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The Neumann Laplacian on spaces of continuous functions

Markus Biegert[i]

Department of Applied Analysis, University of Ulm, Germany.
markus@mathematik.uni-ulm.de

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Abstract. If $\Omega \subset \mathbb{R}^N$ is an open set, one can always define the Laplacian with Neumann boundary conditions Δ_Ω^N on $L^2(\Omega)$. It is a self-adjoint operator generating a C_0-semigroup on $L^2(\Omega)$. Considering the part $\Delta_{\Omega,c}^N$ of Δ_Ω^N in $C(\overline{\Omega})$, we ask under which conditions on Ω it generates a C_0-semigroup.

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Introduction

The question whether or not the Neumann Laplacian on $C(\overline{\Omega})$ generates a C_0-semigroup depends only on the range condition (3) in Proposition 3. It is shown by Fukushima and Tomisaki [5] that the equivalent conditions of Proposition 3 are satisfied if the boundary of Ω is Lipschitz continuous. And in fact, more general assumptions are given (Ω is allowed to have Hölder cusps). However, no counter-examples seem to be known in the literature showing that $\Delta_{\Omega,c}^N$ may not be the generator of a C_0-semigroup. In this note we first consider the one-dimensional case. Here it is actually possible to characterize those open sets for which $\Delta_{\Omega,c}^N$ is a generator. Of course, this is true if Ω is an interval. But for arbitrary open sets it is equivalent to Ω beeing the union of disjoint open intervals B_j $(j \in J)$ such that $\operatorname{dist}(B_j, \Omega \backslash B_j) > 0$ for all $j \in J$. This gives us counter-examples in $\mathbb{R}$ which are not connected. In Section 2 we construct a two-dimensional connected, bounded and open set Ω such that $\Delta_{\Omega,c}^N$ is not a generator. Actually, Ω can be taken a square minus a segment. It is noteworthy that this Ω is Dirichlet regular and therefore the Dirichlet Laplacian generates a C_0-semigroup on $C_0(\Omega)$ [1, p.401].

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Let $\mathcal{E}$ be the bilinear form on $L^2(\Omega)$ given by

$$\begin{aligned} D(\mathcal{E}) &:= H^1(\Omega), \\ \mathcal{E}(u,\varphi) &:= \int_\Omega \nabla u \nabla \varphi \, dx. \end{aligned}$$

The Neumann-Laplacian Δ_Ω^N is the selfadjoint operator on $L^2(\Omega)$ associated to the form $\mathcal{E}$, i.e.

$$\begin{aligned} D(\Delta_\Omega^N) &:= \{u \in H^1(\Omega) \mid \exists v \in L^2(\Omega) : -\mathcal{E}(u,\varphi) = (v \mid \varphi)_{L^2(\Omega)} \ \forall \varphi \in H^1(\Omega)\} \\ \Delta_\Omega^N u &:= v. \end{aligned}$$

By $\Delta_{\Omega,c}^N$ we denote the part of Δ_Ω^N in $C(\overline{\Omega})$, i.e.

$$\begin{aligned} D(\Delta_{\Omega,c}^N) &:= \{u \in D(\Delta_\Omega^N) \cap C(\overline{\Omega}) \mid \Delta_\Omega^N u \in C(\overline{\Omega})\} \\ \Delta_{\Omega,c}^N u &:= \Delta_\Omega^N u. \end{aligned}$$

1 Lemma (The maximum principle for Δ_Ω^N). *Let Ω be an open subset of $\mathbb{R}^N$ with arbitrary boundary and $u \in D(\Delta_\Omega^N)$. Then*

$$\operatorname{ess\,inf}_\Omega[u - \lambda\Delta_\Omega^N u] \le u(x) \le \operatorname{ess\,sup}_\Omega[u - \lambda\Delta_\Omega^N u] \tag{1}$$

for all positive λ and almost all $x \in \Omega$.

Proof. See [3, Théorème IX.30, p.192]. QED

A consequence of Lemma 1 is the dissipativity of $\Delta_{\Omega,c}^N$.

