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The Pytkeev property and the Reznichenko property in function spaces

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Abstract. For a Tychonoff space X we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. Characterizations of sequentiality and countable tightness of $C_p(X)$ in terms of X were given by Gerlits, Nagy, Pytkeev and Arhangel'skii. In this paper, we characterize the Pytkeev property and the Reznichenko property of $C_p(X)$ in terms of X. In particular we note that if $C_p(X)$ over a subset X of the real line is a Pytkeev space, then X is perfectly meager and has universal measure zero.

Keywords: Function space, topology of pointwise convergence, sequential, countable tightness, Pytkeev space, weakly Fréchet-Urysohn, ω-cover, ω-shrinkable, ω-grouping property, the Menger property, the Rothberger property, the Hurewicz property, universal measure zero, perfectly meager, property (γ).

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Introduction

In this paper we assume that all topological spaces are Tychonoff spaces. Unexplained notions and terminology are the same as in [4]. We denote by $[X]^{<\omega}$ the set of finite subsets of a set X. The letter $\mathbb{N}$ is the set of natural numbers.

For a space X we denote by $C_p(X)$ the space of all real-valued continuous functions on X with the topology of pointwise convergence. Basic open sets of $C_p(X)$ are of the form

$$[x_1, x_2, \ldots, x_k; U_1, U_2, \ldots, U_k] = \{f \in C_p(X) : f(x_i) \in U_i, i = 1, 2, \ldots, k\}$$

where $x_i \in X$ and each U_i is an open subset of the real line. Since $C_p(X)$ is a topological vector space, it is homogeneous.

It is known that some topological properties of $C_p(X)$ can be well characterized by topological properties of X [3]. The purpose of the paper is to characterize the Pytkeev property and the Reznichenko property of $C_p(X)$ in terms of X.

We give some definitions to state known results. A space X is said to be *strictly Fréchet* if $x \in \bigcap_{n\in\omega} \overline{A}_n, A_n \subset X$, implies the existence of a sequence $\{x_n\}_{n\in\omega}$ such that $x_n \in A_n$ and $\{x_n\}_{n\in\omega}$ converges to x. A space X is said to be *Fréchet* if $x \in \overline{A}, A \subset X$, implies the existence of a sequence $\{x_n\}_{n\in\omega}$ such that $x_n \in A$ and $\{x_n\}_{n\in\omega}$ converges to x. A space X is said to be *sequential* if for each non-closed set $A \subset X$ there exist a point $x \in \overline{A} - A$ and a sequence $\{x_n\}_{n\in\omega}$ such that $x_n \in A$ and $\{x_n\}_{n\in\omega}$ converges to x. A space X has *countable tightness* if $A \subset X$ and $x \in \overline{A}$, then there exists a countable set $B \subset A$ such that $x \in \overline{B}$. Obviously;

strictly Fréchet $\to$ Fréchet $\to$ sequential $\to$ countable tightness.

None of these implications is reversible.

A cover $\mathcal{U}$ of a space X is called an *ω-cover* of X if for every $F \in [X]^{<\omega}$ there exists a $U \in \mathcal{U}$ with $F \subset U$. An open ω-cover is an ω-cover of open subsets. A space X is said to have *property (γ)* if for each open ω-cover $\mathcal{U}$ of X there exists a sequence $\{U_n\}_{n\in\omega} \subset \mathcal{U}$ with $X = \bigcup_{n\in\omega} \bigcap_{m\geq n} U_n$.

1 Theorem. *[1] [16] For a space X the following are equivalent.*

(1) $C_p(X)$ has countable tightness;

(2) Each finite power of X is Lindelöf;

(3) Every open ω-cover of X has a countable ω-subcover.

The implications (1) $\to$ (2) and (2) $\to$ (1) is due to Pytkeev and Arhangel'skii respectively. The equivalence (2) $\leftrightarrow$ (3) is due to [8, p. 156].

2 Theorem. *[7] [8] For a space X the following are equivalent.*

(1) $C_p(X)$ is strictly Fréchet;

(2) $C_p(X)$ is Fréchet;

(3) $C_p(X)$ is sequential;

(4) X has property (γ).

For a compact space X, $C_p(X)$ is sequential iff X is scattered [8, Corollary, p. 158]. There exists an uncountable subset of the real line with property (γ) in ZFC + CH [8, p. 160].

