbb{N}}\mathbb{N}$ is bounded (with respect to eventual domination);

  D: The set of $X \subseteq \mathbb{R}$ such that no Borel image of X in $^{\mathbb{N}}\mathbb{N}$ is dominating.

For a collection $\mathcal{J}$ of separable metrizable spaces, let $\mathsf{non}(\mathcal{J})$ denote the minimal cardinality of a separable metrizable space which is not a member of $\mathcal{J}$. We also call $\mathsf{non}(\mathcal{J})$ *the critical cardinality* of the class $\mathcal{J}$. The critical cardinalities of the above classes are $\mathfrak{p}$, $\mathfrak{b}$, and $\mathfrak{d}$, respectively. These classes have the interesting property that they transfer the cardinal inequalities $\mathfrak{p} \leq \mathfrak{b} \leq \mathfrak{d}$ to the inclusions $\mathsf{P} \subseteq \mathsf{B} \subseteq \mathsf{D}$.

11 Definition. For each countable cover of X enumerated bijectively as $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ we associate a function $h_{\mathcal{U}} : X \to P(\mathbb{N})$, defined by $h_{\mathcal{U}}(x) = \{n : x \in U_n\}$.

$\mathcal{U}$ is a large cover of X if, and only if, $h_{\mathcal{U}}[X] \subseteq P_{\infty}(\mathbb{N})$. As we assume that X is infinite and is not a member of any of our covers, we have that each ω-cover of X is a large cover of X. The following lemma is a key observation for the rest of this section. Note that $h_{\mathcal{U}}$ is a Borel function whenever $\mathcal{U}$ is a Borel cover of X, and $h_{\mathcal{U}}$ is continuous whenever all elements of $\mathcal{U}$ are *clopen*.

12 Lemma ([15]). *Assume that $\mathcal{U}$ is a countable large cover of X.*

(1) $\mathcal{U}$ is an ω-cover of X if, and only if, $h_{\mathcal{U}}[X]$ is centered.

(2) $\mathcal{U}$ contains a γ-cover of X if, and only if, $h_{\mathcal{U}}[X]$ has a pseudointersection.

(3) $\mathcal{U}$ is a τ-cover of X if, and only if, $h_{\mathcal{U}}[X]$ is linearly quasiordered by $\subseteq^{*}$.

Moreover, if $f : X \to P(\mathbb{N})$ is any function, and $\mathcal{A} = \{O_n\}_{n\in\mathbb{N}}$ is the clopen cover of $P(\mathbb{N})$ such that $x \in O_n \Leftrightarrow n \in x$, then for $\mathcal{U} = \{f^{-1}[O_n]\}_{n\in\mathbb{N}}$ we have that $f = h_{\mathcal{U}}$.

This Lemma implies that $\mathsf{P} = \binom{\mathcal{B}_{\Omega}}{\mathcal{B}_{\Gamma}}$ [13].

13 Corollary. $\mathsf{P} = \binom{\mathcal{B}_{\Omega}}{\mathcal{B}_{\mathrm{T}}} \cap \binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}}$.

It is natural to define the following notion.

T: The set of $X \subseteq \mathbb{R}$ such that no Borel image of X in $P_{\infty}(\mathbb{N})$ is a tower.

14 Theorem. $\mathsf{T} = \binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}}$.

PROOF. See [15] for the *clopen* version of this theorem (a straightforward usage of Lemma 12). The proof for the Borel case is similar. $\boxed{QED}$

15 Corollary. $\mathsf{non}(\binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}}) = \mathsf{non}(\binom{\mathrm{T}}{\Gamma}) = \mathfrak{t}$.

PROOF. By Theorem 14, $\mathfrak{t} \le \mathsf{non}(\binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}})$. In [15] we defined $\mathcal{T}$ to be the collection of sets for which every countable *clopen* τ-cover contains a γ-cover, and showed that $\mathsf{non}(\mathcal{T}) = \mathfrak{t}$. But $\binom{\mathcal{B}_{\mathrm{T}}}{\mathcal{B}_{\Gamma}} \subseteq \binom{\mathrm{T}}{\Gamma} \subseteq \mathcal{T}$. $\boxed{QED}$

Clearly, $\mathsf{P} \subseteq \mathsf{T}$. The cardinal inequality $\mathfrak{p} \le \mathfrak{t} \le \mathfrak{b}$ suggests pushing this further by showing that $\mathsf{T} \subseteq \mathsf{B}$; unfortunately this is false. Sets which are continuous images of Borel sets are called *analytic*.

16 Theorem. *Every analytic set satisfies T. In particular, ${}^{\mathbb{N}}\mathbb{N} \in \mathsf{T}$.*

PROOF. According to [15], no continuous image of an analytic set is a tower. In particular, towers are not analytic subsets of $P_{\infty}(\mathbb{N})$. Since Borel images of analytic sets are again analytic sets, we have that every analytic set satisfies T. $\boxed{QED}$

The following equivalences hold [13]:

- $\mathsf{S}_1(\mathcal{B}_{\Gamma}, \mathcal{B}_{\Gamma}) = \mathsf{B}$,

- $S_1(\mathcal{B}_\Gamma, \mathcal{B}) = D$;

- $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) = P$.

Theorem 16 rules out an identification of T with any of the selection principles. However, we get the following characterization of $S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)$ in terms of Borel images.

17 Theorem. $S_1(\mathcal{B}_T, \mathcal{B}_\Gamma) = T \cap B$.

PROOF. By the Cancellation Laws and Theorem 14,

$$S_1(\mathcal{B}_T, \mathcal{B}_\Gamma) = \binom{\mathcal{B}_T}{\mathcal{B}_\Gamma} \cap S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma) = T \cap B.$$

<div align="right">QED</div>

3.3 The property $\binom{\mathcal{B}_\Omega}{\mathcal{B}_T}$

For a subset Y of $P_\infty(\mathbb{N})$ and $a \in P_\infty(\mathbb{N})$, define

$$Y \upharpoonright a = \{y \cap a : y \in Y\}.$$

If all sets in $Y \upharpoonright a$ are infinite, we say that $Y \upharpoonright a$ is a *large restriction* of Y.

18 Theorem. *For a set X of real numbers, the following are equivalent:*

(1) X satisfies $\binom{\mathcal{B}_\Omega}{\mathcal{B}_T}$

(2) For each Borel image Y of X in $P_\infty(\mathbb{N})$, if Y is centered, then there exists a large restriction of Y which is linearly quasiordered by $\subseteq^*$.

PROOF. $1 \Rightarrow 2$: Assume that $\Psi : X \to P_\infty(\mathbb{N})$ is a Borel function, and let $Y = \Psi[X]$. Assume that Y is centered, and consider the collection $\mathcal{A} = \{O_n\}_{n \in \mathbb{N}}$ where $O_n = \{a : n \in a\} \cap Y$ for each $n \in \mathbb{N}$. If the set $a = \{n : Y = O_n\}$ is infinite, then a is a pseudointersection of Y and we are done. Otherwise, by removing finitely many elements from $\mathcal{A}$ we get that $\mathcal{A}$ is an ω-cover of Y.

Setting $U_n = \Psi^{-1}[O_n]$ for each n, we have that $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is a Borel ω-cover of X, which thus contains a τ-cover $\{U_{a_n}\}_{n \in \mathbb{N}}$ of X. Let $a = \{a_n\}_{n \in \mathbb{N}}$, and define a cover $\mathcal{V} = \{V_n\}_{n \in \mathbb{N}}$ of X by

$$V_n = \begin{cases} U_n & n \in a \\ \emptyset & \text{otherwise} \end{cases}$$

Then $\mathcal{V}$ is a τ-cover of X, and by Lemma 12, $\Psi[X] \upharpoonright a = h_{\mathcal{U}}[X] \upharpoonright a = h_{\mathcal{V}}[X]$ is linearly quasiordered by $\subseteq^*$.

$2 \Rightarrow 1$: Assume that $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$ is an ω-cover of X. By Lemma 12, $h_{\mathcal{U}}[X]$ is centered. Let $a = \{a_n\}_{n \in \mathbb{N}}$ be a large restriction of $h_{\mathcal{U}}[X]$ which is linearly quasiordered by $\subseteq^*$, and define $\mathcal{V}$ as in $1 \Rightarrow 2$. Then $h_{\mathcal{U}}[X] \upharpoonright a = h_{\mathcal{V}}[X]$. Thus all elements in $h_{\mathcal{V}}[X]$ are infinite (i.e., $\mathcal{V}$ is a large cover of X), and $h_{\mathcal{V}}[X]$ is linearly quasiordered by $\subseteq^*$ (i.e., $\mathcal{V}$ is a τ-cover of X). Then $\mathcal{V} \setminus \{\emptyset\} \subseteq \mathcal{U}$ is a τ-cover of X. $\boxed{QED}$

19 Remark. Replacing the "Borel sets" by "clopen sets" and "Borel functions" by "continuous functions" in the last proof we get that the following properties are equivalent for a set X of reals:

(1) Every countable *clopen* ω-cover of X contains a τ-cover of X.