2 Lemma (Dissipativity). *The operator $\Delta_{\Omega,c}^N$ is dissipative.*

Proof. Let $u \in D(\Delta_{\Omega,c}^N)$. By Lemma 1 we have the estimate

$$\|u\|_{C(\overline{\Omega})} \le \|u - \lambda\Delta_{\Omega,c}^N u\|_{C(\overline{\Omega})} \ \ \forall \lambda \ge 0, \tag{2}$$

which gives the dissipativity. QED

3 Proposition. *Let $\Omega \subset \mathbb{R}^N$ be an open and bounded set with arbitrary boundary. Then the following statements are equivalent:*

(1) $\Delta_{\Omega,c}^N$ generates a C_0-semigroup.

(2) $\Delta_{\Omega,c}^N$ generates a C_0-semigroup of contractions.

(3) $R(1,\Delta_\Omega^N)C(\overline{\Omega}) \subset C(\overline{\Omega})$ and $D(\Delta_{\Omega,c}^N)$ is dense in $C(\overline{\Omega})$.

($\star$) *In this case we have $e^{t\Delta_{\Omega,c}^N} = e^{t\Delta_\Omega^N}|_{C(\overline{\Omega})}$.*

PROOF. (1) $\Rightarrow$ ($\star$): Clear, since $C(\overline{\Omega}) \hookrightarrow L^2(\Omega)$.
(1) $\Rightarrow$ (2): Follows from ($\star$) and the fact, that

$$\|e^{t\Delta_\Omega^N} u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\Omega)} \; \forall u \in L^\infty(\Omega).$$

(2) $\Rightarrow$ (3): Since $\Delta_{\Omega,c}^N$ is densely defined and dissipative the Lumer-Phillips Theorem [6, p.83] implies that $\mathrm{rg}(1 - \Delta_{\Omega,c}^N) = C(\overline{\Omega})$ and hence $1 \in \rho(\Delta_{\Omega,c}^N)$. For $f \in C(\overline{\Omega})$ let $u_1 := R(1, \Delta_\Omega^N)f$ and $u_2 := R(1, \Delta_{\Omega,c}^N)f$. Then

$$(1 - \Delta_\Omega^N)u_1 = (1 - \Delta_\Omega^N)u_2$$

which shows that $u_1 = u_2 \in C(\overline{\Omega})$.
(3) $\Rightarrow$ (2) Since $R(1, \Delta_\Omega^N)C(\overline{\Omega}) \subset D(\Delta_{\Omega,c}^N)$ we have $\mathrm{rg}(1 - \Delta_{\Omega,c}^N) = C(\overline{\Omega})$ and therefore the Lumer-Phillips Theorem implies (2). QED

We have seen that Proposition 3 gives a characterisation when $\Delta_{\Omega,c}^N$ is the generator of a C_0-semigroup, but it is not so easy to verify condition (3). The following theorem, proved by Fukushima and Tomisaki, gives a sufficient condition. The interested reader can find the general assumptions on Ω as condition (A) in [5, Section 3]. We will state this result as simple as possible.

4 Theorem (Density and Invariance). *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Then $D(\Delta_{\Omega,c}^N)$ is dense in $C(\overline{\Omega})$ and $R(1, \Delta_\Omega^N)C(\overline{\Omega}) \subset C(\overline{\Omega})$, i.e. $\Delta_{\Omega,c}^N$ generates a C_0-semigroup on $C(\overline{\Omega})$.*

1 The One-Dimensional Case

If $\Omega \subset \mathbb{R}$ is bounded, then we can even give a **sufficient** and **necessary** condition on Ω, such that $\Delta_{\Omega,c}^N$ is the generator of a C_0-semigroup.

