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A ruler and segment-transporter constructive axiomatization of plane hyperbolic geometry

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Abstract. We formulate a universal axiom system for plane hyperbolic geometry in a first-order language with one sort of individual variables, *points* (lower-case), containing three individual constants, a_0, a_1, a_2, standing for three non-collinear points, with $\Pi(a_0a_1) = \pi/3$, one quaternary operation symbol $\tilde{\iota}$, with $\tilde{\iota}(abcd) = p$ to be interpreted as 'p is the point of intersection of lines $\overline{ab}$ and $\overline{cd}$, provided that lines $\overline{ab}$ and $\overline{cd}$ are distinct and have a point of intersection, an arbitrary point, otherwise', and two ternary operation symbols, $\varepsilon_1(abc)$ and $\varepsilon_2(abc)$, with $\varepsilon_i(abc) = d_i$ (for $i = 1, 2$) to be interpreted as 'd_1 and d_2 are two distinct points on line $\overline{ac}$ such that $ad_1 \equiv ad_2 \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise'.

Keywords: Hyperbolic geometry, constructive axiomatization, quantifier-free axiomatization.

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Introduction

J. Strommer [11] showed that in hyperbolic geometry all constructions, which are possible based on Hilbert's Axioms I-IV can be carried out with a ruler and a segment transporter, if two limiting parallel lines are given. The segment-transporter is an instrument that lays off on a given ray a segment congruent to a given segment. A hyperbolic plane, i. e. a Hilbert plane satisfying the axiom of limiting parallels, is uniquely characterized by its abstract field constructed by means of Hilbert's end-calculus. With coordinates from this field one can develop non-Euclidean trigonometry ([4, Ch. 7, §41-43]). Using hyperbolic trigonometry, M. N. Gafurov [1] showed that in a hyperbolic plane two limiting parallel lines can be constructed using a ruler and a gauge, if the opening of the gauge is such that a segment of length x, with $\tanh x = 1/2$, is constructible. (Gafurov's gauge is an instrument that lays off a segment of fixed length on a given ray). We will show that these results turn out to be relevant for constructive axiomatizations of elementary hyperbolic geometry.

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An axiomatization formulated in a first-order language in which the axioms are universal statements is called a *constructive* axiomatization. Such axiomatizations of hyperbolic geometry are but only ‘fragments’ ([8]), since inside first-order logic the Löwenheim-Skolem theorem does not allow a characterization up to isomorphism of the classical Beltrami-Klein model. Several authors formulated axioms systems for hyperbolic geometry inside first-order logic (see [8] for an overview). In 1938, K. Menger [5] observed that hyperbolic geometry can be axiomatized based on point-line incidence alone. Building on Menger’s and his students’ work H. L. Skala [10] produced a first-order axiom system for hyperbolic geometry formulated in a bi-sorted language, with individual variables for *points* and *lines*, and a single binary relation — as a primitive notion, with $P|l$ to be read as ‘point P is incident with line l’.

Recently, starting from Skala’s axiom system, V. Pambuccian [7] showed that plane hyperbolic geometry over Euclidean ordered fields can be constructively axiomatized in a first-order language $\mathcal{L}$ with two sorts of individual variables, *points* and *lines*, containing three individual constants standing for three non-collinear points, two binary operation symbols, φ and ι, and two binary operation symbols, $\pi_1(P,l)$ and $\pi_2(P,l)$. In this axiom system, $\varphi(A,B) = l$ is interpreted as ‘l is the line joining A and B, if $A \neq B$, an arbitrary line, otherwise’; $\iota(g,h) = P$ is interpreted as ‘P is the point of intersection of g and h, when g and h are distinct lines and have a point of intersection, an arbitrary point, otherwise’; and for $i = 1, 2$, $\pi_i(P,l) = g_i$ is interpreted as ‘g_1 and g_2 are the two limiting parallel lines from P to l, if P is not on l, otherwise g_i is an arbitrary line’. The operations π_1, π_2 may be interpreted as an instrument that constructs the limiting parallel lines through a point to a line not incident with the point.

The purpose of this paper is to provide a constructive axiomatization of plane hyperbolic geometry over Euclidean ordered fields in a language that corresponds to constructions with a ruler and an instrument that we shall call *segment - transporter*. The ruler constructs new points by intersecting two lines $\overline{ab}$ and $\overline{cd}$, while the segment - transporter constructs the points of intersection of a circle with a line passing through the center of the circle, but is *not* capable of selecting one of the two points, such as the point rightmost on the ray $\overrightarrow{cd}$. Our universal axiom system for hyperbolic geometry is formulated in $\mathcal{L}_0$, a first-order language with individual variables for *points* (lower-case), one quaternary operation symbol $\tilde{\iota}$ and two ternary operation symbols, ε_1 and ε_2 as primitive notions, with $\tilde{\iota}(abcd) = p$ to be interpreted as ‘p is the point of intersection of lines $\overline{ab}$ and $\overline{cd}$, if the lines $\overline{ab}$ and $\overline{cd}$ are distinct and have a point of intersection, an arbitrary point, otherwise’, and $\varepsilon_i(abc) = d_i$ (for $i = 1, 2$) to be interpreted as ‘d_1 and d_2 are two distinct points on line $\overline{ac}$ such that $ad_i \equiv ab$, provided that $a \neq c$, an arbitrary point, otherwise'. The language $\mathcal{L}_0$ contains also three individual constants, a_0, a_1, a_2, to be interpreted as three non-collinear points, with $\Pi(a_0a_1) = \pi/3$, where $\Pi(a_0a_1)$ is the angle of parallelism of the segment a_0a_1.

The paper is organized as follows: in section 2, using Gafurov's and Strommer's results, we define inside the language $\mathcal{L}_0$ the operations $\tilde{\pi}_1(pab)$, $\tilde{\pi}_2(pab)$, where $\tilde{\pi}_i(pab) = p_i$, $i = 1, 2$, will be interpreted as 'p_i are points on the two limiting parallel rays from point p to line $\overline{ab}$'; in section 3 we formulate our axiom system in two steps: first, we show how to rephrase most of the Pambuccian axioms in our language $\mathcal{L}_0$, and then we state the axioms that will give the desired interpretations for our primitive operations ε_1, ε_2, and $\tilde{\iota}$; finally, in section 4 we prove the adequacy of our system.