(2) For each continuous image Y of X in $P_\infty(\mathbb{N})$, if Y is centered, then there exists a large restriction of Y which is linearly quasiordered by $\subseteq^*$.

We do not know whether the *open* version of this result is true.

3.4   The property $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$

20 Definition. A family $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$ satisfies the *excluded middle* property if there exists $g \in {}^{\mathbb{N}}\mathbb{N}$ such that:

(1) For all $f \in Y$, $g \not\leq^* f$;

(2) For each $f, h \in Y$, one of the situations $f(n) \leq g(n) < h(n)$ or $h(n) \leq g(n) < f(n)$ is possible only for finitely many n.

21 Theorem. *For a set X of real numbers, the following are equivalent:*

(1) X satisfies $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$;

(2) Every Borel image of X in ${}^{\mathbb{N}}\mathbb{N}$ satisfies the excluded middle property.

PROOF. $1 \Rightarrow 2$: For each n, the collection $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, where $U_m^n = \{f \in {}^{\mathbb{N}}\mathbb{N} : f(n) \leq m\}$, $m \in \mathbb{N}$, is an open γ-cover of ${}^{\mathbb{N}}\mathbb{N}$. Assume that Ψ is a Borel function from X to ${}^{\mathbb{N}}\mathbb{N}$. By standard arguments we may assume that $\Psi^{-1}[U_m^n] \neq X$ for all n and m. Then the collections $\mathcal{U}_n = \{\Psi^{-1}[U_m^n] : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel γ-covers of X. For all n, the sequence $\{U_m^n\}_{m \in \mathbb{N}}$ is monotonically increasing with respect to $\subseteq$, therefore—as large subcovers of τ-covers are also τ-covers—we may use $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$ instead of $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})$ to get a τ-cover $\mathcal{U} = \{\Psi^{-1}[U_{m_n}^n]\}_{n \in \mathbb{N}}$ for X.

Let $g \in {}^{\mathbb{N}}\mathbb{N}$ be such that $g(n) = m_n$ for all n. For all $x \in X$, as $\mathcal{U}$ is a large cover of X, we have that $\Psi(x) \in U_{g(n)}^n$ (that is, $\Psi(x)(n) \leq g(n)$) for infinitely many n. Let $\preccurlyeq$ be the linear quasiordering of X induced by the τ-cover $\mathcal{U}$. Then

for all $x, y \in X$, either $x \preccurlyeq y$ or $y \preccurlyeq x$. In the first case we get that for all but finitely many n $\Psi(x)(n) \leq g(n) \rightarrow \Psi(y)(n) \leq g(n)$, and in the second case we get the same assertion with x and y swapped. This shows that $\Psi[X]$ satisfies the excluded middle property.

$2 \Rightarrow 1$: Assume that $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel covers of X which do not contain a finite subcover. Replacing each U_m^n with the Borel set $\bigcup_{k \leq m} U_k^n$ we may assume that the sets U_m^n are monotonically increasing with m. Define a function Ψ from X to $^{\mathbb{N}}\mathbb{N}$ so that for each x and n:

$$\Psi(x)(n) = \min\{m : x \in U_m^n\}.$$

Then Ψ is a Borel map, and so $\Psi[X]$ satisfies the excluded middle property. Let $g \in {}^{\mathbb{N}}\mathbb{N}$ be a witness for that. Then the sequence $\mathcal{U} = \{U_{g(n)}^n\}_{n \in \mathbb{N}}$ is a τ-cover of X: For each $x \in X$ we have that $g \not<^* \Psi(x)$, thus $\mathcal{U}$ is a large cover of X. Moreover, for all $x, y \in X$, we have by the excluded middle property that at least one of the assertions $\Psi(x)(n) \leq g(n) < \Psi(y)(n)$ or $\Psi(y)(n) \leq g(n) < \Psi(y)(n)$ is possible only for finitely many n. Then the first assertion implies that $x \preccurlyeq y$, and the second implies $y \preccurlyeq x$ with respect to $\mathcal{U}$. $\boxed{QED}$

22 Remark. The analogue *clopen* version of Theorem 21 also holds. We do not know whether there exist an analogue characterization of $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$ (the *open* version) in terms of continuous images.

4   Critical cardinalities

23 Theorem. $\mathsf{non}(\mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Omega})) = \mathsf{non}(\mathsf{S}_{fin}(\mathrm{T}, \Omega)) = \mathfrak{d}$.

PROOF. $\mathsf{S}_{fin}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega}) \subseteq \mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Omega}) \subseteq \mathsf{S}_{fin}(\mathrm{T}, \Omega) \subseteq \mathsf{S}_{fin}(\Gamma, \Omega)$, and according to [7] and [13], $\mathsf{non}(\mathsf{S}_{fin}(\mathcal{B}_{\Omega}, \mathcal{B}_{\Omega})) = \mathsf{non}(\mathsf{S}_{fin}(\Gamma, \Omega)) = \mathfrak{d}$. $\boxed{QED}$

24 Theorem. $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_{\mathrm{T}}, \mathcal{B}_{\Gamma})) = \mathsf{non}(\mathsf{S}_1(\mathrm{T}, \Gamma)) = \mathfrak{t}$.

PROOF. By Theorem 5, $\mathsf{S}_1(\mathrm{T}, \Gamma) = \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_1(\Gamma, \Gamma)$, thus by Corollary 15, $\mathsf{non}(\mathsf{S}_1(\mathrm{T}, \Gamma)) = \min\{\mathsf{non}(\binom{\mathrm{T}}{\Gamma}), \mathsf{non}(\mathsf{S}_1(\Gamma, \Gamma))\} = \min\{\mathfrak{t}, \mathfrak{b}\} = \mathfrak{t}$. The proof for the Borel case is similar. $\boxed{QED}$

25 Definition. $\mathfrak{x}$ is the minimal cardinality of a family $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$ which does not satisfy the excluded middle property.

Therefore $\mathfrak{b} \leq \mathfrak{x} \leq \mathfrak{d}$. A family $Y \subseteq P_{\infty}(\mathbb{N})$ is *splitting* if for each infinite $a \subseteq \mathbb{N}$ there exists $s \in Y$ which *splits* a, that is, such that the sets $a \cap s$ and $a \setminus s$ are infinite. $\mathfrak{s}$ is the minimal size of a splitting family. In [14] it is proved that $\mathfrak{x} = \max\{\mathfrak{s}, \mathfrak{b}\}$.

26 Theorem. $\mathsf{non}(\mathsf{U}_{fin}(\mathcal{B}_{\Gamma}, \mathcal{B}_{\mathrm{T}})) = \mathsf{non}(\mathsf{U}_{fin}(\Gamma, \mathrm{T})) = \mathfrak{x}$.

PROOF. By Theorem 21, $\mathsf{non}(\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T})) = \mathfrak{x}$. Thus, our theorem will follow from the inclusion $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T}) \subseteq \mathsf{U}_{fin}(\Gamma, \mathrm{T})$ once we prove that non $(\mathsf{U}_{fin}(\Gamma, \mathrm{T})) \le \mathfrak{x}$. To this end, consider a family $Y \subseteq {}^\mathbb{N}\mathbb{N}$ of size $\mathfrak{x}$ which does not satisfy the excluded middle property, and consider the monotone γ-covers $\mathcal{U}_n$, $n \in \mathbb{N}$, of ${}^\mathbb{N}\mathbb{N}$ defined in the proof of Theorem 21. Then, as in that proof, we cannot extract from these covers a τ-cover of Y. Thus, Y does not satisfy $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$.   QED

27 Definition. Let $\kappa_{\omega\tau}$ be the minimal cardinality of a centered set $Y \subseteq P_\infty(\mathbb{N})$ such that for no $a \in P_\infty(\mathbb{N})$, the restriction $Y \upharpoonright a$ is large and linearly quasiordered by $\subseteq^*$.

It is easy to see (either from the definitions or by consulting the involved selection properties) that $\kappa_{\omega\tau} \le \mathfrak{d}$ and $\mathfrak{p} = \min\{\kappa_{\omega\tau}, \mathfrak{t}\}$. In [14] it is proved that in fact $\kappa_{\omega\tau} = \mathfrak{p}$.