There exists some interesting classes of spaces between the class of sequential spaces and the class of spaces with countable tightness. A space X is said to be *subsequential* if it is homeomorphic to a subspace of a sequential space. Not each subspace of a sequential space is sequential. For a space X and $x \in X$, a family $\mathcal{N}$ of subsets of X is called a *π-network at x* if every neighborhood of x contains some element of $\mathcal{N}$. According to [14], a space X is called a *Pytkeev space* if $x \in \overline{A} - A$ and $A \subset X$ imply the existence of a countable π-network at x of infinite subsets of A. This property was introduced in [17] and every subsequential space is a Pytkeev space [17]. According to Reznichenko, a space

X is called a *weakly Fréchet-Urysohn* space (or X has the *Reznichenko property*) if $x \in \overline{A} - A$ and $A \subset X$ imply the existence of a countable infinite disjoint family $\mathcal{F}$ of finite subsets of A such that for every neighborhood V of x the family $\{F \in \mathcal{F} : F \cap V = \emptyset\}$ is finite. Every Pytkeev space is weakly Fréchet-Urysohn [14, Corollary 1.2]. Obviously every weakly Fréchet-Urysohn space has countable tightness.

subsequential → Pytkeev → weakly Fréchet-Urysohn → countable tightness

None of these implications is reversible.

For subsequential spaces, Pytkeev spaces and weakly Fréchet-Urysohn spaces, see [6], [14] and [5].

1 The Pytkeev property

An open ω-cover $\mathcal{U}$ of a space X is said to be *non-trivial* if $X \notin \mathcal{U}$. Note that for a non-trivial open ω-cover $\mathcal{U}$ of X and every $F \in [X]^{<\omega}$ the family $\{U \in \mathcal{U} : F \subset U\}$ is infinite. An open ω-cover $\mathcal{U}$ of X is said to be *ω-shrinkable* if for each $U \in \mathcal{U}$ there exists a closed set $C(U)$ of X such that $C(U) \subset U$ and $\{C(U) : U \in \mathcal{U}\}$ is an ω-cover of X. Some of $C(U)$'s may be empty.

For an $F \in [X]^{<\omega}$ and an open subset U of the real line containing the real number 0, we set $[F; U] = \{f \in C_p(X) : f(F) \subset U\}$. The family of the form $[F; U]$ is a neighborhood base of f_0, where f_0 is the constant map to the real number 0.

3 Theorem. *For a space X the following are equivalent.*

(1) $C_p(X)$ is a Pytkeev space;

(2) If $\mathcal{U}$ is an ω-shrinkable non-trivial open ω-cover of X, then there is a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of subfamilies of $\mathcal{U}$ such that $|\mathcal{U}_n| = \omega$ and $\{\bigcap \mathcal{U}_n\}_{n\in\omega}$ is an ω-cover of X.

Proof. (1) → (2) : For every $U \in \mathcal{U}$, choose a closed set $C(U)$ in X such that $C(U) \subset U$ and $\{C(U) : U \in \mathcal{U}\}$ is an ω-cover of X. Since $C_p(X)$ has countable tightness, X is Lindelöf (in particular normal) by Theorem 1. So we can take a zero-set $Z(U)$ and a cozero-set $V(U)$ with $C(U) \subset Z(U) \subset V(U) \subset U$. Without loss of generality, we may assume that for distinct $U, U' \in \mathcal{U}$ $Z(U)$ and $Z(U')$ are distinct, and every $Z(U)$ is non-empty. For every $U \in \mathcal{U}$, take a continuous map $f_U : X \to [0,1]$ such that $f_U^{-1}(0) = Z(U), f_U^{-1}(1) = X - V(U)$. Let $A = \{f_U : U \in \mathcal{U}\}$. Note that for distinct $U, U' \in \mathcal{U}$ f_U and $f_{U'}$ are distinct. Obviously $f_0 \in \overline{A} - A$. By the condition (1), there is a sequence $\{A_n\}_{n\in\omega}$ of subsets of A such that $|A_n| = \omega$ and $\{A_n\}_{n\in\omega}$ is a π-network at f_0. Take a subfamily $\mathcal{U}_n \subset \mathcal{U}$ such that $|\mathcal{U}_n| = \omega$ and $A_n = \{f_U : U \in \mathcal{U}_n\}$. Let $F \in [X]^{<\omega}$

and consider the neighborhood $[F;(-1,1)]$ of f_0. Then there is an $n\in\omega$ with $A_n\subset[F;(-1,1)]$. This means $F\subset\bigcap\{V(U):U\in\mathcal{U}_n\}\subset\bigcap\{U:U\in\mathcal{U}_n\}$. Thus $\{\bigcap\mathcal{U}_n\}_{n\in\omega}$ is an ω-cover of X.