5 Lemma. *For each ball $\mathcal{B} := B(x_0, \rho) \subset \mathbb{R}^N$ there exists $v \in C^2(\overline{\mathcal{B}})$ such that $v = \Delta v = 1$ on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ and the normal derivative $\partial v/\partial n = 0$ on $\partial\mathcal{B}$.*

PROOF. For $z \in \mathcal{B}$ and $r(z) := |z - x_0|$ we set

$$v(z) := 1 + c(r^2 - \rho^2)^2$$

where $c := 1/(8\rho^2)$. It is easy to verify that v satisfies the desired properties. QED

6 Definition. We call a bounded open set $\Omega \subset \mathbb{R}^N$ **simple**, if Ω is the union of disjoint balls B_j ($j \in J$) such that

$$\mathrm{dist}(B_j, \Omega \backslash B_j) > 0 \quad \forall j \in J.$$

7 Theorem. *Let $\Omega \subset \mathbb{R}^N$ be a bounded set which is the union of disjoint open balls B_j $(j \in J)$. Then we have the following equivalence:*

$$\boxed{R(1,\Delta_\Omega^N)C(\overline{\Omega}) \subset C(\overline{\Omega}) \Leftrightarrow \Omega \text{ is simple.}}$$

Proof. $\Rightarrow$: If Ω is not simple, then there exists $k_0 \in J$, a sequence $(k_n) \subset J\backslash\{k_0\}$, $y_0 \in \partial B_{k_0}$ and $y_n \in \partial B_{k_n}$, such that $y_n \to y_0$ as $n \to \infty$. For B_{k_0} we choose a function v with the properties in Lemma 5. Then for u defined on Ω by $u(x) := v(x)\chi_{\overline{B_{k_0}}}$ one has $u \in D(\Delta_\Omega^N)$ and $(u - \Delta_\Omega^N u) \in C(\overline{\Omega})$ but $u \notin C(\overline{\Omega})$. In fact, one has $0 = u(y_n) \to 0 \neq 1 = u(y_0)$.
$\Leftarrow$: The only problem is to show the continuity in those points x_0 on the boundary $\partial\Omega$, for which $x_0 \notin \partial B_j$ $\forall j \in J$. Let x_0 be such a point, $f \in C(\overline{\Omega})$ and $u := R(1,\Delta_\Omega^N)f$. Without loss the generality we assume that $f(x_0) = 0$. For a fixed $\varepsilon > 0$ there exists a $\delta_1 > 0$, such that $|f(x)| < \varepsilon$ $\forall x \in B(x_0,\delta_1)$. We set

$$O := \bigcup B_j$$

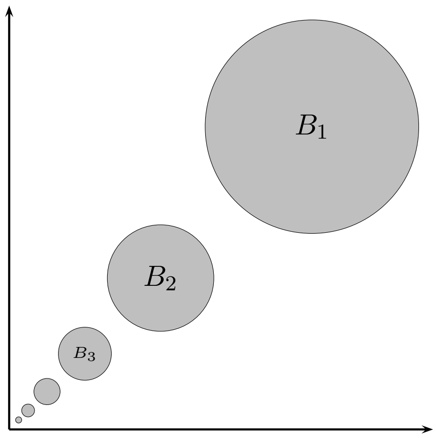
where the union is taken over all B_j $(j \in J)$ such that $B_j \subset B(x_0,\delta_1)$. Then there exists $\delta_2 \in (0,\delta_1)$ such that $B(x_0,\delta_2) \cap O = B(x_0,\delta_2) \cap \Omega$. In fact, one can take $\delta_2 := \min\{\delta_1/2, \operatorname{dist}(x_0,\Omega\backslash O)\}$. Clearly, $|u(x)| = |R(1,\Delta_O^N)f| \le \|f\|_{C(\overline{O})} \le \varepsilon$ $\forall x \in \Omega \cap B(x_0,\delta_2)$, i.e. for each sequence $x_n \in \Omega$ which converges to x_0, one has that $u(x_n)$ converges to 0.
Without the assumption that $f(x_0) = 0$, one has that $u(x_n)$ converges to $f(x_0)$. QED

8 Theorem. *Let $\Omega \subset \mathbb{R}$ be a bounded open set. Then the Neumann-Laplacian $\Delta_{\Omega,c}^N$ generates a C_0-semigroup (of contractions) on $C(\overline{\Omega})$ if and only if Ω is simple.*