28 Lemma. $\mathsf{non}(\binom{\mathcal{B}_\Omega}{\mathcal{B}_\mathrm{T}}) = \mathsf{non}(\binom{\Omega}{\mathrm{T}}) = \mathfrak{p}$.

PROOF. Let $\mathcal{P}_{\omega\tau}$ denote the property that every *clopen* ω-cover contains a γ-cover. Then $\binom{\mathcal{B}_\Omega}{\mathcal{B}_\mathrm{T}} \subseteq \binom{\Omega}{\mathrm{T}} \subseteq \mathcal{P}_{\omega\tau}$. By Theorem 18 and Remark 19, $\mathsf{non}(\binom{\mathcal{B}_\Omega}{\mathcal{B}_\mathrm{T}}) = \mathsf{non}(\mathcal{P}_{\omega\tau}) = \kappa_{\omega\tau} = \mathfrak{p}$.   QED

29 Theorem. $\mathsf{non}(\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\mathrm{T})) = \mathsf{non}(\mathsf{S}_{fin}(\Omega, \mathrm{T})) = \mathfrak{p}$.

PROOF. By Corollary 4 and Theorem 23,

$$
\begin{aligned}
\mathsf{non}(\mathsf{S}_{fin}(\Omega, \mathrm{T})) &= \min\{\mathsf{non}(\binom{\Omega}{\mathrm{T}}), \mathsf{non}(\mathsf{S}_{fin}(\mathrm{T}, \Omega))\} = \\
&= \min\{\kappa_{\omega\tau}, \mathfrak{d}\} = \kappa_{\omega\tau} = \mathfrak{p}. \quad (1)
\end{aligned}
$$

The proof for the Borel case is the same.   QED

5 Topological variants of the Minimal Tower problem

Let $\mathfrak{c}$ denote the size of the continuum. The following inequalities are well known [4]:

$$\mathfrak{p} \le \mathfrak{t} \le \mathfrak{b} \le \mathfrak{d} \le \mathfrak{c}.$$

For each pair except $\mathfrak{p}$ and $\mathfrak{t}$, it is well known that a strict inequality is consistent.

30 Problem (*Minimal Tower*). Is it provable that $\mathfrak{p} = \mathfrak{t}$?

This is one of the major and oldest problems of infinitary combinatorics. Allusions to this problem can be found in Rothberger's works (see, e.g., [10]).

We know that $\mathsf{S}_1(\Omega, \Gamma) \subseteq \mathsf{S}_1(\mathrm{T}, \Gamma)$, and that $\mathsf{non}(\mathsf{S}_1(\Omega, \Gamma)) = \mathfrak{p}$, and $\mathsf{non}(\mathsf{S}_1(\mathrm{T}, \Gamma)) = \mathfrak{t}$. Thus, if $\mathfrak{p} < \mathfrak{t}$ is consistent, then it is consistent that

$S_1(\Omega, \Gamma) \neq S_1(T, \Gamma)$. Thus the following problem, which was suggested to us by Scheepers, is a logical lower bound on the difficulty of the Minimal Tower problem.

31 Problem ([15]). Is it consistent that $S_1(\Omega, \Gamma) \neq S_1(T, \Gamma)$?

We also have a Borel variant of this problem.

32 Problem. Is it consistent that $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) \neq S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)$?

We will solve both of these problems.

For a class $\mathcal{J}$ of sets of real numbers with $\cup \mathcal{J} \notin \mathcal{J}$, the *additivity number* of $\mathcal{J}$ is the minimal cardinality of a collection $\mathcal{F} \subseteq \mathcal{J}$ such that $\cup \mathcal{F} \notin \mathcal{J}$. The additivity number of $\mathcal{J}$ is denoted $\mathsf{add}(\mathcal{J})$.

33 Lemma ([15]). *Assume that $Y \subseteq P_\infty(\mathbb{N})$ is linearly ordered by $\subseteq^*$, and for some $\kappa < \mathfrak{t}$, $Y = \bigcup_{\alpha < \kappa} Y_\alpha$ where each Y_α has a pseudointersection. Then Y has a pseudointersection.*

34 Theorem. $\mathsf{add}(\binom{\mathcal{B}_T}{\mathcal{B}_\Gamma}) = \mathsf{add}(\binom{T}{\Gamma}) = \mathfrak{t}$.

PROOF. By Theorem 14 and Lemma 33, we have that $\mathsf{add}(\binom{\mathcal{B}_T}{\mathcal{B}_\Gamma}) = \mathsf{add}(T) = \mathfrak{t}$. The proof that $\mathsf{add}(\binom{T}{\Gamma}) = \mathfrak{t}$ is not as elegant and requires a back-and-forth usage of Lemma 12. Assume that $\kappa < \mathfrak{t}$, and let X_α, $\alpha < \kappa$, be sets satisfying $\binom{T}{\Gamma}$. Let $\mathcal{U}$ be a countable open τ-cover of $X = \bigcup_{\alpha < \kappa} X_\alpha$. Then $h_\mathcal{U}[X] = \bigcup_{\alpha < \kappa} h_\mathcal{U}[X_\alpha]$ is linearly quasiordered by $\subseteq^*$. Since each X_α satisfies $\binom{T}{\Gamma}$, for each α $\mathcal{U}$ contains a γ-cover of X_α, that is, $h_\mathcal{U}[X_\alpha]$ has a pseudo-intersection. By Lemma 33, $h_\mathcal{U}[X]$ has a pseudo-intersection, that is, $\mathcal{U}$ contains a γ-cover of X. $\boxed{QED}$

35 Theorem. $\mathsf{add}(S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)) = \mathfrak{t}$.

PROOF. By Theorem 5, $S_1(\mathcal{B}_T, \mathcal{B}_\Gamma) = \binom{\mathcal{B}_T}{\mathcal{B}_\Gamma} \cap S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, and according to [1], $\mathsf{add}(S_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)) = \mathfrak{b}$. By Theorem 34, we get that

$$\mathsf{add}(S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)) \geq \min\{\mathfrak{t}, \mathfrak{b}\} = \mathfrak{t}.$$

On the other hand, by Theorem 24 we have

$$\mathsf{add}(S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)) \leq \mathsf{non}(S_1(\mathcal{B}_T, \mathcal{B}_\Gamma)) = \mathfrak{t}.$$

$\boxed{QED}$

In [12] Scheepers proves that $S_1(\Gamma, \Gamma)$ is closed under taking unions of size less than the distributivity number $\mathfrak{h}$. Consequently, we get that $\mathsf{add}(S_1(T, \Gamma)) = \mathfrak{t}$ [1]. As it is consistent that $S_1(\Omega, \Gamma)$ is not closed under taking finite unions [5], we get a positive solution to Problem 31. We will now prove something stronger: Consistently, no class between $S_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ and $\binom{\Omega}{T}$ (inclusive) is closed under taking finite unions. This solves Problem 31 as well as Problem 32.

36 Theorem (CH). *There exist sets of reals A and B satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, such that $A \cup B$ does not satisfy $\binom{\Omega}{\mathrm{T}}$. In particular, $\mathsf{S}_1(\mathrm{T}, \Gamma) \neq \mathsf{S}_1(\Omega, \Gamma)$, and $\mathsf{S}_1(\mathcal{B}_\mathrm{T}, \mathcal{B}_\Gamma) \neq \mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.*

PROOF. By a theorem of Brendle [3], assuming CH there exists a set of reals X of size continuum such that all subsets of X satisfy P. (Recall that $\mathsf{P} = \mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.)

As P is closed under taking Borel (continuous is enough) images, we may assume that $X \subseteq [0, 1]$. For $Y \subseteq [0, 1]$, write $Y + 1 = \{y + 1 : y \in Y\}$ for the translation of Y by 1. As $|X| = \mathfrak{c}$ and only $\mathfrak{c}$ many out of the $2^\mathfrak{c}$ many subsets of X are Borel, there exists a subset Y of X which is not F_σ neither G_δ. By a theorem of Galvin and Miller [5], for such a subset Y the set $(X \setminus Y) \cup (Y + 1)$ does not satisfy $\mathsf{S}_1(\Omega, \Gamma)$. Set $A = X \setminus Y$ and $B = Y + 1$. Then A and B satisfy $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, and $A \cup B$ does not satisfy $\mathsf{S}_1(\Omega, \Gamma) = \binom{\Omega}{\mathrm{T}} \cap \binom{\mathrm{T}}{\Gamma}$. By Theorem 34, $A \cup B$ satisfies $\binom{\mathrm{T}}{\Gamma}$ and therefore it does not satisfies $\binom{\Omega}{\mathrm{T}}$. But by Theorem 35, the set $A \cup B$ satisfies $\mathsf{S}_1(\mathcal{B}_\mathrm{T}, \mathcal{B}_\Gamma)$.   QED

6 Special elements

6.1 The Cantor set C

Let $C \subseteq \mathbb{R}$ be the canonic middle-third Cantor set.