1 Definitions of operations and relations

In this section we will define the operations $\tilde{\pi}_1$, $\tilde{\pi}_2$ in the language $\mathcal{L}_0$. We will also show that these definitions are valid in hyperbolic geometry if the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations, and a_0, a_1, a_2 are three non-collinear points such that $\Pi(a_0a_1) = \pi/3$. We start by defining the notions of collinearity and 'two lines coincide', and then translate in $\mathcal{L}_0$ two simple constructions in neutral geometry.

(i) $C(abc) \leftrightarrow (\bigvee_{i=1}^{2} \varepsilon_i(abc) = b) \vee a = c$

may be read as 'a, b, c are collinear';

(ii) $\overline{C}(abc) \leftrightarrow a \neq b \wedge b \neq c \wedge c \neq a \wedge (\bigvee_{i=1}^{2} \varepsilon_i(abc) = b)$

may be read as 'a, b, c are three different collinear points';

(iii) $\eta(abcd) \leftrightarrow a \neq b \wedge c \neq d \wedge C(acb) \wedge C(adb)$

may be read as ' lines $\overline{ab}$ and $\overline{cd}$ coincide';

(iv) $\sigma(ab) = p \leftrightarrow (a = b \wedge p = b) \vee (a \neq b \wedge (\bigvee_{i=1}^{2} p = \varepsilon_i(abb) \wedge p \neq b))$

may be interpreted as '$\sigma(ab)$ is the reflection of b in a';

(v) $\mu_c(ab) = m \leftrightarrow (a = b \wedge m = a) \vee (\neg C(abc) \wedge m = \tilde{\iota}(c\tilde{\iota}(a\sigma(bc)\sigma(ac)b)ab))$
$\vee(C(abc) \wedge a \neq b \wedge m = c)$

$\mu_c(ab)$ will be used only if $c \neq b \wedge (\bigvee_{i=1}^{2} \varepsilon_i(cab) = b)$. It may be interpreted as '$\mu_c(ab)$ is the midpoint of the segment ab, provided $a \neq b$, and $m = a$ if $a = b$', and will be used only in the presence of a point c equidistant from a and b.

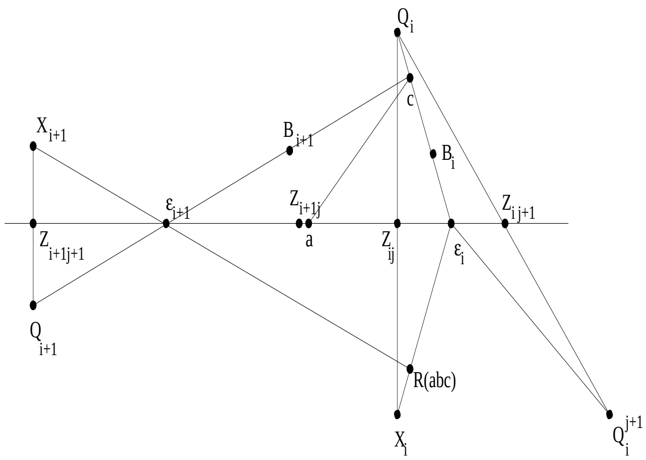

Figure 1. Definition of $R(abc)$

For the definition of the operation 'reflection of a point in a line' we will modify the construction of $F(abc)$ in [6], where $F(abc)$ may be read as the 'footpoint of the perpendicular line from c to line $\overline{ab}$'. To do this we have to take into account that ε_1, ε_2 are orientation-blind operations (Figure 1). For i, $j \in \{1, 2\}$ let: $\varepsilon_i = \varepsilon_i(acb)$, $B_i(abc) = \mu_a(c\varepsilon_i(acb))$, $Q_i(abc) = \varepsilon_1(\varepsilon_i(acb)ac)$, $Z_{ij}(abc) = \varepsilon_j(\varepsilon_i(acb)B_i(abc)a)$, $Q_i^j(abc) = \sigma(Z_{ij}(abc)Q_i(abc))$. We have:

(vi) $\neg C(abc) \to [X_i(abc) = d \leftrightarrow \bigvee_{1\le j,k\le 2}(\varepsilon_k(\varepsilon_i(acb)Q_i^j(abc)a) = a \wedge d = Q_i^j(abc))]$, for $i = 1, 2$,

(vii) $R(abc) = c' \leftrightarrow (\neg C(abc) \wedge c' = \tilde{\iota}(X_1(abc)\varepsilon_1(abc)X_2(abc)\varepsilon_2(abc)) \vee (a \neq b \wedge C(abc) \wedge c' = c) \vee (a = b \wedge c' = \sigma(ac))$,

which may be read as '$R(abc)$ is the reflection of c in line ab'.

To see that for three non-collinear points a, b, c the definition of $R(abc)$ holds in neutral geometry, we will use the following abbreviations: for $i, j \in \{1, 2\}$ let: $\varepsilon_i := \varepsilon_i(acb)$, $B_i := B_i(abc)$, $Q_i := Q_i(abc)$, $Z_{ij} := Z_{ij}(abc)$, $Q_i^j := Q_i^j(abc)$, and $X_i := X_i(abc)$. B_i may be read as the midpoint of segment $c\varepsilon_i$; Q_i is a point on line $\overline{c\varepsilon_i}$ such that $\varepsilon_i Q_i \equiv \varepsilon_i a$; Z_{ij} are points on line $\overline{ab}$ with $B_i\varepsilon_i \equiv Z_{ij}\varepsilon_i$; Q_i^j is the reflection of Q_i in Z_{ij}; X_i is one of the points Q_i^1, Q_i^2, which satisfies $X_i\varepsilon_i \equiv \varepsilon_i a$. The points Q_i, $i = 1, 2$, could lie either on ray $\overrightarrow{c\varepsilon_i}$ or on ray $\overrightarrow{\varepsilon_i c}$. We note that the construction of $R(abc)$ is independent of the position of Q_1 and Q_2. We may assume w. l. o. g. that Q_i is on the ray $\overrightarrow{\varepsilon_i c}$. For a unique $j \in \{1, 2\}$, the triangle $\Delta Q_i^j \varepsilon_i Z_{ij}$ is congruent to the triangle $\Delta a B_i \varepsilon_i$. For that j the line

$\overline{Q_i^j Z_{ij}}$ is perpendicular to $\overline{ab}$ and $Q_i^j\varepsilon_i \equiv a\varepsilon_i$. Thus $X_i = Q_i^j$ and the point of intersection of lines $\overline{X_1\varepsilon_1}$ and $\overline{X_2\varepsilon_2}$ is the symmetric point of c with respect to line $\overline{ab}$.