(2) $\to$ (1) : Assume $A\subset C_p(X)$ and $f_0\in\overline{A}-A$. For every $n\in\mathbb{N}$ and $f\in A$ let $U_n(f)=\{x\in X:-1/n<f(x)<1/n\}$ and $\mathcal{U}_n=\{U_n(f):f\in A\}$. If $\{n\in\mathbb{N}:X\in\mathcal{U}_n\}$ is infinite, we can find a sequence in A which converges to f_0. Then the conclusion is trivial. So we may assume that there is an $n_0\in\mathbb{N}$ such that $\mathcal{U}_n$ is non-trivial for each $n\geq n_0$. We see that $\mathcal{U}_n$ is an ω-shrinkable open ω-cover of X. Indeed for every $n\in\mathbb{N}$ and $f\in A$ let $Z_n(f)=\{x\in X:-1/2n\leq f(x)\leq 1/2n\}$. Then obviously $Z_n(f)$ is closed in X and $Z_n(f)\subset U_n(f)$. Let $F\in[X]^{<\omega}$. The neighborhood $[F;(-1/2n,1/2n)]$ of f_0 contains an $f\in A$. Then $F\subset Z_n(f)$. Thus for each $n\geq n_0$, $\mathcal{U}_n$ is an ω-shrinkable non-trivial open ω-cover of X. By applying to $\mathcal{U}_n(n\geq n_0$) the condition (2), there is a sequence $\{\mathcal{V}_{nm}:m\in\mathbb{N}\}$ of subfamilies of $\mathcal{U}_n$ such that $|\mathcal{V}_{nm}|=\omega$ and $\{\bigcap\mathcal{V}_{nm}:m\in\mathbb{N}\}$ is an ω-cover of X. For each $n\geq n_0$ there is a sequence $\{A_{nm}:m\in\mathbb{N}\}$ of countably infinite subsets of A such that $\mathcal{V}_{nm}=\{U_n(f):f\in A_{nm}\}$. It is easy to see that the family $\mathcal{A}=\{A_{nm}:n\geq n_0,m\in\mathbb{N}\}$ is a π-network at f_0. QED

For the sake of simplicity, we call the condition (2) in the above theorem *property* (π). Property (γ) obviously implies property (π).

For $A\subset X$ and $\mathcal{V}$ a family of subsets of X, we put $\mathcal{V}|A=\{V\cap A:V\in\mathcal{V}\}$.

4 Lemma. *(1) Let X be the union of an increasing sequence $\{X_n\}_{n\in\omega}$ of subspaces of X. If each X_n satisfies property (π), then so does X.*

(2) Let X be a continuous image of Y. If Y has property (π), then so does X.

Proof. (1) Let $\mathcal{U}$ be an ω-shrinkable non-trivial open ω-cover of X. Let $\mathcal{V}_n=\mathcal{U}|X_n$. Obviously $\mathcal{V}_n$ is an ω-shrinkable open ω-cover of X_n. Assume that the set $\{n\in\omega:X_n\in\mathcal{V}_n\}$ is infinite. Then there are sequences $n_0<n_1<\cdots$ and $U_0,U_1,\cdots\in\mathcal{U}$ such that $X_{n_i}\subset U_i, i\in\omega$. Since $\mathcal{U}$ is non-trivial, the family $\{U_i:i\in\omega\}$ is infinite. Let $\mathcal{U}_i=\{U_n:n\geq i\}$ for each $i\in\omega$. Then $|\mathcal{U}_i|=\omega$ and $\{\bigcap\mathcal{U}_i\}_{i\in\omega}$ is an ω-cover of X. Therefore , without loss of generality, we may assume that each $\mathcal{V}_n$ is non-trivial. For each $n\in\omega$ take a sequence $\{\mathcal{V}_{nm}\}_{m\in\omega}$ of subfamilies of $\mathcal{V}_n$ such that $|\mathcal{V}_{nm}|=\omega$ and $\{\bigcap\mathcal{V}_{nm}\}_{m\in\omega}$ is an ω-cover of X_n. Take a subfamily $\mathcal{U}_{nm}\subset\mathcal{U}$ such that $\mathcal{V}_{nm}=\mathcal{U}_{nm}|X_n$. The collection $\{\mathcal{U}_{nm}:n,m\in\omega\}$ is a desired one.