Proof. Assume that Ω is simple. Then $D(\Delta_{\Omega,c}^N)$ is dense in $C(\overline{\Omega})$. In fact, let Ω be the union of disjoint balls B_j $(j \in J)$, $f \in C(\overline{\Omega})$ and $\varepsilon > 0$. Since the function f is continuous on $\overline{\Omega}$ there exists $\delta > 0$ such that $|f(x) - f(y)| < \varepsilon$ whenever $x, y \in \overline{\Omega}$ with $|x - y| < \delta$. Using the fact that $D(\Delta_{B_j,c}^N)$ is dense in $C(\overline{B_j})$ we can choose a function $f_j \in D(\Delta_{B_j,c}^N)$ such that $\|f_j - f|_{B_j}\|_{C(\overline{B_j})} < \varepsilon$. If the length of the interval B_j is less than δ then the function f_j is given by $f_j(x) := (\sup_{B_j} f - \inf_{B_j} f)/2$. Let $\tilde{f}$ be given by $\tilde{f}(x) := f_j(x)$ if $x \in B_j$. Then $\tilde{f}$ and $\Delta\tilde{f}$ are continuous on $C := \bigcup_{j\in J} \overline{B_j}$. Moreover, for every $x_0 \in \overline{\Omega} \setminus C$ and every sequence $(x_n) \subset \overline{\Omega}$ converging to x_0 one has $\lim_n \tilde{f}(x_n) = f(x_0)$ and $\lim_n \Delta\tilde{f}(x_n) = 0$, showing that $\tilde{f} \in D(\Delta_{\Omega,c}^N)$. Moreover, one has $\|\tilde{f} - f\|_{C(\overline{\Omega})} \le \varepsilon$. Now we can apply Proposition 3 and Theorem 7 to conclude the assertion. QED

9 Examples.

- $\Omega_1 := (0,1) \cup (1,2)$ is **not** simple.
- For $k \in \mathbb{N}$ let $I_k := (2^{-2k-1}, 2^{-2k})$ and $I_{-k} := (-2^{-2k}, -2^{-2k-1})$.
  Then $\Omega_2 := \bigcup_{k\in\mathbb{N}} I_k$, $\Omega_3 := \bigcup_{k\in\mathbb{N}} I_{-k}$ and $\Omega_4 := \Omega_2 \cup \Omega_3$ are simple, but they do **not** have Lipschitz boundaries.
- $\Omega_5 := (-1,0) \cup \Omega_2$ is **not** simple.
- Let $x_0 \in \mathbb{R}^N \backslash \{0\}$ and $l := |x_0|$. We set $B_k := B(x_0 \cdot 2^{1-k}, l \cdot 2^{-1-k})$.
  Then $\Omega_6 := \bigcup_{k\in\mathbb{N}} B_k$ is simple and $\Omega_7 := B(-x_0, l) \cup \Omega_6$ is **not** simple.



Ω_6: For $N = 2$ and $x_0 = (1,1)$

2 Counterexamples

We have seen some examples $\Omega \subset \mathbb{R}^N$, where the operator $R(1, \Delta_\Omega^N)$ does not leave the space $C(\overline{\Omega})$ invariant. In these examples the set Ω was never connected. Now we give an example of a connected set in $\mathbb{R}^2$, such that

$$\boxed{R(1, \Delta_\Omega^N) C(\overline{\Omega}) \not\subset C(\overline{\Omega})}$$

For this example we need the following definition

10 Definition. Let $a, b \in \mathbb{R}^2$, $a = (a_1, a_2)$ and $b = (b_1, b_2)$ such that $a < b$, i.e. $a_1 < b_1$ and $a_2 < b_2$. By $R(a,b)$ we denote the rectangle

$$R(a,b) := \{x \in \mathbb{R}^2 \mid a < x < b\}$$

and by $N(R(a,b))$ the space of functions $u \in C^2(\overline{R(a,b)})$ such that the following two conditions are satisfied:

(1) $\partial u/\partial x(a_1, y) = \partial u/\partial x(b_1, y) = 0 \ \forall y \in [a_2, b_2]$

(2) $\partial u/\partial y(x, a_2) = \partial u/\partial y(x, b_2) = 0 \ \forall x \in [a_1, b_1]$

11 Lemma. *We consider the rectangle* $\Omega := R((a,c),(b,d))$. *If* $u \in N(\Omega)$ *and* $f = u - \Delta u$, *then the following holds*

$$\int_\Omega u\varphi + \int_\Omega \nabla u \nabla \varphi = \int_\Omega f\varphi \quad \forall \varphi \in D(\mathbb{R}^2) \tag{3}$$

Remark: Since Ω has Lipschitz boundary, equation (3) holds for all $\varphi \in H^1(\Omega)$.