It is not very hard to see that the next five operation definitions hold in absolute geometry if the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations and a_0, a_1, a_2 are three non-collinear points.

(viii) $a \neq b \wedge F(abc) = q \leftrightarrow (\neg C(abc) \wedge q = \tilde{\iota}(abR(abc)c)) \vee (C(abc) \wedge q = c)$

may be read as '$F(abc)$ is the footpoint of the perpendicular from point c to line $\overline{ab}$';

(ix) $a \neq b \wedge P(ab) = p \leftrightarrow \bigvee_{k=0}^{2}(\neg C(aba_k) \wedge p = \mu_a(R(aba_k)\sigma(aa_k)))$,

may be read as '$P(ab)$ is a point on the perpendicular from point a to line $\overline{ab}$';

(x) $M(ab) = m \leftrightarrow (a = b \wedge m = a)$

$\vee(a \neq b \wedge (\bigvee_{1\leq i,j\leq 2} m = \tilde{\iota}(ab\varepsilon_i(abP(ab))\varepsilon_j(baP(ba))))$,

may be read as '$M(ab)$ is the midpoint of segment ab';

(xi) $x \neq y \wedge T_i(pqxy) = t_i \leftrightarrow (p = q \wedge t_i = x) \vee (p \neq q \wedge x = q \wedge t_i = \varepsilon_i(xpy))$

$\vee (p \neq q \wedge x \neq q \wedge t_i = \varepsilon_i(x\sigma(M(qx)\varepsilon_1(qpx))y))$, for $i = 1, 2$

may be read as '$T_1(pqxy)$, $T_2(pqxy)$ are two points on line $\overline{xy}$ such that the segments $xT_1(pqxy)$ and $xT_2(pqxy)$ are congruent to segment pq'.

The next operation definition rephrases in the language $\mathcal{L}_0$ of S. Guber's [3] ruler and gauge construction in absolute geometry of transport of an angle to a half ray. If $b \neq x$, let $H(pbx) := R(M(bx)P(M(bx)b)p)$, to be read as 'the reflection of p in the perpendicular bisector of segment bx, and $G(abxy) := M(H(abx)T_1(abxy))$, to be read as 'the midpoint of the segment determined by the reflection of point a in the perpendicular bisector of segment bx and one of the points on line $\overline{xy}$ obtained by laying off at x the segment ab'.

(xii) $x \neq y \wedge \neg C(abc) \rightarrow [A(abcxy) = v \leftrightarrow (b = x \wedge v = R(xM(a\varepsilon_1(xay))c)$

$\vee(b \neq x \wedge v = R(xG(abxy)H(cbx)))]$

may be read, if a, b, c are not collinear and $x \neq y$, as '$A(abcxy)$ is a point such that $bc \equiv xA(abcxy)$ and $\angle A(abcxy)xT_1(abxy) \equiv \angle cba$'.

To see that these definitions hold in absolute geometry let a, b, c be three non-collinear points and x, y two distinct points, with $b \neq x$ (for $b = x$ we

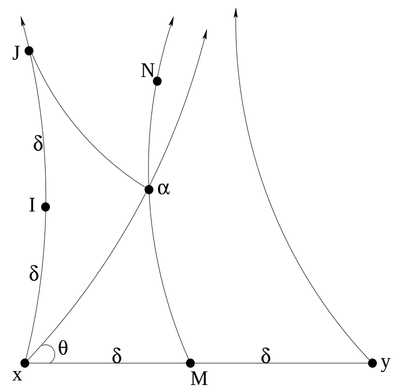

Figure 2. Definition of operation $\alpha(xy)$

reason analogously). The following abbreviations $H_a := H(abx)$, $H_c := H(cbx)$, $G := G(abxy)$, $A := A(abcxy)$, $T := T_1(abxy)$ may then be read as: H_a, H_c are the reflections of points a and c in the perpendicular bisector of segment bx, T is a point on line $\overline{xy}$ such that $ab \equiv xT$, and G is the midpoint of segment H_aT. The line $\overline{xG}$ is the bisector of angle $\angle H_axT$ and the perpendicular bisector of H_aT. Since $\angle cab \equiv \angle H_cxH_a$ it follows that $\angle H_cxH_a \equiv \angle AxT$. Thus the definition of A holds in hyperbolic geometry.

The abbreviations $I(xy)$, $J(xy)$, $N(xy)$, and operation $\alpha(xy)$ below are used to translate in $\mathcal{L}_0$ Gafurov's [1] construction of a pair of limiting parallel lines:

$x \neq y \wedge I(xy) = p \leftrightarrow p = \varepsilon_1(xM(xy)P(xy))$,

$x \neq y \wedge J(xy) = q \leftrightarrow \bigvee_{i=1}^{2}(\varepsilon_i(I(xy)xx) \neq x \wedge q = \varepsilon_i(I(xy)xx))$,

$x \neq y \wedge N(xy) = n \leftrightarrow n = P(M(xy)y)$.

(xiii) $x \neq y \wedge \alpha(xy) = r \leftrightarrow r = F(M(xy)N(xy)J(xy))$

may be read as '$\alpha(xy)$ is the footpoint of the perpendicular from point x to the perpendicular bisector of segment xy'.

The operation $\alpha(xy)$ will be used only when the angle of parallelism of the segment xy is $\pi/3$. In this case, the line $\overline{x\alpha(xy)}$ is limiting parallel to the perpendicular line at y to $\overline{xy}$.

We show that in hyperbolic geometry, if the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations, a_0, a_1, a_2 are three non-collinear points, and x, y are two points such that $\Pi(xy) = \pi/3$, then $\alpha(xy)$ has the desired interpretation. We will use the following abbreviations: $M := M(xy)$, $I := I(xy)$, $J := J(xy)$, $N := N(xy)$, $\alpha := \alpha(xy)$, which may be read as: M is the midpoint of segment xy; I, J are distinct points on the perpendicular line at x to line $\overline{xy}$ and $IJ \equiv$

$Ix \equiv xM$; N is a point on the perpendicular at M to line $\overline{xy}$; α is the footpoint of the perpendicular from point J to the perpendicular bisector of segment xy. Let 2δ and γ be the hyperbolic lengths of the segments xy and $M\alpha$, respectively. Let θ denote the radian measure of angle $\angle yx\alpha$. In the Lambert quadrilateral $JxM\alpha$ we have $\tanh\gamma = \tanh 2\delta/\cosh\delta$ ([2] p.415). From the right triangle $\Delta xM\alpha$ we obtain $\tan\theta = \tanh\gamma/\sinh\delta$. It follows that $\tan\theta = 2/\cosh 2\delta$. Since $\tan\Pi(xy) = 1/\sinh xy$ and $\Pi(xy) = \pi/3$, we have that $\sinh 2\delta = 1/\sqrt{3}$. Hence $\theta = \pi/3$. It follows that $\angle yx\alpha$ is the angle of parallelism of the segment xy and $\overline{x\alpha(xy)}$ is limiting parallel to the perpendicular line at y to $\overline{xy}$.