(2) This is a routine. We omit the proof. QED

5 Corollary. *If $C_p(X)$ is a Pytkeev space, then so is $C_p(X)^\omega$.*

Proof. Assume that $C_p(X)$ is a Pytkeev space, in other words X satisfies property (π). For each $n\in\omega$ let X_n be the copy of X, $Z_n=X_0\oplus\cdots\oplus X_n$

and $Y = \oplus_{n\in\omega} X_n$, where $\oplus$ is the topological sum of spaces. Since $C_p(X)^\omega$ is homeomorphic to $C_p(Y)$, we have only to show that Y satisfies property (π).

First we note that Z_1 satisfies property (π). Let $\mathcal{U}$ be an ω-shrinkable non-trivial open ω-cover of Z_1 and ψ_i the map from X into Z_1 defined by $\psi_i(x) = x \in X_i$ for $x \in X$, where i =0, 1. Let $\{C(U) : U \in \mathcal{U}\}$ be an ω-shrinking for $\mathcal{U}$. Then obviously $\mathcal{V} = \{\psi_0^{-1}(U) \cap \psi_1^{-1}(U) : U \in \mathcal{U}\}$ is a non-trivial open ω-cover of X and the family $\{\psi_0^{-1}(C(U)) \cap \psi_1^{-1}(C(U)) : U \in \mathcal{U}\}$ is an ω-shrinking for $\mathcal{V}$. Hence there is a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of subfamilies of $\mathcal{V}$ such that $|\mathcal{V}_n| = \omega$ and $\{\bigcap \mathcal{V}_n\}_{n\in\omega}$ is an ω-cover of X. For each $n \in \omega$ we take a countably infinite subfamily $\mathcal{U}_n \subset \mathcal{U}$ such that $\mathcal{V}_n = \{\psi_0^{-1}(U) \cap \psi_1^{-1}(U) : U \in \mathcal{U}_n\}$. Then it is easy to see that $\{\bigcap \mathcal{U}_n\}_{n\in\omega}$ is an ω-cover of Z_1.

Similarly we can see that each $Z_n, n \geq 2$, satisfies property (π). Now, by applying Lemma 4 (1), Y satisfies property (π). QED

6 Lemma. *If $C_p(X)$ is a Pytkeev space, then X is zero-dimensional.*

Proof. For $x \in X$ and an open set $U \subset X$ with $x \in U$, take a continuous map $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(y) = 1$ for any $y \in X - U$. If the map f is onto, then naturally $C_p([0, 1])$ is embeddable into $C_p(X)$. Hence $C_p([0, 1])$ is a Pytkeev space. But, as shown in [13], $C_p([0, 1])$ is not a Pytkeev space. Therefore the map f is not onto. Let $r \in [0, 1] - f(X)$. Then $f^{-1}([0, r])$ is a clopen set in X with $x \in f^{-1}([0, r]) \subset U$. QED

According to [12], a space X is said to have the *Menger property* if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ such that each $\mathcal{V}_n \subset \mathcal{U}_n$ is finite and $\bigcup_{n\in\omega} \mathcal{V}_n$ is a cover of X. It is known that each finite power of X has the Menger property iff for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open ω-covers of X there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ such that each $\mathcal{V}_n \subset \mathcal{U}_n$ is finite and $\bigcup_{n\in\omega} \mathcal{V}_n$ is an ω-cover of X. Every space with the Menger property is Lindelöf.

7 Lemma. *Let X be a space such that each finite power of X has the Menger property. Then every open ω-cover of X is ω-shrinkable.*