37 Proposition. *Cantor's set C does not satisfy $\mathsf{S}_{fin}(\Gamma, \mathrm{T})$.*

PROOF. Had it satisfied this property, we would have by Theorem 16 that $C \in \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Gamma, \mathrm{T}) = \mathsf{S}_1(\Gamma, \Gamma)$, contradicting [7].   QED

Thus C satisfies $\mathsf{S}_{fin}(\mathrm{T}, \Omega)$ and $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$, and none of the other new properties.

6.2 A special Lusin set

In Subsection 3.3 of [1] we construct, using $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, special Lusin sets of size $\mathfrak{c}$ which satisfy $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. The meta-structure of the proof is as follows. At each stage of this construction we define a set Y_α^* which is a union of less that $\mathsf{cov}(\mathcal{M})$ many meager sets, and choose an element $x_\alpha \in G_\alpha \setminus Y_\alpha^*$ where G_α is a basic open subset of $^\mathbb{N}\mathbb{Z}$.

38 Theorem. *If $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$, then there exists a Lusin set satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ but not $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$.*

PROOF. We modify the aforementioned construction so to make sure that the resulting Lusin set L does not satisfy the excluded middle property. As we do not need to use any group structure, we will work in $^\mathbb{N}\mathbb{N}$ rather than $^\mathbb{N}\mathbb{Z}$.

39 Lemma. *Assume that A is an infinite set of natural numbers, and $f \in {}^{\mathbb{N}}\mathbb{N}$. Then the sets*

$$
\begin{aligned}
M_{f,A} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [g \le f] \cap A \text{ is finite}\} \\
\tilde{M}_{f,A} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [f < g] \cap A \text{ is finite}\}
\end{aligned}
$$

are meager subsets of ${}^{\mathbb{N}}\mathbb{N}$.

PROOF. For each k, the sets

$$
\begin{aligned}
N_k &= \{g \in {}^{\mathbb{N}}\mathbb{N} : (\forall n > k)\ n \in A \to f(n) < g(n)\} \\
\tilde{N}_k &= \{g \in {}^{\mathbb{N}}\mathbb{N} : (\forall n > k)\ n \in A \to g(n) \le f(n)\}
\end{aligned}
$$

are nowhere dense in ${}^{\mathbb{N}}\mathbb{N}$. Now, $M_{f,A} = \bigcup_{k \in \mathbb{N}} N_k$, and $\tilde{M}_{f,A} = \bigcup_{k \in \mathbb{N}} \tilde{N}_k$. $\boxed{QED}$

Consider an enumeration $\langle f_{2\alpha} : \alpha < \mathfrak{c} \rangle$ of ${}^{\mathbb{N}}\mathbb{N}$ which uses only even ordinals. At stage α for α even, let Y_α^* be the set defined in [1], and let $\tilde{Y}_\alpha^*$ be the union of Y_α^* and the two meager sets

$$
\begin{aligned}
M_{f_\alpha, \mathbb{N}} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [g \le f_\alpha] \text{ is finite}\} \\
\tilde{M}_{f_\alpha, \mathbb{N}} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [f_\alpha < g] \text{ is finite}\}.
\end{aligned}
$$

Then $\tilde{Y}_{\alpha^*}$ is a union of less than $\mathsf{cov}(\mathcal{M})$ many meager sets. Choose $x_\alpha \in G_\alpha \setminus \tilde{Y}_\alpha^*$. In step $\alpha + 1$ of the construction let $Y_{\alpha+1}^*$ be defined as in [1], and let $\tilde{Y}_{\alpha+1}^*$ be the union of $Y_{\alpha+1}^*$ with the meager sets

$$
\begin{aligned}
M_{f_\alpha, [f_\alpha < x_\alpha]} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [g \le f_\alpha < x_\alpha] \text{ is finite}\} \\
\tilde{M}_{f_\alpha, [x_\alpha \le f_\alpha]} &= \{g \in {}^{\mathbb{N}}\mathbb{N} : [x_\alpha \le f_\alpha < g] \text{ is finite}\}.
\end{aligned}
$$

Now choose $x_{\alpha+1} \in G_{\alpha+1} \setminus \tilde{Y}_{\alpha+1}^*$. Then x_α and $x_{\alpha+1}$ witness that f_α does not avoid middles in the resulting set $L = \{x_\alpha : \alpha < \mathfrak{c}\}$. Consequently, L does not satisfy $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$.

The proof that L satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$ is as in [1]. $\boxed{QED}$

6.3 Sierpinski sets

If a Sierpinski set satisfies $\binom{\mathrm{T}}{\Gamma}$, then it satisfies $\mathsf{S}_1(\mathrm{T}, \Gamma)$ but not $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. Such a result would give another solution to Problem 31. However, as towers are null in the usual measure on ${}^{\mathbb{N}}\{0, 1\}$, it is not straightforward to construct a Sierpinski set which is a member of $\binom{\mathrm{T}}{\Gamma}$.

40 Problem. Does there exist a Sierpinski set satisfying $\binom{\mathrm{T}}{\Gamma}$?

6.4   Unsettled implications

The paper [13] ruled out the possibility that any selection property for the open case implies any selection property for the Borel case. Some implications are ruled out by constructions of [7] and [13]. Several other implications are eliminated due to critical cardinality considerations.

41 Problem. Which implications can be added to the diagram in Figure 2 and to the corresponding Borel diagram?

A summary of all unsettled implications appears in [18]. As a first step towards solving Problem 41, one may try to answer the following.

42 Problem. What are the critical cardinalities of the remaining classes?

Part 2: Variations on the theme of τ-covers

7 τ^*-covers

The notion of a τ^*-cover is a more flexible variant of the notion of a τ-cover.

43 Definition. A family $Y \subseteq P_\infty(\mathbb{N})$ is *linearly refinable* if for each $y \in Y$ there exists an infinite subset $\hat{y} \subseteq y$ such that the family $\hat{Y} = \{\hat{y} : y \in Y\}$ is linearly quasiordered by $\subseteq^*$.

A cover $\mathcal{U}$ of X is a *τ^*-cover* of X if it is large, and $h_\mathcal{U}[X]$ (where $h_\mathcal{U}$ is the function defined before Lemma 12) is linearly refinable.

For $x \in X$, we will write $x_\mathcal{U}$ for $h_\mathcal{U}(x)$, and $\hat{x}_\mathcal{U}$ for the infinite subset of $x_\mathcal{U}$ such that the sets $\hat{x}_\mathcal{U}$ are linearly quasiordered by $\subseteq^*$.

If $\mathcal{U}$ is a countable τ-cover, then $h_\mathcal{U}[X]$ is linearly quasiordered by $\subseteq^*$ and in particular it is linearly refinable. Thus every countable τ-cover is a τ^*-cover. The converse is not necessarily true. Let T^* ($\mathcal{B}_{\mathrm{T}^*}$) denote the collection of all countable open (Borel) τ^*-covers of X. Then $\mathrm{T} \subseteq \mathrm{T}^* \subseteq \Omega$ and $\mathcal{B}_\mathrm{T} \subseteq \mathcal{B}_{\mathrm{T}^*} \subseteq \Omega$.

Often problems which are difficult in the case of usual τ-covers become solvable when shifting to τ^*-covers. We will give several examples.

7.1   Refinements

One of the major tools in the analysis of selection principles is to use refinements and de-refinements of covers. In general, the de-refinement of a τ-cover is not necessarily a τ-cover.

44 Lemma. *Assume that $\mathcal{U} \in \mathrm{T}^*$ refines a countable open cover $\mathcal{V}$ (that is, for each $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$). Then $\mathcal{V} \in \mathrm{T}^*$.*

The analogous assertion for countable Borel covers also holds.

PROOF. Fix a bijective enumeration $\mathcal{U} = \{U_n\}_{n \in \mathbb{N}}$. Let $\hat{x}_{\mathcal{U}}$, $x \in X$, be as in the definition of τ^*-covers. For each n let $V_n \in \mathcal{V}$ be such that $U_n \subseteq V_n$. We claim that $\mathcal{W} = \{V_n : n \in \mathbb{N}\} \in \mathrm{T}^*$. As $\mathcal{W}$ is an ω-cover of X, it is infinite; fix a bijective enumeration $\{W_n\}_{n \in \mathbb{N}}$ of $\mathcal{W}$. For each n define $S_n = \{k : U_k \subseteq W_n\}$, and $\tilde{S}_n = S_n \setminus \bigcup_{m < n} S_m$. For each $x \in X$ define $\hat{x}_{\mathcal{W}}$ by:

$$n \in \hat{x}_{\mathcal{W}} \Leftrightarrow \tilde{S}_n \cap \hat{x}_{\mathcal{U}} \neq \emptyset.$$

Then each $\hat{x}_{\mathcal{W}}$ is a subset of $x_{\mathcal{W}}$.