We now come to the final step. We will use Strommer's [11] construction of a limiting parallel line through a point to a given line when in the hyperbolic plane there is already given a pair of limiting parallel lines. We will need the following abbreviations to be used only when $\neg C(abc)$ and $x \neq y$:

$D(pabxy) = T_1(pF(abp)yx)$, $E(pabxy) = T_2(pF(abp)yx)$,
$S(pabxy) = R(y\alpha(xy)D(pabxy))$, $W(pabxy) = R(yP(yx)S(pabxy))$,
$U(pabxy) = R(x\alpha(xy)E(pabxy))$, and
$V(pabxy) = F(U(pabxy)W(pabxy)D(pabxy))$. Then

(xiv) $\neg C(abp) \wedge x \neq y \rightarrow \Psi_1(pabxy) := A(V(pabxy)D(pabxy)ypF(abp))$

(xv) $\neg C(abp) \wedge x \neq y \rightarrow \Psi_2(pabxy) := R(pF(abp)\Psi_1(pabxy))$

may be read as 'p, a, b are three distinct non-collinear points, $\Psi_1(pabxy)$, $\Psi_2(pabxy)$ are points on each of the two limiting parallel lines from point p to line $\overline{ab}$', and will be used only when the distance between x and y is 2δ.

If the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations, and a_0, a_1, a_2 are three non-collinear points, x, y are two points such that $\Pi(xy) = \pi/3$, then in hyperbolic geometry the operations $\Psi_1(pabxy)$, $\Psi_2(pabxy)$ construct points on the limiting parallel lines from p to $\overline{ab}$. Let $F := F(abp)$, $D := D(pabxy)$, $E := E(pabxy)$, $S := S(pabxy)$, $W := W(pabxy)$, $U := U(pabxy)$, $V := V(pabxy)$. These abbreviations may be read as: F is the footpoint of the perpendicular from p to line $\overline{ab}$; D and E are points on $\overline{xy}$ such that $Dy \equiv Ey \equiv Fp$; S is the reflection of D in the line $\overline{x\alpha(xy)}$; W is the reflection of S in the perpendicular line at y to $\overline{xy}$; U is the reflection of E in the line $\overline{x\alpha(xy)}$; and V is the footpoint of the perpendicular from D to line $\overline{UW}$. From the interpretation of $\alpha(xy)$ in (xiii) above, the line $\overline{x\alpha(xy)}$ and the perpendicular line at y to $\overline{xy}$ are limiting parallel. Let Σ be their common rimpoint. If $\Upsilon = \sigma_3 \circ \sigma_2 \circ \sigma_1$, where σ_1 is the reflection in the perpendicular line at y to $\overline{xy}$, σ_2 the reflection in the line $\overline{x\alpha(xy)}$, and σ_3 the reflection in the line $\overline{D\Sigma}$, then $\Upsilon^{-1}(D) = W$. As Υ is also a reflection in a line (by the three-reflection theorem), we have $\Upsilon(D) = \Upsilon^{-1}(D)$.

Hence W and U are symmetric with respect to the line $\overline{D\Sigma}$. Thus D, V and Σ are collinear. It follows that the angle $\angle VDy$ is the angle of parallelism of the segment Dy. Since $Dy \equiv Fp$, when we transport the angle $\angle VDy$ back to pF with vertex at p, the line $\overline{p\Psi_1(pabxy)}$ is limiting parallel to $\overline{ab}$. By reflecting the point $\Psi_1(pabxy)$ in $\overline{pF}$ we obtain that $\overline{p\Psi_2(pabxy)}$ is the other limiting parallel line. Finally, we define:

(xvi) $\neg C(abp) \rightarrow \tilde{\pi}_i(pab) := \Psi_i(paba_0a_1)$, for $i = 1, 2$,

$\tilde{\pi}_i(pab)$ may be read when a, b, p are non-collinear as '$\tilde{\pi}_1(pab)$, $\tilde{\pi}_2(pab)$ are points on the two limiting parallel lines from point p to line $\overline{ab}$'.

We have shown above that the definitions of $\tilde{\pi}_i(pab)$ in $\mathcal{L}_0$ are valid in hyperbolic geometry if the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations, and a_0, a_1, a_2 are three non-collinear points, with $\Pi(a_0a_1) = \pi/3$.

2 The axiom system

To define our axiom system we start with $\Sigma = \{C1, \ldots, C25, \mathbf{pas}, \mathbf{pap}, \mathbf{des}\}$ from [7]. Let $\Sigma' = \Sigma \backslash \{C1, C2, C10\}$. We will first rephrase the axioms in Σ' in the language $\mathcal{L}_0$. The individual variables that were interpreted as *points* in $\mathcal{L}$ will be interpreted as *points* in $\mathcal{L}_0$ as well, but will be denoted by lower case letters.

A careful reading of the axioms in Σ and the abbreviations used to state them shows that in each axiom, with the exception of C10, every line occurs as a line determined by two distinct points, i.e. as $\varphi(a, b)$, with $a \neq b$; every limiting parallel line as the parallel from a point to a line determined by two distinct points, i. e. as $\pi_k(a, \varphi(b, c))$, where $b \neq c$; every intersection of two lines occurs only as the intersection of lines determined by two pairs of points, i. e. as $\iota(\varphi(a, b), \varphi(c, d))$. In $\mathcal{L}_0$ we do not have an analogue of the operation φ, but we do have anologues of π_k and ι. Henceforth, every occurrence in Σ' of the form $\pi_k(a, \varphi(b, c))$ and $\iota(\varphi(a, b), \varphi(c, d))$ will be replaced with $\tilde{\pi}_k(abc)$ and $\tilde{\iota}(abcd)$, respectively.