Proof. Let $\mathcal{U}$ be an open ω-cover of X. Since each finite power of X is Lindeöf, every open ω-cover of X has a countable ω-subcover, recall Theorem 1. And the union of countably many cozero-sets is a cozero-set. Therefore, without loss of generality, we may assume that $\mathcal{U}$ is a countable non-trivial open ω-cover of X consisting of cozero-sets of X. Let $\mathcal{U} = \{U_n : n \in \omega\}$. For each $n \in \omega$, take sequences $\{Z_{nj}\}_{j\in\omega}$ of zero-sets of X and $\{U_{nj}\}_{j\in\omega}$ of cozero-sets of X satisfying $Z_{nj} \subset U_{nj} \subset Z_{nj+1}$ and $U_n = \cup_{j\in\omega} Z_{nj}$. For each $n \in \omega$, let $\mathcal{V}_n = \{U_{kj} : k \geq n, j \in \omega\}$. Since $\mathcal{U}$ is non-trivial, each $\mathcal{V}_n$ is an ω-cover of X. Take a finite subfamily $\mathcal{W}_n \subset \mathcal{V}_n$ such that $\mathcal{W} = \bigcup_{n\in\omega} \mathcal{W}_n$ is an ω-cover of X. For each $n \in \omega$, let $l(n)$ =max$\{j : U_{nj} \in \mathcal{W}\}$ if such a j exists, otherwise let

$l(n)$ be an arbitrary natural number. Then the family $\{Z_{n\,l(n)+1} : n \in \omega\}$ is an ω-shrinking of $\mathcal{U}$. QED

A subset A of a space X is said to be *perfectly meager* [15] if for each perfect set P the set $P \cap A$ is meager in P (i. e. $P \cap A$ is the union of countably many nowhere dense subsets of P). A subset A of a space X is said to have *universal measure zero* if for each Borel measure μ on X there exists a Borel set B with $A \subset B$ and $\mu(B) = 0$, where a Borel measure means a countably additive, atomless (i. e. $\mu(\{x\}) = 0$ for each $x \in X$), finite measure. For a subset A of the real line, A has universal measure zero iff for any Borel measure μ on A, $\mu(A)$=0 [15, p. 212]. Every countable subset of a space X is obviously perfectly meager and has universal measure zero. There exists an uncountable subset of the real line which is perfectly meager and has universal measure zero [15, Theorem 5.3]. In the following, we note that each subset X of the real line whose $C_p(X)$ is a Pytkeev space is perfectly meager and has universal measure zero.

8 Lemma. *Let X be a separable metric space and μ a Borel measure on X. If $C_p(X)$ is a Pytkeev space, then $\mu(X) = 0$.*

Proof. We assume that X is infinite. Let $\widetilde{X}$ be a metrizable compactification of X. Let $\tilde{\mu}$ be the Borel measure on $\widetilde{X}$ defined by $\tilde{\mu}(B) = \mu(B \cap X)$, where B is a Borel set of $\widetilde{X}$. For each $n \in \mathbb{N}$ we define a family $\mathcal{U}_n$ as follows. For each $F \in [\widetilde{X}]^{<\omega}$ take an open set $U(F)$ of $\widetilde{X}$ satisfying $F \subset U(F), X - U(F) \neq \emptyset$ and $\tilde{\mu}(U(F)) \leq 1/n$. Let $\mathcal{U}_n = \{U(F) : F \in [\widetilde{X}]^{<\omega}\}$. Each $\mathcal{U}_n$ is an open ω-cover of $\widetilde{X}$. Since $\widetilde{X}$ is compact (hence each finite power of $\widetilde{X}$ has the Menger property), there is a finite subfamily $\mathcal{V}_n \subset \mathcal{U}_n$ such that $\mathcal{V} = \bigcup_{n\in\mathbb{N}} \mathcal{V}_n$ is an ω-cover of $\widetilde{X}$. By Lemma 7, $\mathcal{V}$ is ω-shrinkable. Hence the family $\{V \cap X : V \in \mathcal{V}\}$ is an ω-shrinkable non-trivial open ω-cover of X. By Theorem 3, there is a sequence $\mathcal{W}_n \subset \mathcal{V}$ such that each $\mathcal{W}_n$ is infinite and $X \subset \bigcup_{n\in\mathbb{N}}(\bigcap \mathcal{W}_n)$. For each $m, n \in \mathbb{N}$ we may assume $|\mathcal{W}_n \cap \mathcal{V}_m| \leq 1$. Then $\tilde{\mu}(\bigcap \mathcal{W}_n) = 0$. Thus $\mu(X) = \tilde{\mu}(\bigcup_{n\in\mathbb{N}}(\bigcap \mathcal{W}_n)) = 0$. QED

9 Proposition. *Let X be a subset of the real line. If $C_p(X)$ is a Pytkeev space, then X has universal measure zero and is perfectly meager.*

Proof. That X has universal measure zero is direct by the above Lemma.