Proof. By Fubini's theorem it follows immediately that

$$\int_\Omega \nabla u \nabla \varphi =$$

$$\int_c^d \int_a^b \frac{\partial u}{\partial x}(x,y) \cdot \frac{\partial \varphi}{\partial x}(x,y)\, dx\, dy + \int_a^b \int_c^d \frac{\partial u}{\partial y}(x,y) \cdot \frac{\partial \varphi}{\partial y}(x,y)\, dy\, dx =$$

$$\int_c^d \frac{\partial u}{\partial x}(x,y)\varphi(x,y)\Big|_{x=a}^{b} dy - \int_c^d \int_a^b \frac{\partial^2 u}{\partial x^2}(x,y) \cdot \varphi(x,y)\, dx\, dy \ +$$

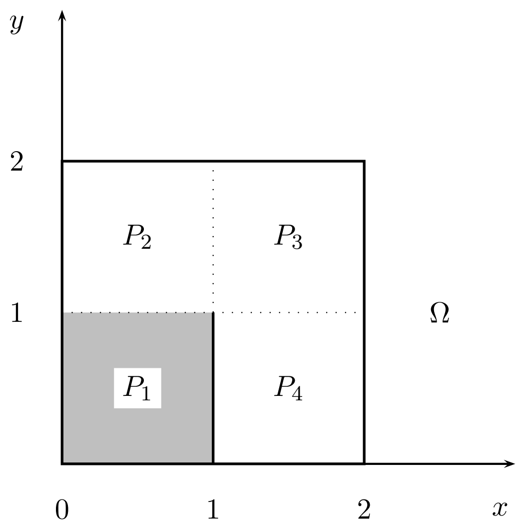
$$\int_a^b \frac{\partial u}{\partial y}(x,y)\varphi(x,y)\Big|_{y=c}^{d} dx - \int_a^b \int_c^d \frac{\partial^2 u}{\partial y^2}(x,y) \cdot \varphi(x,y)\, dy\, dx \ = - \int_\Omega \Delta u \cdot \varphi$$

QED

12 Example.
Let $\Omega \subset \mathbb{R}^2$ be given by $\Omega := R((0,0),(2,2)) \setminus \{(1,y) \in \mathbb{R}^2 | 0 < y \leq 1\}$. We denote by P_1, P_2, P_3 and P_4 the rectangles

$$P_2 := R((0,1),(1,2)),\ P_3 := R((1,1),(2,2)),$$

$$P_1 := R((0,0),(1,1)),\ P_4 := R((1,0),(2,1)).$$



(1) Let $u : [0,1] \to \mathbb{R}$ be a function in $C^2([0,1],\mathbb{R})$ with the properties

- $u(0) = u''(0) = 1$
- $u(1) = u''(1) = 1$
- $u'(0) = u'(1) = 0.$

For example we may take $u(x) := 1/(4\pi^2) \cdot [-\cos(2\pi x) + 4\pi^2 + 1]$.

(2) Let $A, L : [0,1] \to \mathbb{R}$ be functions in $C^2([0,1],\mathbb{R})$ with the properties

- $A^{(k)}(0) = A^{(k)}(1) = L^{(k)}(0) = L^{(k)}(1) = 0$ for $k = 0,\dots,2$.
- $L''(y) = L(y) - A''(y)$
- $A + L \not\equiv 0$

For example, we may take

$$A(y) := -y^{10} + 5y^9 + 80y^8 - 350y^7 + 555y^6 - 419y^5 + 150y^4 - 20y^3$$

$$L(y) := 20y^3(1-y)^5 - 50y^4(1-y)^4 + 20y^5(1-y)^3$$

For $(x,y) \in P_1$ we set $g(x,y) := u(x) \cdot A(y) + L(y)$ and for $(x,y) \in \Omega$

$$v(x,y) := \begin{cases} g(x,y) & \text{if } (x,y) \in P_1 \\ 0 & \text{else.} \end{cases}$$

In the first step we observe that $v|_{P_k} \in N(P_k)$ for $k = 1,\dots,4$. In fact, for $k = 2,3,4$ it is clear and for $k = 1$ this is equivalent to $g \in N(P_1)$.