The axioms in Σ' are expressed in terms of the following notions and their negations: 'three points are collinear' and 'two lines are equal'. To ensure that we have the correct translations in $\mathcal{L}_0$, we will replace $\lambda(a, b, c)$ with $C(abc)$ and $\varphi(a, b) = \varphi(c, d)$ with $\eta(abcd)$. For example, every term of the form $\pi_k(a, \varphi(b, c))$ $= \pi_j(x, \varphi(y, z))$ will be replaced with $\eta(a\tilde{\pi}_k(abc)x\tilde{\pi}_j(xyz))$. Let Σ'' denote the axioms in Σ' rephrased in $\mathcal{L}_0$ as indicated.

Axiom C21 in [7] states that: if a, b, p are three non-collinear points, denote by Π_a and Π_b the ends which are incident with rays $\overrightarrow{pa}$ and $\overrightarrow{pb}$ respectively, let c, q, u, v, x be the intersection points of $a\Pi_b$ and $b\Pi_a$, ab and pc, $q\Pi_a$ and

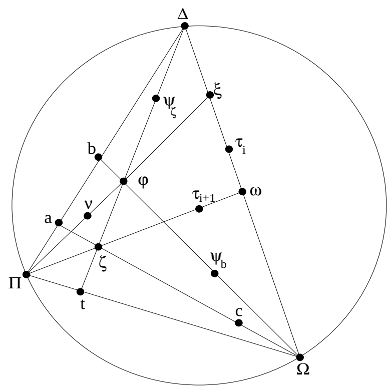

Figure 3. Definition of operation $\xi(abc)$

pb, $b\Pi_a$ and $q\Pi_b$, uv and pc, respectively. Then x lies on $\Pi_a\Pi_b$. We denote by $\omega(pab)$ the point x given by axiom C21 rephrased in $\mathcal{L}_0$.

To state axiom G8 of our system we will need the abbreviation $\varrho(p_1, q_1; p_2, q_2)$ from [7], which may be read as 'the rays $\overrightarrow{p_1q_1}$ and $\overrightarrow{p_2q_2}$ have a rimpoint in common'. In addition, we will use the operation $\xi(abc)$ defined below. First, for each i, j, k, $l \in \{1, 2\}$ let:

$\tau_i(abc) := \tilde{\pi}_i(\omega(abc)ab)$, $\zeta_i(abc) := \tilde{\iota}(ac\omega(abc)\tau_{i+1}(abc))$,
$\psi^i_j(xabc) := \tilde{\pi}_j(x\omega(abc)\tau_i(abc))$,
$\varphi^i_{jk}(abc) := \tilde{\iota}(\zeta_i(abc)\psi^i_j(\zeta_i(abc)abc)b\psi^i_k(babc))$,
$\nu^i_{jkl}(abc) := \tilde{\pi}_l(\varphi^i_{jk}(abc)ab)$. We define:

(xvii) $\neg C(abc) \to [\xi(abc) = y \leftrightarrow \bigvee_{1\leq i\leq 2}[(\varrho(\omega(abc), \tau_i(abc); a, b)$
$\vee\, \varrho(\tau_i(abc), \omega(abc); a, b))$
$\wedge\, (\bigvee_{1\leq j,k,l\leq 2}(y = \tilde{\iota}(\varphi^i_{jk}(abc)\nu^i_{jkl}(abc)\omega(abc)\tau_i(abc))$
$\wedge\, C(ab\psi^i_{k+1}(babc)) \wedge C(ac\psi^i_{j+1}(\zeta_i(abc)abc))$
$\wedge\, C(\varphi^i_{jk}(abc)\zeta_i(abc)\nu^i_{jk(l+1)}(abc))))]$, which may be read as:

(*) 'if a, b, c are three distinct non-collinear points and Δ, Ω, Π are the endpoints of rays $\overrightarrow{ab}$, $\overrightarrow{ac}$, $\overrightarrow{ba}$; and $\omega(abc)$ is a point on $\overline{\Delta\Omega}$, then $\xi(abc)$ is a point on $\overline{\Delta\Omega}$ such that $\omega(abc)\xi(abc) \equiv ab$'.

The definition of operation $\xi(abc)$ and axiom G8 below follow from Menger's definition of *directed congruence* and [9, Satz 4.62 (p. 305-309)]. Two directed pairs $< a, b >$ and $< c, d >$ are directed congruent if the segments ab and cd

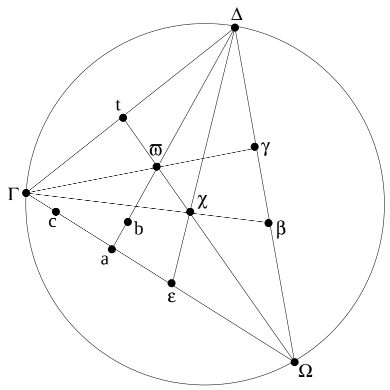

Figure 4. Axiom G8

are congruent, and either the rays $\overrightarrow{ab}$ and $\overrightarrow{cd}$ or the rays $\overrightarrow{ba}$ and $\overrightarrow{dc}$ are limiting parallel. We show that the interpretation (*) of $\xi(abc)$ holds in hyperbolic geometry, if $\tilde{\iota}$, $\tilde{\pi}_1$, $\tilde{\pi}_2$ have the desired interpretation and a_0, a_1, a_2 are three non-collinear points. We consider the Beltrami-Klein inner-disc model. For three non-collinear points a, b, c, for some i, j, k, $l \in \{1, 2\}$, $\tau_i := \tau_i(abc)$ is a point on $\overline{\Delta\Omega}$ such that Δ is the rimpoint of either ray $\overrightarrow{\omega\tau_i}$ or ray $\overrightarrow{\tau_i\omega}$; $\zeta := \zeta_i(abc)$ is the point of intersection of line $\overline{ac}$ with the parallel from ω to line $\overline{ab}$, which is not incident with Δ; $\psi_b := \psi^i_k(babc)$ is a point on the limiting parallel from b to $\overline{\varpi\tau_i}$, which is incident with Ω; $\psi_\zeta := \psi^i_j(\zeta_i(abc)abc)$ is a point on the limiting parallel from ζ to $\overline{\varpi\tau_i}$, which is not incident with Ω; $\varphi := \varphi^i_{jk}(abc)$ is the point of intersection of lines $\overline{b\psi_b}$ and $\overline{\zeta\psi_\zeta}$; $\nu := \nu^i_{jkl}(abc)$ is a point on the parallel from φ to $\overline{ab}$, which is incident with Π; $\xi := \xi(abc)$ is the intersection point of $\overline{\varpi\tau_i}$ and $\overline{\varphi\nu}$. Let t be the point of intersection of $\overline{\zeta\varphi}$ with $\overline{\Pi\Omega}$. Using cross ratios we have $(ab, \Delta\Pi) = (\zeta\varphi, \Delta t)$ and $(\zeta\varphi, \Delta t) = (\omega\xi, \Delta\Omega)$. It follows that $(ab, \Delta\Pi) = (\omega\xi, \Delta\Omega)$ and thus $ab \equiv \omega\xi$.