We see that X is perfectly meager. Recall that X is zero-dimensional by Lemma 6. As described in [11, p. 516], for a subset Y of a complete separable metric space is perfectly meager iff every dense in itself subset of Y is meager in itself. So we have only to see that every dense in itself subset of X is meager in itself. Let A be a dense in itself subset of X and let B be the closure of A in X. Since B is a closed subset of the zero-dimensional separable metric space X, it is a retract of X (i. e. there is a continuous map r of X onto B such that $r|B$ is the identity map), see [4, 6.2.B]. By Lemma 4(2), B has property

(π). Malykhin showed in [13, Theorem 1.5] that if a space Y is a dense in itself separable metric space and $C_p(Y)$ is a Pytkeev space, then Y is meager in itself. Hence B is meager in itself. Since A is dense in B, A is meager in itself. QED

We conclude this section with some open questions.

Question 1. If each finite power of X is Lindelöf, is any open ω-cover of X ω-shrinkable? If this question were positive, we could delete "ω-shrinkable" in Theorem 3 (2). The author does not know if any open ω-cover of the space of irrational numbers is ω-shrinkable?

Question 2. If $C_p(X)$ is a Pytkeev space, does each finite power of X have the Menger property? If this question were positive, we could delete "ω-shrinkable" in Theorem 3 (2). It is known in [2] that each finite power of X has the Menger property iff $C_p(X)$ has countable fan tightness. A space X is said to have *countable fan tightness* if $x \in \bigcap_{n\in\omega} \overline{A}_n, A_n \subset X$, implies that for each $n \in \omega$ there exists a finite set $B_n \subset A_n$ such that $x \in \overline{\bigcup_{n\in\omega} B_n}$.

Question 3. Let $C_p(X)$ be a Pytkeev space. Is $C_p(X)$ sequential (or subsequential)? As noted in [13], if X is compact, it is true.

2 The Reznichenko property

A space X is said to have the *Rothberger property* if for every sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X there exist $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \omega\}$ is a cover of X [18]. The Rothberger property is sometimes called property C''. A space X is said to have *countable strong fan tightness* if $x \in \bigcap_{n\in\omega} \overline{A}_n, A_n \subset X$, implies that there exist $x_n \in A_n$ such that $x \in \overline{\{x_n : n \in \omega\}}$. It is known [19] that $C_p(X)$ has countable strong fan tightness iff each finite power of X has the Rothberger property (i. e. property C'').

A space X is said to have the *ω-grouping property* [9] if for each open ω-cover $\mathcal{U}$ of X there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of pairwise disjoint finite subfamilies of $\mathcal{U}$ such that for each $F \in [X]^{<\omega}$ $\{n \in \omega : F \subset U$ for some $U \in \mathcal{U}_n\}$ is cofinite in ω. A space X is said to have the *Hurewicz property* [9] if for each sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of open covers of X there is a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of finite families such that for each $n \in \omega$ $\mathcal{V}_n \subset \mathcal{U}_n$ and for each $x \in X$, $\{n \in \omega : x \in V$ for some $V \in \mathcal{V}_n\}$ is cofinite in ω. These notions were used to show the following theorem.

10 Theorem. *[9, Main Theorem, Theorem 1.4] For a space X the following are equivalent.*

(1) $C_p(X)$ is weakly Fréchet-Urysohn and has countable strong fan tightness;

(2) X has the ω-grouping property and each finite power of X has the Rothberger property;

(3) Each finite power of X has both the Hurewicz property and the Rothberger property.

There is another result describing when $C_p(X)$ is weakly Fréchet-Urysohn.

11 Theorem. *[10, Theorem 19] For a space X the following are equivalent.*

(1) $C_p(X)$ is weakly Fréchet-Urysohn and has countable fan tightness;

(2) Each finite power of X has the Hurewicz property.

We characterize spaces X such that $C_p(X)$ is *only* weakly Fréchet-Urysohn (without countable fan tightness or countable strong fan tightness). As a consequence we obtain that the ω-grouping property of X implies the weak Fréchet-Urysohn property of $C_p(X)$ (compare with Theorem 10).

12 Theorem. *For a space X the following are equivalent.*

(1) $C_p(X)$ is weakly Fréchet-Urysohn;

(2) If $\mathcal{U}$ is an ω-shrinkable non-trivial open ω-cover of X, then there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of pairwise disjoint finite subfamilies of $\mathcal{U}$ such that for each $F \in [X]^{<\omega}$ $\{n \in \omega : F \subset U$ for some $U \in \mathcal{U}_n\}$ is cofinite in ω.