(1) We show that $g \in C^2(\overline{P_1})$:
Since $u, A, L \in C^2([0,1])$ and $g(x,y) = u(x) \cdot A(y) + L(y)$ this is trivial.

(2) We show that $\partial g/\partial x(0,y) = \partial g/\partial x(1,y) = 0 \ \forall y \in [0,1]$:
We have $\partial g/\partial x(x,y) = u'(x) \cdot A(y)$ and $u'(0) = u'(1) = 0$.

(3) We show that $\partial g/\partial y(x,0) = \partial g/\partial y(x,1) = 0 \ \forall x \in [0,1]$:
We have $\partial g/\partial y(x,y) = u(x) \cdot A'(y) + L'(y)$ with
$A'(0) = A'(1) = L'(0) = L'(1) = 0$.

Moreover $D^\alpha g(x,1) = D^{\alpha_1} u(x) \cdot D^{\alpha_2} A(1) + \{D_2^\alpha L(1)\} \cdot \chi_{\{0\}}(\alpha_1)$ and therefore $D^\alpha g(x,1) = 0$ for all α with $|\alpha| \le 2$. This shows, that $v \in C^2(\Omega)$. Let $y_0 \in (0,1)$ be such that $A(y_0) \neq -L(y_0)$, then it follows

- $\lim\limits_{(x,y)\in P_1 \to (1,y_0)} v(x,y) = u(1)A(y_0) + L(y_0) = A(y_0) + L(y_0) \neq 0.$

- $\lim_{(x,y)\in P_4 \to (1,y_0)} v(x,y) = 0$

Therefore $v \notin C(\overline{\Omega})$. Now, we show that $v \in D(\Delta_\Omega^N)$. By Lemma 11, $v|_{P_1} \in N(P_1)$ and for any $\varphi \in H^1(\Omega)$, we have

$$\int_\Omega \nabla v \nabla \varphi = \int_{P_1} \nabla v \nabla \varphi = -\int_{P_1} \Delta v \; \varphi = -\int_\Omega \Delta v \; \varphi$$

Since $\Delta v \in L^2(\Omega)$ one has $v \in D(\Delta_\Omega^N)$.

In the third step we set $f := v - \Delta_\Omega^N v = v - \Delta v$ and we show that $f \in C(\overline{\Omega})$. Since $v \in C^2(\Omega)$ it is sufficient to show the continuity on the boundary of Ω. The only problem lies on the segment $\{1\} \times [0,1]$. Let $(1, y_0)$ be a fixed point on this segment and take a sequence $(x_n, y_n)_{n\in\mathbb{N}} \in P_1$ which converges to $(1, y_0)$. Then

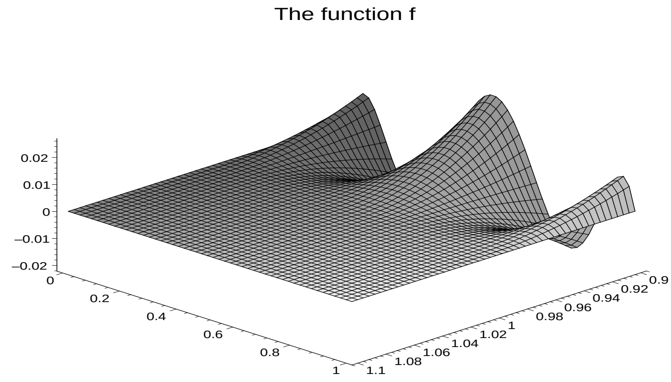
$$f(x_n, y_n) = u(x_n)A(y_n) + L(y_n) - u''(x_n)A(y_n) - u(x_n)A''(y_n) - L''(y_n) \to$$