The last abbreviations to be used in the statement of axiom G8 are the following:

$\beta_i(abc) := \omega(ab\varepsilon_i(abc))$, $\gamma_i(abc) := \xi(ab\varepsilon_i(abc))$,
$\varpi^i_j(abc) := \tilde{\iota}(ab\gamma_i(abc)\tilde{\pi}_j(\gamma_i(abc)ac))$,
$\chi^i_{kj}(abc) := \tilde{\iota}(\beta_i(abc)\tilde{\pi}_k(\beta_i(abc)ac)\varepsilon_i(abc)\tilde{\pi}_l(\varepsilon_i(abc)\beta_i(abc)\gamma_i(abc)))$,
which may be read when $\neg C(abc)$ as:

(**) ‘let Δ, Ω, Γ be the endpoints of rays $\overrightarrow{ab}$, $\overrightarrow{a\varepsilon}$, $\overrightarrow{\varepsilon a}$; for each $i \in \{1, 2\}$, if we denote $\varepsilon := \varepsilon_i(acb)$, $\beta := \beta_i(abc)$, $\gamma := \gamma_i(abc)$, then ε is a point on $\overline{ac}$ such that $a\varepsilon \equiv ab$, β is a point on $\overline{\Delta\Omega}$, and γ is a point on $\overline{\Delta\Omega}$ such that

$ab \equiv \beta\gamma$; for some $j \in \{1,2\}$ $\varpi := \varpi_j^i(abc)$ is the point of intersection of line $\overline{ab}$ with the parallel from γ to $\overline{ac}$, which is not incident with Δ; for some $k, l \in \{1,2\}$ $\chi := \chi_{kl}^i(abc)$ is the point of intersection of the parallel from β to $\overline{ac}$, which is not incident with Ω, with the parallel from ε to $\overline{\beta\gamma}$, not incident with Ω'.

We complete our axiom system by adding the following axioms:[1]

1 G.$_i$ $\varepsilon_i(aac) = a$

2 G.$_i$ $a \neq c \wedge \varepsilon_i(abc) = a \rightarrow a = b$

3 G. $a \neq b \wedge a \neq c \rightarrow \varepsilon_1(abc) \neq \varepsilon_2(abc)$

4 G. $a \neq b \rightarrow \vee_{i=1}^{2} \varepsilon_i(abb) = b$

5 G.$_i$ $a \neq b \wedge a \neq c \rightarrow \vee_{j=1}^{2} \varepsilon_j(\varepsilon_i(abc)ac) = a$

6 G.$_i$ $a \neq b \wedge a \neq c \rightarrow \vee_{j=1}^{2} \varepsilon_j(c\varepsilon_i(abc)a) = \varepsilon_i(abc)$

7 G. $a \neq b \wedge p \neq q \wedge C(apb) \wedge C(aqb) \rightarrow C(paq)$

8 G.$_i$ $\neg C(abc) \rightarrow \bigvee_{1\leq j,k,l\leq 2} \varrho(\varpi_j^i(abc), \chi_{kl}^i(abc); a, \varepsilon_i(acb))$

9 G.$_p$ $a \neq b \wedge a \neq c \wedge C(abc) \wedge \neg C(aba_p) \rightarrow \bigvee_{1\leq k,j\leq 2} \varepsilon_k(a\varepsilon_1(abc)a_p)$
$= \varepsilon_j(a\varepsilon_2(abc)a_p)$

10 G. $R(a_0R(a_0a_1\alpha(a_0a_1))R(a_0\alpha(a_0a_1)R(a_0a_1\alpha(a_0a_1)))) = \alpha(a_0a_1)$

Informally, these axioms can be phrased as follows:

G1 states that by laying off the segment aa on line $\overline{ac}$ at a in both directions we get a.

G2 states that: if laying off the segment ab on line $\overline{ac}$ at a one of the new points is a, then $a = b$.

G3 states that: laying off the segment ab on line $\overline{ac}$ at a we get two distinct points.

G4 states that: laying off the segment ab on line $\overline{ab}$ at a one of the new points coincides with b.

G5 states that: if x_1, x_2 are the points obtained by laying off the segment ab on line $\overline{ac}$ at a, then laying off each of the segments x_ia on line $\overline{x_ic}$ at x_i one of the new points obtained is a.

G6 states that: if x_1, x_2 are the two points obtained by laying off the segment ab on line $\overline{ac}$ at a, then laying off each of the segments cx_i on line $\overline{ca}$ at c one of the new points obtained is x_i.

G7 states that: if points a, p, b are collinear and points a, q, b are collinear then p, a, q are collinear.

G8 states (using the abbreviations in (**) above) that: if a, b, c are three distinct non-collinear points, then the rays $\overrightarrow{\varpi\chi}$ and $\overrightarrow{a\varepsilon}$ have a common rimpoint.

[1]The index i is in $\{1,2\}$, while $p \in \{0,1,2\}$.

G9 states that: if a, b, c are collinear points and a, b, a_p are non-collinear points; $b_i = \varepsilon_i(abc)$, $i = 1, 2$, are the points obtained by laying off the segment ab on the line $\overline{ac}$; $x_k = \varepsilon_k(ab_1a_p)$, $k = 1, 2$, are the points obtained by laying off ab_1 on the line $\overline{aa_p}$; $y_j = \varepsilon_j(ab_2a_p)$, $j = 1, 2$, are the points obtained by laying off ab_2 on the line $\overline{aa_p}$; then for some k, $j \in \{1, 2\}$, $x_k = y_j$.

G10 states that the radian measure of angle $\angle\alpha(a_0a_1)a_0a_1$ is $\pi/3$.