Each $\hat{x}_{\mathcal{W}}$ is infinite: For each $W_{n_1}, \ldots, W_{n_k}$ choose $x_i \notin W_{n_i}$, $i = 1, \ldots, k$. Then $\{x, x_1, \ldots, x_k\} \nsubseteq W_{n_i}$ for all $i = 1, \ldots, k$. As $\mathcal{U}$ is an τ^*-cover of X, there exists $m \in \hat{x}_{\mathcal{U}}$ such that $\{x, x_1, \ldots, x_k\} \subseteq U_m$. Consider the (unique) n such that $m \in \tilde{S}_n$. Then $U_m \subseteq W_n$; therefore $W_n \notin \{W_{n_1}, \ldots, W_{n_k}\}$, and in particular $n \notin \{n_1, \ldots, n_k\}$. As $m \in \tilde{S}_n \cap \hat{x}_{\mathcal{U}}$, we have that $n \in \hat{x}_{\mathcal{W}}$.

The sets $\hat{x}_{\mathcal{W}}$ are linearly quasiordered by $\subseteq^*$: Assume that $a, b \in X$. We may assume that $\hat{a}_{\mathcal{U}} \subseteq^* \hat{b}_{\mathcal{U}}$. As $\lim_n \min \tilde{S}_n \to \infty$, we have that $S_n \cap \hat{a}_{\mathcal{U}} \subseteq S_n \cap \hat{b}_{\mathcal{U}}$ for all but finitely many n.

This shows that $\mathcal{W}$ is a τ^*-cover of X. Now, $\mathcal{V}$ is an extension of $\mathcal{W}$ by at most countably many elements. It is easy to see that an extension of a τ^*-cover by countably many open sets is again a τ^*-cover, see [17]. $\boxed{QED}$

The first consequence of this important Lemma is that

$$\mathsf{S}_{fin}(\mathfrak{U}, \mathrm{T}^*) \, (\mathsf{S}_{fin}(\mathfrak{U}, \mathcal{B}_{\mathrm{T}^*})) \quad \text{implies} \quad \mathsf{U}_{fin}(\mathfrak{U}, \mathrm{T}^*) \, (\mathsf{U}_{fin}(\mathfrak{U}, \mathcal{B}_{\mathrm{T}^*}))$$

that is, the analogue of Lemma 1 holds.

45 Corollary. *Assume that $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each $\mathcal{F}_n$ is finite, is a τ^*-cover of a space X. Then either $\cup \mathcal{F}_n = X$ for some n, or else $\mathcal{V} = \{\cup \mathcal{F}_n\}_{n \in \mathbb{N}}$ is also a τ^*-cover of X.*

PROOF. $\mathcal{U}$ refines $\mathcal{V}$. $\boxed{QED}$

7.2 Equivalences

All equivalences mentioned in Subsection 1.4 hold for τ^*-covers as well. In particular, the analogue of Theorem 5 holds (with a similar proof).

46 Corollary. *The following equivalences hold:*

(1) $\mathsf{S}_1(\mathrm{T}^*, \Gamma) = \mathsf{S}_{fin}(\mathrm{T}^*, \Gamma)$;

(2) $\mathsf{S}_1(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_{\Gamma}) = \mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_{\Gamma})$.

In fact, in the Borel case we get more equivalences in the case of τ^*-covers than in the case of τ-covers – see Subsection 7.4.

7.3   Continuous images

We now solve the problems mentioned in Remarks 19 and 22 in the case of τ^*-covers.

47 Theorem. *The following properties are equivalent for a set X of reals:*

(1) *X satisfies $\binom{\Omega}{\mathrm{T}^*}$;*

(2) *For each continuous image Y of X in $P_\infty(\mathbb{N})$, if Y is centered, then Y is linearly refinable.*

PROOF. $1 \Rightarrow 2$: The proof for this is similar to the proof of $1 \Rightarrow 2$ in Theorem 18.

$2 \Rightarrow 1$: Assume that $\mathcal{U}$ is an ω-cover of X. Replacing each member of $\mathcal{U}$ with all finite unions of Basic clopen subsets of it, we may assume that all members of $\mathcal{U}$ are clopen (to unravel this assumption we will use the fact that T^* is closed under de-refinements).

Thus, $h_\mathcal{U}$ is continuous and $Y = h_\mathcal{U}[X]$ is centered. Consequently, Y is linearly refinable, that is, $\mathcal{U}$ is a τ^*-cover of X. $\boxed{QED}$

48 Remark. The analogue assertion (to Theorem 47) for the Borel case, where open covers are replaced by Borel covers and continuous image is replaced by Borel image, also holds and can be proved similarly.

As in [14], we will use the notation

$$[f \le h] := \{n : f(n) \le g(n)\}.$$

Then a subset $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$ satisfies the excluded middle property if, and only if, there exists a function $h \in {}^{\mathbb{N}}\mathbb{N}$ such that the collection

$$\{[f \le h] : f \in Y\}$$

is a subset of $P_\infty(\mathbb{N})$ and is linearly quasiordered by $\subseteq^*$.

49 Definition. We will say that a subset $Y \subseteq {}^{\mathbb{N}}\mathbb{N}$ satisfies the *weak excluded middle property* if there exists a function $h \in {}^{\mathbb{N}}\mathbb{N}$ such that the collection $\{[f \le h] : f \in Y\}$ is linearly refinable.

Recall that $\mathsf{U}_{fin}(\Gamma, \mathrm{T}^*) = \mathsf{U}_{fin}(\mathcal{O}, \mathrm{T}^*)$.

50 Theorem. *For a zero-dimensional set X of real numbers, the following are equivalent:*

(1) *X satisfies $\mathsf{U}_{fin}(\mathcal{O}, \mathrm{T}^*)$;*

(2) *Every continuous image of X in ${}^{\mathbb{N}}\mathbb{N}$ satisfies the weak excluded middle property.*

PROOF. We make the needed changes in the proof of Theorem 2.1 in [16].

$2 \Rightarrow 1$: Assume that $\mathcal{U}_n$, $n \in \mathbb{N}$, are open covers of X which do not contain finite subcovers. For each n, replacing each member of $\mathcal{U}_n$ with all of its basic clopen subsets we may assume that all elements of $\mathcal{U}_n$ are clopen, and thus we may assume further that they are disjoint. For each n enumerate $\mathcal{U}_n = \{U_m^n\}_{m \in \mathbb{N}}$. As we assume that the elements U_m^n, $m \in \mathbb{N}$, are disjoint, we can define a function Ψ from X to $^{\mathbb{N}}\mathbb{N}$ by

$$\Psi(x)(n) = m \Leftrightarrow x \in U_m^n.$$

Then Ψ is continuous. Therefore, $Y = \Psi[X]$ satisfies the weak excluded middle property. Let $h \in ^{\mathbb{N}}\mathbb{N}$, and for each $f \in Y$, $A_f \subseteq [f \leq h]$ be such that $\{A_f : f \in Y\}$ is linearly quasiordered by $\subseteq^*$.

For each n set

$$\mathcal{F}_n = \{U_k^n : k \leq h(n)\}.$$

We claim that $\mathcal{U} = \{\cup \mathcal{F}_n\}_{n \in \mathbb{N}}$ is a τ^*-cover of X. We will use the following property.

($\star$) For each finite subset F of X and each $n \in \bigcap_{x \in F} A_{\Psi(x)}$, $F \subseteq \cup \mathcal{F}_n$.

Let $\{\cup \mathcal{F}_{k_n}\}_{n \in \mathbb{N}}$ be a bijective enumeration of $\mathcal{U}$, and let $f \in ^{\mathbb{N}}\mathbb{N}$ be such that for each n, $\cup \mathcal{F}_n = \cup \mathcal{F}_{k_{f(n)}}$. For each $x \in X$ set $\hat{x}_{\mathcal{U}} = f[A_{\Psi(x)}]$. We have the following.

$\hat{x}_{\mathcal{U}}$ is a subset of $x_{\mathcal{U}}$: Assume that $f(n) \in \hat{x}_{\mathcal{U}}$, where $n \in A_{\Psi(x)}$. Then $x \in \cup \mathcal{F}_n = \cup \mathcal{F}_{k_{f(n)}}$, therefore $f(n) \in x_{\mathcal{U}}$.