Proof. (1) $\to$ (2): Let $\mathcal{U}$ be an ω-shrinkable non-trivial open ω-cover of X. By the same idea as in the proof (1) $\to$ (2) of Theorem 3, for each $U \in \mathcal{U}$ we can take a continuous map $f_U : X \to [0,1]$ such that $f_U^{-1}(0) \subset U, X - U \subset f_U^{-1}(1)$ and $\{f_U^{-1}(0) : U \in \mathcal{U}\}$ is an ω-cover of X. Since $\{f_U^{-1}(0) : U \in \mathcal{U}\}$ is an ω-cover of X, we may assume that for distinct $U, U' \in \mathcal{U}$ f_U and $f_{U'}$ are distinct. Let $A = \{f_U : U \in \mathcal{U}\}$. Obviously $f_0 \in \overline{A} - A$, where f_0 is the constant map to the real number 0. Since $C_p(X)$ is weakly Fréchet-Urysohn, there exists a sequence $\{\mathcal{U}_n\}_{n\in\omega}$ of pairwise disjoint finite subfamilies of $\mathcal{U}$ such that for each neighborhood V of f_0 the family $\{F_n : F_n \cap V = \emptyset\}$ is finite, where $F_n = \{f_U : U \in \mathcal{U}_n\}$. It is easy to check that the sequence $\{\mathcal{U}_n\}_{n\in\omega}$ is a desired one.

(2) $\to$ (1): Assume $A \subset C_p(X)$ and $f_0 \in \overline{A} - A$. For every $n \in \mathbb{N}$ and $f \in A$ let $U_n(f) = \{x \in X : -1/n < f(x) < 1/n\}$ and $\mathcal{U}_n = \{U_n(f) : f \in A\}$. By the same reason as in the proof (2) $\to$ (1) of Theorem 3, we may assume that each $\mathcal{U}_n$ is non-trivial (i. e. $U_n(f) \neq X$ for each $n \in \mathbb{N}$ and $f \in A$). Since $\mathcal{U}_1$ is ω-shrinkable by the same reason as in the proof (2) $\to$ (1) of Theorem 3, there exists a sequence $\{\mathcal{V}_n\}_{n\in\omega}$ of pairwise disjoint finite subfamilies of $\mathcal{U}_1$ such that for each $F \in [X]^{<\omega}$ $\{n \in \omega : F \subset U$ for some $U \in \mathcal{V}_n\}$ is cofinite in ω. For each $n \in \omega$ we set $\mathcal{V}_n = \{U_1(f) : f \in H_n\}$, where H_n is a finite subset of A. Then the family $\{H_n\}_{n\in\omega}$ is disjoint and for each $F \in [X]^{<\omega}$ $\{n \in \omega : [F;(-1,1)] \cap H_n \neq \emptyset\}$ is cofinite in ω. We set $H = \bigcup\{H_n : n \in \omega\}, J_0 = \bigcup\{H_{2n} : n \in \omega\}$ and $J_1 = \bigcup\{H_{2n+1} : n \in \omega\}$. Obviously $f_0 \in \overline{J_0 \cup (A-H)}$ or $f_0 \in \overline{J_1 \cup (A-H)}$. Let $f_0 \in \overline{J_1 \cup (A-H)}$ and enumerate as $\{H_{2n} : n \in \omega\} = \{A_{1\,n}\}_{n\in\omega}$. Since $\mathcal{U}_2' = \{U_2(f) : f \in J_1 \cup (A-H)\}$

is an ω-shrinkable non-trivial open ω-cover of X, by the same procedure as above, there exists a disjoint family $\{A_{2\,n} : n \in \omega\}$ of finite subsets of $J_1 \cup (A - H)$ such that for each $F \in [X]^{<\omega}$ $\{n \in \omega : [F; (-1/2, 1/2)] \cap A_{2\,n} \neq \emptyset\}$ is cofinite in ω. and $f_0 \in \overline{A - \bigcup\{A_{m\,n} : n \in \omega, m = 1, 2\}}$. By repeating this operation, we have a disjoint family $\{A_{m\,n} : m \in \mathbb{N}, n \in \omega\}$ of finite subsets of A such that for each $m \in \mathbb{N}$ and $F \in [X]^{<\omega}$ $\{n \in \omega : [F; (-1/m, 1/m)] \cap A_{m\,n} \neq \emptyset\}$ is cofinite in ω. Now let $A_n = \bigcup\{A_{i\,j} : i + j = n\}$ for each $n \in \mathbb{N}$. It is not difficult to see that the disjoint family $\{A_n : n \in \mathbb{N}\}$ is a desired one. QED

13 Corollary. *If X has the ω-grouping property, then $C_p(X)$ is weakly Fréchet-Urysohn.*

By the same argument as in Corollary 5, we obtain the following.