Let $\Sigma_0 = \Sigma'' \cup \{\text{G1}, \ldots, \text{G10}\}$. We note that the actual number of axioms in Σ_0 when expressed in the language $\mathcal{L}_0$ is in fact much larger. Sine the definition of $\pi_i(xab)$ involves operations defined by cases (such as R, M, T_i, A, etc.), each axiom containing $\pi_i(xab)$ and R will be split into several axioms by listing conjunctions of relevant combinations of conditions as the antecedent and then state the consequent (the axiom). Σ_0 is an axiom system for hyperbolic geometry. From [7] it follows that if $\tilde{\iota}$, $\tilde{\pi}_1$, $\tilde{\pi}_2$ have the desired interpretations, then the axioms in Σ'' hold in hyperbolic geometry. If ε_1, ε_2 have the desired interpretation, then G1 - G7 and G9 hold in fact in absolute geometry. To see that axiom G8 holds in hyperbolic geometry, if $\tilde{\iota}$, $\tilde{\pi}_1$, $\tilde{\pi}_2$ have the desired interpretations, we will consider the Beltrami-Klein inner disc model. Let a, b, c be three distinct non-collinear points and $i \in \{1, 2\}$ fixed. Using the abbreviations in (**), let t be the intersection of $\overline{\varpi\Omega}$ with $\overline{\Gamma\Delta}$. We have $ab \equiv a\varepsilon$, and as the interpretation of operation ξ holds as intended in (*), $ab \equiv \beta\gamma$. Hence the (directed) segments $a\varepsilon$ and $\gamma\beta$ are congruent. Let χ_1 be the intersection of $\overline{\varepsilon\Delta}$ with $\overline{t\Omega}$, χ_2 the intersection of $\overline{\beta\Gamma}$ with $\overline{t\Omega}$. Then the following cross-ratios are equal: $(a\varepsilon, \Omega\Gamma) = (\varpi\chi_1, \Omega t)$ and $(\gamma\beta, \Omega\Delta) = (\varpi\chi_2, \Omega t)$. Since $(a\varepsilon, \Omega\Gamma) = (\gamma\beta, \Omega\Delta)$ we must then have $\chi_1 = \chi_2 = \chi$. Thus $\overline{\varpi\chi}$ is incident with Ω. ([9] p. 308-309). Finally, we note that we have shown in Section 2 that in hyperbolic geometry, if ε_1, ε_2 have the desired interpretation and a_0, a_1, a_2 are three non-collinear points such that $\Pi(a_0a_1) = \pi/3$, then $\angle a_1a_0\alpha(a_0a_1) = \pi/3$. If we reflect the point $\alpha(a_0a_1)$ in line $\overline{a_0a_1}$, then reflect $R(a_0a_1\alpha(a_0a_1))$ in $\overline{a_0\alpha(a_0a_1)}$, and finally reflect this new point in $\overline{a_0R(a_0a_1\alpha(a_0a_1))}$ we get back $\alpha(a_0a_1)$. Thus G10 holds.

3 Adequacy of the axiom system

Pambuccian [7] proved that Σ with the incidence predicate defined by

$$P|g \leftrightarrow (\exists Q)\, P \neq Q \wedge g = \varphi(P, Q), \tag{1}$$

implies the Skala axioms from [10]. Axiom C10 in [7], a universal axiom containing generic lines as individual variables, is necessary only to prove what we will call statement L: for every line g there exist two distinct points P and Q such that $g = \varphi(P, Q)$. Then Skala's axiom A2: 'each line is on at least one point' is an immediate consequence. Thus, it is in fact shown that the axiom system $\Sigma \setminus$

$\{C10\} \cup \{(1), L\}$ implies the Skala axioms. Since our axioms are expressed in $\mathcal{L}_0$, which contains only one-sort of individual variables, *points*, we first need to define the notion of *line*. A pair of two distinct points a and b, $a \neq b$, will be called a line and be denoted by $\overline{ab}$. The incidence predicate, denoted by $p|\overline{ab}$, to be read 'point p is incident with line $\overline{ab}$', is defined by:

$$p|\overline{ab} \leftrightarrow a \neq b \wedge C(apb). \tag{2}$$

We define what it means for two lines $\overline{ab}$ and $\overline{cd}$ to be equal or coincide by setting:

$$\overline{ab} = \overline{cd} \leftrightarrow a \neq b \wedge c \neq d \wedge (\forall)x\, C(axb) \leftrightarrow C(cxd) \tag{3}$$

To prove the adequacy of our system we show that $\Sigma \setminus \{C10\} \cup \{(1), L\}$ follows from $\Sigma_0 \cup \{(2), (3)\}$. Let a, b, and c be three collinear points, i.e. $C(abc)$ holds. We show that C is symmetric. If $a = c$ and $a \neq b$, by G1 we have $C(aab)$, by G4 $C(baa)$, while $C(aba)$ holds by definition. If $a \neq c$ and $a \neq b$, then for some $i \in \{1, 2\}$, $\epsilon_i(abc) = b$. By G5 $C(bac)$, by G6 $C(cba)$ is true. A repeated application of G5 and G6 shows that $C(abc)$ is symmetric.

Let p and q be two distinct points. We show that there exists a unique line incident with both p and q. By G1 we have $C(ppq)$, by G4 $C(pqq)$. Thus p and q are incident with line $\overline{pq}$. If $a \neq b$, and $\overline{ab}$ is another line incident with p and q, by G7 $C(paq)$. Hence, a is incident with line $\overline{pq}$. By symmetry of C and G7, $b|\overline{pq}$. If now x is a point incident with $\overline{pq}$, then $C(pxq)$. Since we also have $C(paq)$ the antecedent of G7 is true, and thus $C(apx)$ holds. By the symmetry of C, $C(xap)$ is true. $C(xbp)$ holds also. Applying again G7 we get $C(axb)$. We have shown that:

$$a \neq b \wedge p \neq q \wedge a|\overline{pq} \wedge b|\overline{pq} \wedge x|\overline{pq} \rightarrow x|\overline{ab}. \tag{4}$$

It follows that the points p and q are incident with the line $\overline{ab}$. Hence, if x is a point incident with $\overline{ab}$, by (4) we have $x|\overline{pq}$. Thus the lines $\overline{ab}$ and $\overline{pq}$ coincide. Moreover, if $\eta(abcd)$ holds, then by (4) $a|\overline{cd}$ and $b|\overline{cd}$. Applying (4) again we obtain that the lines $\overline{ab}$ and $\overline{cd}$ coincide. Conversely, if $\overline{ab} = \overline{cd}$, then $C(acb)$ and $C(adb)$ hold. We obtain that $\eta(abcd)$ is equivalent to $\overline{ab} = \overline{cd}$.