$\hat{x}_{\mathcal{U}}$ is infinite: Assume that $f[A_{\Psi(x)}] = \{f(n_1), \ldots, f(n_k)\}$ where $n_1, \ldots, n_k \in A_{\Psi(x)}$. For each $i \leq k$ choose $x_i \notin \cup \mathcal{F}_{k_{f(n_i)}}$, and set $F = \{x, x_1, \ldots, x_k\}$. Then for all $i \leq k$ $F \not\subseteq \cup \mathcal{F}_{k_{f(n_i)}}$. Choose $n \in \bigcap_{a \in F} A_{\Psi(x)}$. By property ($\star$), $F \subseteq \cup \mathcal{F}_n = \cup \mathcal{F}_{k_{f(n)}}$, therefore $f(n) \notin \{f(n_1), \ldots, f(n_k)\}$. But $n \in A_{\Psi(x)}$, thus $f(n) \in \hat{x}_{\mathcal{U}}$, a contradiction.

As the sets $A_{\Psi(x)}$ are linearly quasiordered by $\subseteq^*$, so are the sets $\hat{x}_{\mathcal{U}} = f[A_{\Psi(x)}]$.

$1 \Rightarrow 2$: Since Ψ is continuous, $Y = \Psi[X]$ also satisfies $\mathsf{U}_{fin}(\mathcal{O}, \mathrm{T}^*)$. Consider the basic open covers $\mathcal{U}_n = \{U_m^n\}_{m \in \mathbb{N}}$ defined by $U_m^n = \{f \in Y : f(n) = m\}$. Then there exist finite $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that either $Y = \cup \mathcal{F}_n$ for some n, or else $\mathcal{V} = \{\cup \mathcal{F}_n : n \in \mathbb{N}\}$ is a τ^*-cover of Y.

The first case can be split into two sub-cases: If there exists an infinite set $A \subseteq \mathbb{N}$ such that $Y = \cup \mathcal{F}_n$, then for each $n \in A$ the set $\{f(n) : f \in Y\}$ is finite, and we can define

$$h(n) = \begin{cases} \max\{f(n) : f \in Y\} & n \in A \\ 0 & \text{otherwise} \end{cases}$$

so that $A \subseteq [f \leq h]$ for each $f \in Y$, and we are done. Otherwise $Y = \cup\mathcal{F}_n$ for only finitely many n, therefore we may replace each $\mathcal{F}_n$ satisfying $Y = \cup\mathcal{F}_n$ with $\mathcal{F}_n = \emptyset$, so we are in the second case.

The second case is the interesting one. $\mathcal{V} = \{\cup\mathcal{F}_n : n \in \mathbb{N}\}$ is a τ^*-cover of Y – fix a bijective enumeration $\{\cup\mathcal{F}_{k_n}\}_{n\in\mathbb{N}}$ of $\mathcal{V}$ and witnesses $\hat{f}_\mathcal{V}$, $f \in Y$, for that. Define $h(n) = \max\{m : U_m^n \in \mathcal{F}_n\}$ for each n. Then the subsets $\{k_n : n \in \hat{f}_\mathcal{V}\}$ of $[f \leq h]$, $f \in Y$, are infinite and linearly quasiordered by $\subseteq^*$. This shows that Y is linearly refinable. $\boxed{QED}$

7.4 Borel images

Define the following notion.

T^*: The set of $X \subseteq \mathbb{R}$ such that for each linearly refinable Borel image Y of X in $P_\infty(\mathbb{N})$, Y has a pseudointersection.

By the usual method we get the following.

51 Lemma. $\mathsf{T}^* = \binom{\mathcal{B}_{\mathsf{T}^*}}{\mathcal{B}_\Gamma}$.

Clearly T^* implies T.

52 Lemma. $\mathsf{non}(\mathsf{T}^*) = \mathfrak{t}$.

PROOF. It is easy to see that $\mathsf{non}(\mathsf{T}^*)$ is the minimal size of a linearly refinable family $Y \subseteq P_\infty(\mathbb{N})$ which has no pseudointersection. We will show that $\mathfrak{t} \leq \mathsf{non}(\mathsf{T}^*)$. Assume that $Y \subseteq P_\infty(\mathbb{N})$ is a linearly refinable family of size less than $\mathfrak{t}$, and let $\hat{Y}$ be a linear refinement of Y. As $|\hat{Y}| \leq |Y| < \mathfrak{t}$, $\hat{Y}$ has a pseudointersection, which is in particular a pseudointersection of Y. $\boxed{QED}$

An application of Lemma 51 and the Cancellation Laws implies the following.

53 Theorem. $\mathsf{S}_1(\mathcal{B}_{\mathsf{T}^*}, \mathcal{B}_\Gamma) = \mathsf{T}^* \cap \mathsf{B}$.

We do not know whether $\mathsf{T}^* = \mathsf{T}$. In particular, we have the following (recall Theorem 16).

54 Problem. Is it true that every analytic set of reals satisfies T^*? Does $^\mathbb{N}\mathbb{N} \in \mathsf{T}^*$?

We do not know whether $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\mathsf{T}) = \mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathsf{T})$ or not. This can be contrasted with the following result.

55 Theorem. *For a set X of real numbers, the following are equivalent:*

(1) X satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_{\mathsf{T}^*})$,

(2) X satisfies $\mathsf{S}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_{\mathsf{T}^*})$,

(3) X satisfies $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_{\mathsf{T}^*})$;

(4) Every Borel image of X in $^{\mathbb{N}}\mathbb{N}$ satisfies the weak excluded middle property.

PROOF. Clearly $1 \Rightarrow 2 \Rightarrow 3$.

$3 \Rightarrow 4$: This can be proved like $1 \Rightarrow 2$ in Theorem 50.

$4 \Rightarrow 1$: Assume that $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$, $n \in \mathbb{N}$, are Borel γ-covers of X. We may assume that these covers are pairwise disjoint. Define a function $\Psi : X \to {}^{\mathbb{N}}\mathbb{N}$ so that for each x and n:

$$\Psi(x)(n) = \min\{k : (\forall m \geq k)\ x \in U_m^n\}.$$

Then Ψ is a Borel map, and so $Y = \Psi[X]$ satisfies the weak excluded middle property. Let $h \in {}^{\mathbb{N}}\mathbb{N}$ and $A_f \subseteq [f \leq h]$, $f \in Y$, be witnesses for that. Set $\mathcal{U} = \{U_{h(n)}^n\}_{n \in \mathbb{N}}$. For each $x \in X$ set $\hat{x}_{\mathcal{U}} = A_{\Psi(x)}$. Then $\hat{x}_{\mathcal{U}}$ is infinite and $\hat{x}_{\mathcal{U}} \subseteq [\Psi(x) \leq h] \subseteq x_{\mathcal{U}}$ for each $x \in X$, and the sets $\hat{x}_{\mathcal{U}}$ are linearly quasiordered by $\subseteq^*$. QED

7.5 Critical cardinalities

The argument of Theorem 23 implies that

$$\mathsf{non}(\mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_\Omega)) = \mathsf{non}(\mathsf{S}_{fin}(\mathrm{T}^*, \Omega)) = \mathfrak{d}.$$

56 Theorem. $\mathsf{non}(\mathsf{S}_1(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_\Gamma)) = \mathsf{non}(\mathsf{S}_1(\mathrm{T}^*, \Gamma)) = \mathfrak{t}$.

PROOF. By Theorem 53,

$$\mathsf{non}(\mathsf{S}_1(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_\Gamma)) = \min\{\mathsf{non}(\mathrm{T}^*), \mathsf{non}(\mathcal{B})\} = \min\{\mathfrak{t}, \mathfrak{b}\} = \mathfrak{t}.$$

On the other hand, $\mathsf{S}_1(\mathrm{T}^*, \Gamma)$ implies $\mathsf{S}_1(\mathrm{T}, \Gamma)$, whose critical cardinality is $\mathfrak{t}$. QED

Define the following properties.

X: The set of $X \subseteq \mathbb{R}$ such that each Borel image of X in $^{\mathbb{N}}\mathbb{N}$ satisfies the excluded middle property.

wX: The set of $X \subseteq \mathbb{R}$ such that each Borel image of X in $^{\mathbb{N}}\mathbb{N}$ satisfies the weak excluded middle property.

Recall that by Theorem 21, $\mathsf{U}_{fin}(\mathcal{B}_\Gamma, \mathcal{B}_\mathrm{T}) = \mathsf{X}$. In Theorem 55 we proved that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_{\mathrm{T}^*}) = \mathsf{wX}$. We do not know whether $\mathsf{wX} = \mathsf{X}$.