14 Corollary. *If $C_p(X)$ is weakly Fréchet-Urysohn, then so is $C_p(X)^\omega$.*

Since a σ-compact space satisfies the ω-grouping property [10, Lemma 1.1], $C_p(X)$ over a σ-compact space X is weakly Fréchet-Urysohn. Weak Fréchet-Urysohn property of $C_p(X)$ over a σ-compact space was first pointed out by Reznichenko, see [14, p. 184]. Thus $C_p([0,1])$ is weakly Fréchet-Urysohn.

Let $X = \omega \cup \{p\}$, where p is an arbitrary point of the Čech-Stone remainder ω^*. As noted in [14, Example 1.6], X is not weakly Fréchet-Urysohn. Let $Y = C_p(X)$. Since Y is separable metrizable, $C_p(Y)$ has countable tightness. But, since $C_p(Y)$ has a subspace homeomorphic to X, $C_p(Y)$ is not weakly Fréchet-Urysohn. There exists a subset L of the real line under CH such that $C_p(L)$ is not weakly Fréchet-Urysohn, see [10, Remarks 1].

Question 4. Can we delete the condition "ω-shrinkable" in Theorem 12 (2)?

Question 5. Let P be the space of irrational numbers. Is $C_p(P)$ weakly Fréchet-Urysohn?

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3 Bibliography

References

[1] A. V. Arhangel'skii: *On some topological spaces that arise in functional analysis*, Russian Math. Surveys **31** (1976) 14–30.

[2] A. V. Arhangel'skii: *Hurewicz spaces, analytic sets and fan tightness of function spaces*, Soviet Math. Doklady **33** (1986) 396–399.

[3] A. V. Arhangel'skii: Topological Function Spaces, Kluwer, 1992.

[4] R. Engelking: General Topology, Helderman Verlag Berlin, 1989.

[5] A. Fedeli, A. Le Donne: *Pytkeev spaces and sequential extensions*, Topology Appl. **117** (2002) 345–348.

[6] S. P. Franklin, M. Rajagopalan: *On subsequential spaces*, Topology Appl. **35** (1990) 1–19.

[7] J. Gerlits: *Some properties of $C(X)$, II*, Topology Appl. **15** (1983) 255–262.

[8] J. Gerlits, Zs. Nagy: *Some properties of $C(X)$, I*, Topology Appl. **14** (1982) 151–161.

[9] Lj. Kočinac, M. Scheepers: *Function spaces and a property of Reznichenko*, Topology Appl. **123** (2002) 135–143.

[10] Lj. Kočinac, M. Scheepers: *Combinatorics of open covers (VII):Groupability*, Fund. Math., **179**, (2003), 131–155.

[11] C. Kuratowski: Topology, Vol. I, Academic Press, 1966.

[12] A. Lelek: *Some cover properties of spaces*, Fund. Math. **64** (1964) 209–218.

[13] V. I. Malykhin: *$C_p(I)$ is not subsequential*, Comment. Math. Univ. Calolinae **40** (1999) 785–788.

[14] V. I. Malykhin, G. Tironi: *Weakly Fréchet-Urysohn and Pytkeev spaces*, Topology Appl. **104** (2000) 181–190.

[15] A. W. Miller: *Special subsets of the real line*, Handbook of Set-theoretic Topology (K. Kunen and J. E. Vaughan eds.), Elsevier Science Publishers B.V., 1984, 201–233.

[16] E. G. Pytkeev: *The tightness of spaces of continuous functions*, Russian Math. Surveys **37** (1982) 176–177.

[17] E. G. Pytkeev: *On maximally resolvable spaces*, Proc. Steklov Institute of Math. **154** (1984) 225–230.

[18] F. Rothberger: *Eine Verschärfung der Eigenschaft C*, Fund. Math. **30** (1938) 50–55.

[19] M. Sakai: *Property C'' and function spaces*, Proc. Amer. Math. Soc. **104** (1988) 917–919.