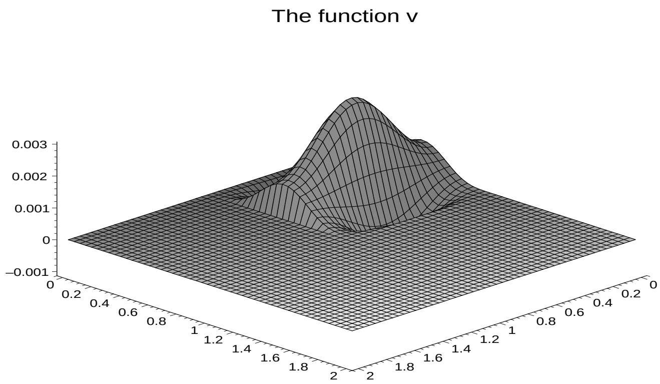
$$A(y_0) + L(y_0) - A(y_0) - A''(y_0) - L''(y_0) = 0$$
$$\Leftrightarrow L''(y_0) = L(y_0) - A''(y_0)$$

Now the function $R(1, \Delta_{\Omega,c}^N)f = v$ is not in $C(\overline{\Omega})$. This finishes the example.

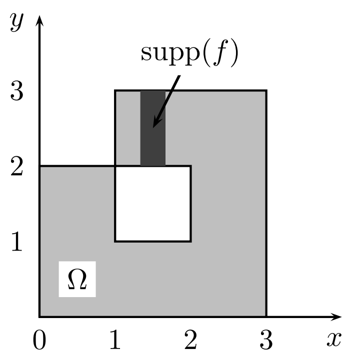
In this example, Ω is connected, Dirichlet regular and satisfies the Uniform Interior Cone Property. We remark, that the Dirichlet Laplacian Δ_0 on $C_0(\Omega)$ generates a C_0-semigroup if and only if Ω is Dirichlet regular - see [1]. But Ω is not too good, since Ω is not a Caratheodory domain, i.e. $\partial\Omega \neq \partial\overline{\Omega}$, and it does not satisfy the Exterior Cone Property.

13 Example. Let $A \subset (0,1)$ be a closed set with empty interior and $\Omega_1 := R \backslash S$, where R is the rectangle $R((0,0),(2,2))$ and $S := \{1\} \times A$. It is easy to see that $H^1(\Omega_1) = H^1(R)$, i.e. S is a removable singularity for H^1, cf. [2]. Therefore the Neumann Laplacian $\Delta_{\Omega_1,c}^N$ generates a C_0-semigroup on $C(\overline{\Omega_1}) = C(\overline{R})$. If in addition $[0,1]\backslash A$ is dense in $[0,1]$, then $H_0^1(\Omega_1) = H_0^1(\Omega) \neq H_0^1(R)$, where Ω is given by Example 12. Since Ω is Dirichlet regular, it follows that the Dirichlet Laplacian $\Delta_{\Omega_1}^D$ generates a C_0-semigroup on $C_0(\Omega_1) = C_0(\Omega)$.





14 Example. Let $\Omega \subset \mathbb{R}^2$ be as follows



For $\delta > 0$ we choose $\varphi_\delta \in C^\infty(\mathbb{R})$ such that

$$\varphi_\delta(x) = \begin{cases} 1 & \text{for } x \in (-\infty, \delta/3) \\ 0 & \text{for } x > \frac{2}{3}\delta \end{cases}$$

Then let $v : \mathbb{R} \to \mathbb{R}$ be given by

$$v_\delta(x) := \begin{cases} \varphi_\delta(x) \cdot \cosh(x) & \text{if } x > 0 \\ 0 & \text{else} \end{cases}$$

We consider the function $u \in C^\infty(\Omega)$ given by

$$u(x, y) := \begin{cases} v_1(x - 1) & \text{if } (x, y) \in (1, 2) \times (2, 3) \\ 0 & \text{else} \end{cases}$$

and the function $f := u - \Delta u$. Then $u \in D(\Delta_\Omega^N)$, $u = R(1, \Delta_\Omega^N)f$ and $f \in C^\infty(\overline{\Omega})$. Since $u \notin C(\overline{\Omega})$, $R(1, \Delta_\Omega^N)C(\overline{\Omega}) \not\subset C(\overline{\Omega})$.

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