57 Problem. Is $\mathsf{non}(\mathsf{wX}) = \mathfrak{x}$?

7.6   Finite powers

In [7] it is observed that if $\mathcal{U}$ is an ω-cover of X, then for each k $\mathcal{U}^k = \{U^k : U \in \mathcal{U}\}$ is an ω-cover of X^k. Similarly, it is observed in [15] that if $\mathcal{U}$ is a τ-cover of X, then for each k $\mathcal{U}^k$ is a τ-cover of X^k. We will need the same assertion for τ^*-covers.

58 Lemma. *Assume that $\mathcal{U}$ is a τ^*-cover of X. Then for each k, $\mathcal{U}^k$ is a τ^*-cover of X^k.*

Proof. Fix k. Let $\mathcal{U} = \{U_n\}_{n\in\mathbb{N}}$ be an enumeration of $\mathcal{U}$, and let $\hat{x}_{\mathcal{U}} \subseteq x_{\mathcal{U}}$, $x \in X$, witness that $\mathcal{U}$ is a τ^*-cover of X. For each $\vec{x} = (x_0, \ldots, x_{k-1}) \in X^k$ define

$$\hat{\vec{x}}_{\mathcal{U}^k} = \bigcap_{i<k} (\hat{x_i})_{\mathcal{U}}.$$

As the sets $\hat{x}_{\mathcal{U}}$ are infinite and linearly quasiordered by $\subseteq^*$, the sets $\hat{\vec{x}}_{\mathcal{U}^k}$ are also infinite and linearly quasiordered by $\subseteq^*$. Moreover, for each $n \in \hat{\vec{x}}_{\mathcal{U}^k}$ and each $i < k$, $n \in (\hat{x_i})_{\mathcal{U}}$, and therefore $x_i \in U_n$ for each $i < k$; thus $\vec{x} \in U_n^k$, as required. QED

In [7] it is proved that the classes $\mathsf{S}_1(\Omega, \Gamma)$, $\mathsf{S}_1(\Omega, \Omega)$, and $\mathsf{S}_{fin}(\Omega, \Omega)$ are closed under taking finite powers, and that none of the remaining classes they considered has this property. Actually, their argument for the last assertion shows that assuming CH, there exist a Lusin set L and a Sierpinski set S such that $L \times L$ and $S \times S$ can be mapped continuously onto the Baire space $\mathbb{N}\mathbb{N}$. Consequently, we have that none of the classes $\mathsf{S}_1(\Gamma, \mathrm{T})$, $\mathsf{S}_{fin}(\Gamma, \mathrm{T})$, $\mathsf{U}_{fin}(\Gamma, \mathrm{T})$, $\mathsf{S}_1(\mathrm{T}, \mathcal{O})$, and their corresponding Borel versions, is closed under taking finite powers. We do not know whether the remaining 7 classes which involve τ-covers are closed under taking finite powers.

59 Theorem. $\mathsf{S}_1(\Omega, \mathrm{T}^*)$ *and* $\mathsf{S}_{fin}(\Omega, \mathrm{T}^*)$ *are closed under taking finite powers.*

Proof. We will prove the assertion for $\mathsf{S}_1(\Omega, \mathrm{T}^*)$; the proof for the remaining assertion is similar. Fix k. In [7] it is proved that for each open ω-cover $\mathcal{U}$ of X^k there exists an open ω-cover $\mathcal{V}$ of X such that the ω-cover $\mathcal{V}^k$ of X^k refines $\mathcal{U}$.

Assume that $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ is a sequence of open ω-covers of X^k. For each n choose an open ω-cover $\mathcal{V}_n$ of X such that $\mathcal{V}_n^k$ refines $\mathcal{U}_n$. Apply $\mathsf{S}_1(\Omega, \mathrm{T}^*)$ to extract elements $V_n \in \mathcal{V}_n$, $n \in \mathbb{N}$, such that $\mathcal{W} = \{V_n\}_{n\in\mathbb{N}} \in \mathrm{T}^*$. By Lemma 58, $\mathcal{W}^k$ is a τ^*-cover of X^k. For each n choose $U_n \in \mathcal{U}_n$ such that $V_n^k \subseteq U_n$. Then by Lemma 44, $\{U_n\}_{n\in\mathbb{N}}$ is a τ^*-cover of X. QED

7.7   Strong properties

Assume that $\{\mathfrak{U}_n\}_{n\in\mathbb{N}}$ is a sequence of collections of covers of a space X, and that $\mathfrak{V}$ is a collection of covers of X. The following selection principle is defined in [17].

$\mathsf{S}_1(\{\mathfrak{U}_n\}_{n\in\mathbb{N}}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ where for each n $\mathcal{U}_n \in \mathfrak{U}_n$, there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

The notion of *strong γ-set*, which is due to Galvin and Miller [5], is a particular instance of the new selection principle, where $\mathfrak{V} = \Gamma$ and for each n $\mathfrak{U}_n = \mathcal{O}_n$, the collection of open n-covers of X (we use here the simple characterization given in [17]). It is well known that the γ-property $\mathsf{S}_1(\Omega, \Gamma)$ does not imply the strong γ-property $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \Gamma)$. It is an open problem whether $\mathsf{S}_1(\Omega, \mathrm{T})$ implies $\mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathrm{T})$.

The following notions are defined in [17]. A collection $\mathfrak{U}$ of open covers of a space X is *finitely thick* if:

(1) If $\mathcal{U} \in \mathfrak{U}$ and for each $U \in \mathcal{U}$ $\mathcal{F}_U$ is a finite family of open sets such that for each $V \in \mathcal{F}_U$, $U \subseteq V \neq X$, then $\bigcup_{U\in\mathcal{U}} \mathcal{F}_U \in \mathfrak{U}$.

(2) If $\mathcal{U} \in \mathfrak{U}$ and $\mathcal{V} = \mathcal{U} \cup \mathcal{F}$ where $\mathcal{F}$ is finite and $X \notin \mathcal{F}$, then $\mathcal{V} \in \mathfrak{U}$.

A collection $\mathfrak{U}$ of open covers of a space X is *countably thick* if for each $\mathcal{U} \in \mathfrak{U}$ and each countable family $\mathcal{V}$ of open subsets of X such that $X \notin \mathcal{V}$, $\mathcal{U} \cup \mathcal{V} \in \mathfrak{U}$.

Whereas T is in general not finitely thick nor countably thick, T^* is both finitely and countably thick [17]. In [17] it is proved that if $\mathfrak{V}$ is countably thick, then $\mathsf{S}_1(\Omega, \mathfrak{V}) = \mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathfrak{V})$. Consequently, $\mathsf{S}_1(\Omega, \mathrm{T}^*) = \mathsf{S}_1(\{\mathcal{O}_n\}_{n\in\mathbb{N}}, \mathrm{T}^*)$.

7.8   Closing on the Minimal Tower problem

Clearly $\mathsf{S}_1(\mathrm{T}^*, \Gamma)$ implies $\mathsf{S}_1(\mathrm{T}, \Gamma)$, and $\mathsf{S}_1(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_\Gamma)$ implies $\mathsf{S}_1(\mathcal{B}_\mathrm{T}, \mathcal{B}_\Gamma)$. So we now have new topological lower bounds on the Minimal Tower problem.

60 Problem.   (1) Is $\mathsf{S}_1(\Omega, \Gamma) = \mathsf{S}_1(\mathrm{T}^*, \Gamma)$?

(2) Is $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma) = \mathsf{S}_1(\mathcal{B}_{\mathrm{T}^*}, \mathcal{B}_\Gamma)$?

We also have a new combinatorial bound.

61 Definition. $\mathfrak{p}^*$ is the minimal size of a centered family in $P_\infty(\mathbb{N})$ which is not linearly refineable.

62 Theorem. *The critical cardinalities of the properties* $\binom{\mathcal{B}_\Omega}{\mathcal{B}_{\mathrm{T}^*}}$ *and* $\binom{\Omega}{\mathrm{T}^*}$ *is* $\mathfrak{p}^*$.

PROOF. This follows from Theorem 47 and Remark 47. $\boxed{QED}$

The following can be proved either directly from the definitions or from the equivalence $\binom{\Omega}{\Gamma} = \binom{\Omega}{T^*} \cap \binom{T^*}{\Gamma}$.

63 Corollary. $\mathfrak{p} = \min\{\mathfrak{p}^*, \mathfrak{t}\}$. *Thus, if $\mathfrak{p} < \mathfrak{t}$ is consistent, then $\mathfrak{p}^* < \mathfrak{t}$ is consistent as well.*

We therefore have the following problem.