In [7] the operation symbol φ has the desired interpretation, i. e. $\varphi(a, b)$ is the line determined by a and b, when $a \neq b$. This implies that $\lambda(a, b, c)$, where $\lambda(a, b, c) \leftrightarrow a = b \vee a = c \vee \varphi(a, b) = \varphi(a, c)$, may be read as '$a$, b, c are collinear'. We will use λ for an equivalent definition in $\mathcal{L}$ of 'point p is incident with the line determined by points a and b, $a \neq b$'. More precisely,

$$p|\varphi(a, b) \leftrightarrow a \neq b \wedge \lambda(a, b, p). \tag{5}$$

From this it follows immediately that if we identify our notion of line $\overline{ab}$ with $\varphi(a,b)$, $a \neq b$, and $p|\overline{ab}$ with $p|\varphi(a,b)$, C is equivalent to λ. Since we proved above that the lines $\overline{ab}$ and $\overline{ba}$ coincide, we obtain axiom C1 in [7], which states in $\mathcal{L}$ that $\varphi(a,b) = \varphi(b,a)$. Assume now that line $\overline{ab}$, $a \neq b$, coincides with line $\overline{cd}$, $c \neq d$, $b \neq c$, and $d \neq b$. Since b is incident with $\overline{cd}$, by symmetry of C we have $C(bdc)$. Hence, d is incident with $\overline{bc}$. As b is incident with $\overline{bc}$, and every line is uniquely determined by two points, we have $\overline{db} = \overline{bc}$. We have thus proved C2 in [7], which states that: $a \neq b \wedge b \neq c \wedge b \neq d \wedge \varphi(a,b) = \varphi(d,c) \rightarrow \varphi(d,b) = \varphi(b,c)$, i. e. if the line $\overline{ab}$ coincides with line $\overline{dc}$, then so do lines $\overline{db}$ and $\overline{bc}$. Since if $a \neq b$ and $c \neq d$, $\eta(abcd)$ may be read as 'lines $\overline{ab}$ and $\overline{cd}$ coincide', $\eta(abcd)$ is equivalent to $\varphi(a,b) = \varphi(c,d)$ in $\mathcal{L}$. Thus our translations of the axioms in Σ' are correct rephrasings in $\mathcal{L}_0$ and may be read as they were intended to be read in $\mathcal{L}$. It is straightforward to see that the axioms in Σ' follow from our system. As the statement L follows from our definition of *line*, we are done. We obtain that $\overline{ab}$ and $p|\overline{ab}$ have the desired interpretation whenever $a \neq b$, and thus C has the desired interpretation. Whenever $\overline{ab}$ and $\overline{cd}$ are two intersecting lines the operation $\tilde{\iota}$ has the desired interpretation. Finally, $\overline{x\tilde{\pi}_k(xab)}$ has the interpretation of one of the limiting parallel lines from point x to line $\overline{ab}$, whenever x is not incident with $\overline{ab}$. Thus $\tilde{\pi}_k(xab)$ has the desired interpretation as a point on one of the limiting parallel lines from x to $\overline{ab}$. (We note that by C8 in [7] $\tilde{\pi}_1(xab)$ and $\tilde{\pi}_2(xab)$ are not on the same parallel.)

Finally, we want to show that ε_1 and ε_2 have the intended interpretation. If a, b, c are three points with $a \neq c$ and $a \neq b$, by G2 $\varepsilon_i(abc) \neq a$, for $i = 1,2$, and by G3, $\varepsilon_1(abc)$ and $\varepsilon_2(abc)$ are distinct. Let $i \in \{1,2\}$ and $\varepsilon := \varepsilon_i(abc)$. By G8, if a, b, c are non-collinear, then ϖ, χ, Ω are collinear. Let t be the intersection of $\overline{\varpi\Omega}$ and $\overline{\Gamma\Delta}$. Since $(\varpi\chi, \Omega t) = (\gamma\beta, \Omega\Delta)$ and $(\varpi\chi, \Omega t) = (a\varepsilon, \Omega\Gamma)$, the segments $\gamma\beta$ and $a\varepsilon_i$ are congruent. As ab and $\gamma\beta$ are congruent, it follows that $a\varepsilon_i \equiv ab$. Thus, if a, b, c are non-collinear points, then $a\varepsilon_i(abc) \equiv ab$, for $i = 1,2$. Axiom C5 (rephrased in $\mathcal{L}_0$) states that $\neg C(a_0a_1a_2)$, i.e. the points a_0, a_1, a_2 are non-collinear. Let a, b, c be three points with $a \neq c$, $a \neq b$ and a, b, c collinear. Then, for some $p \in \{0,1,2\}$, the points a, b, a_p are non-collinear. Otherwise, by G7 and the symmetry of C we would have $C(aa_0a_1)$ and $C(aa_0a_1)$. Then by G7 $C(a_0a_1a_2)$. Thus for some p the antecedent of axiom G9$_p$ holds. Let $b_i = \varepsilon_i(abc)$, $x_k = \varepsilon_k(ab_1a_p)$, and $y_j = \varepsilon_j(ab_2a_p)$, for i, j, $k \in \{1, 2\}$. As a consequence of axiom G8, $ab_1 \equiv ax_1 \equiv ax_2$, $ab_2 \equiv ay_1 \equiv ay_2$. By G9$_p$ for some k and $j \in \{1, 2\}$, $x_k = y_j$. Thus $ab_1 \equiv ab_2$. (As $C(abc)$ holds, for some $i \in \{1, 2\}$ $b_i = \varepsilon_i(abc) = b$.) Now that ε_1, ε_2, $\tilde{\iota}$ have the desired interpretation and a_0, a_1, a_2 are three non-collinear points, R has the desired interpretation by (vi). Then by axiom G10 $\angle a_0a_1\alpha(a_0a_1) = \pi/3$. In the notation of Section 2, definition (xiii), $\tan\theta = 2/\cosh 2\delta$. Since $\theta = \pi/3$, we have $\cosh 2\delta = 2/\sqrt{3}$.

Hence $\tan \Pi(2\delta) = \sqrt{3}$ and $\Pi(a_0 a_1) = \pi/3$. Thus a_0, a_1, a_2 are as desired.

We have thus proved the following:

1 Theorem. *$\mathfrak{M}$ is a model of Σ_0 if and only if $\mathfrak{M}$ is isomorphic to $\mathfrak{K}_2(F)$, the Beltrami-Klein model of 2-dimensional hyperbolic geometry, where F is a Euclidean ordered field and the operations ε_1, ε_2, $\tilde{\iota}$ have the desired interpretations, and a_0, a_1, a_2 are three non-collinear points such that $\Pi(a_0 a_1) = \pi/3$.*

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