64 Problem. Is $\mathfrak{p} = \mathfrak{p}^*$?

7.9   The remaining Borel classes

We are left with Figure 3 for the Borel case.
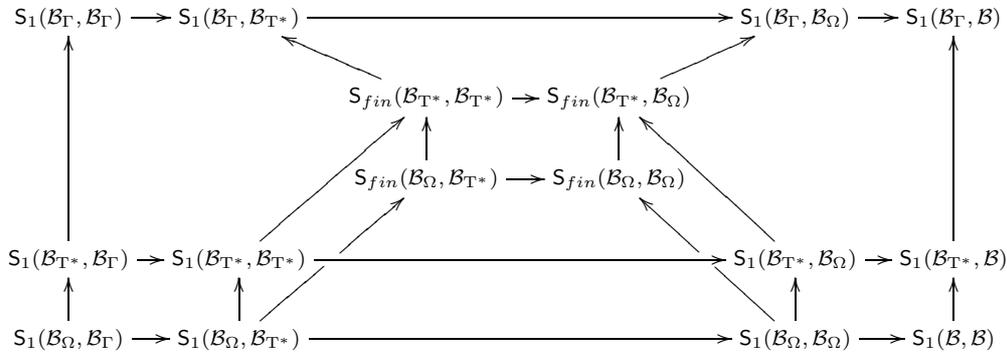
Figure 3.

8   Sequences of compatible τ-covers

When considering *sequences* of τ-covers, it may be convenient to have that the linear quasiorderings they define on X agree, in the sense that there exists a Borel linear quasiordering $\preccurlyeq$ on X which is contained in all of the induced quasiorderings. In this case, we say that the τ-covers are *compatible*. We thus have the following new selection principle:

$\mathsf{S}_1^{\preccurlyeq}(\mathrm{T}, \mathfrak{V})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of countable open *compatible* τ-covers of X there is a sequence $\{U_n\}_{n\in\mathbb{N}}$ such that for each n $U_n \in \mathcal{U}_n$, and $\{U_n\}_{n\in\mathbb{N}} \in \mathfrak{V}$.

The selection principle $\mathsf{S}_{fin}^{\preccurlyeq}(\mathrm{T}, \mathfrak{V})$ is defined similarly. Replacing "open" by "Borel" gives the selection principles $\mathsf{S}_1^{\preccurlyeq}(\mathcal{B}_\mathrm{T}, \mathfrak{V})$ and $\mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_\mathrm{T}, \mathfrak{V})$. The following

implications hold:

$$
\begin{array}{ccccc}
\mathsf{S}_1^{\preccurlyeq}(\mathrm{T},\mathfrak{V}) & \rightarrow & \mathsf{S}_{fin}^{\preccurlyeq}(\mathrm{T},\mathfrak{V}) & \rightarrow & \binom{\mathrm{T}}{\mathfrak{V}} \\
\uparrow & & \uparrow & & \\
\mathsf{S}_1(\mathrm{T},\mathfrak{V}) & \rightarrow & \mathsf{S}_{fin}(\mathrm{T},\mathfrak{V}) & &
\end{array}
$$

and similarly for the Borel case.

For $\mathfrak{V} = \Gamma$ the new notions coincide with the old ones.

65 Proposition. *The following equivalences hold:*

(1) $\mathsf{S}_1(\mathrm{T},\Gamma) = \mathsf{S}_1^{\preccurlyeq}(\mathrm{T},\Gamma) = \mathsf{S}_{fin}(\mathrm{T},\Gamma) = \mathsf{S}_{fin}^{\preccurlyeq}(\mathrm{T},\Gamma);$

(2) $\mathsf{S}_1(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\Gamma}) = \mathsf{S}_1^{\preccurlyeq}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\Gamma}) = \mathsf{S}_{fin}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\Gamma}) = \mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\Gamma}).$

PROOF. We will prove (1); (2) is similar. By Theorem 5, we have the following implications

$$
\begin{array}{ccccc}
\mathsf{S}_1^{\preccurlyeq}(\mathrm{T},\Gamma) & \rightarrow & \mathsf{S}_{fin}^{\preccurlyeq}(\mathrm{T},\Gamma) & \rightarrow & \binom{\mathrm{T}}{\Gamma} \cap \mathsf{S}_{fin}(\Gamma,\Gamma) = \mathsf{S}_1(\mathrm{T},\Gamma) \\
\uparrow & & \uparrow & & \\
\mathsf{S}_1(\mathrm{T},\Gamma) & \rightarrow & \mathsf{S}_{fin}(\mathrm{T},\Gamma) & &
\end{array}
$$

<div align="right">

$\boxed{QED}$

</div>

8.1   The class $\mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_{\mathrm{T}},\mathcal{B}_{\mathrm{T}})$

66 Definition. A *τ-cover of $\langle X,\preccurlyeq\rangle$* is a τ-cover of X such that the induced quasiordering contains $\preccurlyeq$.

67 Lemma. *Let $\langle X,\preccurlyeq\rangle$ be a linearly quasiordered set of reals, and assume that every Borel image of $\preccurlyeq$ in ${}^{\mathbb{N}}\mathbb{N}$ is bounded (with respect to $\leq^*$). Assume that $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ are Borel τ-covers of $\langle X,\preccurlyeq\rangle$. Then there exist finite subsets $\mathcal{F}_n$ of $\mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n$ is a τ-cover of $\langle X,\preccurlyeq\rangle$.*

PROOF. Fix a linear quasiordering $\preccurlyeq$ of X, and assume that $\mathcal{U}_n = \{U_k^n : k \in \mathbb{N}\}$ are Borel τ-covers of $\langle X,\preccurlyeq\rangle$. Define a Borel function Ψ from $\preccurlyeq$ to ${}^{\mathbb{N}}\mathbb{N}$ by:

$$\Psi(x,y)(n) = \min\{k : (\forall m \geq k)\ x \in U_m^n \rightarrow y \in U_m^n\}.$$

$\Psi[\preccurlyeq]$ is bounded, say by g. Now define a Borel function Φ from X to ${}^{\mathbb{N}}\mathbb{N}$ by:

$$\Phi(x)(n) = \min\{k : g(n) \leq k \text{ and } x \in U_k^n\}$$

Note that $\Phi[X]$ is a Borel image of $\preccurlyeq$ in ${}^{\mathbb{N}}\mathbb{N}$, thus it is bounded, say by f. It follows that the sequence $\{U_{g(n)}^n, \ldots, U_{f(n)}^n : g(n) \leq f(n)\}_{n\in\mathbb{N}}$ is large, and is a τ-cover of X.

<div align="right">

$\boxed{QED}$

</div>

According to [13], the property that every Borel image is bounded is equivalent to $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$.

68 Lemma. *Let $\mathcal{P}$ be a collection of spaces which is closed under taking Borel subsets, continuous images (or isometries), and finite unions. Then for each set X of real numbers, the following are equivalent:*

(1) Each Borel linear quasiordering $\preccurlyeq$ of X satisfies $\mathcal{P}$,

(2) X^2 satisfies $\mathcal{P}$.

(3) There exists a Borel linear quasiordering $\preccurlyeq$ of X satisfying $\mathcal{P}$,

PROOF. $1 \Rightarrow 2 \Rightarrow 3$: The set $\preccurlyeq = X^2$ is a linear quasiordering of X.

$3 \Rightarrow 2$: If $\preccurlyeq$ satisfies $\mathcal{P}$, then so does its continuous image $\succcurlyeq = \{(y, x) : x \preccurlyeq y\}$. Thus, $X^2 = \succcurlyeq \cup \preccurlyeq$ satisfies $\mathcal{P}$.

$2 \Rightarrow 1$: $\mathcal{P}$ is closed under taking Borel subsets.   QED

Thus, lemma 67 can be restated as follows.

69 Theorem. *If X^2 satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, then X satisfies $\mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_T, \mathcal{B}_T)$.*

PROOF. The property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ satisfies the assumptions of Lemma 68.
                                                                         QED

70 Problem. Assume that X satisfies $\mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_T, \mathcal{B}_T)$. Is it true that X^2 satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$?

Since $\mathfrak{b} = \mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})$ (in particular, the Continuum Hypothesis) implies that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is not closed under taking squares [13], a positive answer to Problem 70 would imply that the property $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ does not imply $\mathsf{S}_{fin}^{\preccurlyeq}(\mathcal{B}_T, \mathcal{B}_T)$.

ADDED IN PROOF. Problem 42 was almost completely solved in: H. Mildenberger, S. Shelah, and B. Tsaban, *The combinatorics of τ-covers* (see `http://arxiv.org/abs/math.GN/0409068`). There remains exactly one unsettled critical cardinality in the diagram. It follows from the results of that paper that the answer to the question before Theorem 55 is negative.

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