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Ideal properties and integral extension of convolution operators on $L^\infty(G)$

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Abstract. We investigate operator ideal properties of convolution operators C_λ (via measures λ) acting in $L^\infty(G)$, with G a compact abelian group. Of interest is when C_λ is compact, as this corresponds to λ having an integrable density relative to Haar measure μ, i.e., $\lambda \ll \mu$. Precisely then is there an *optimal* Banach function space $L^1(m_\lambda)$ available which contains $L^\infty(G)$ properly, densely and continuously and such that C_λ has a continuous, $L^\infty(G)$-valued, linear extension I_{m_λ} to $L^1(m_\lambda)$. A detailed study is made of $L^1(m_\lambda)$ and I_{m_λ}. Amongst other things, it is shown that C_λ is compact iff the finitely additive, $L^\infty(G)$-valued set function $m_\lambda(A) := C_\lambda(\chi_A)$ is norm σ-additive iff $\lambda \in L^1(G)$, whereas the corresponding optimal extension I_{m_λ} is compact iff $\lambda \in C(G)$ iff m_λ has finite variation. We also characterize when m_λ admits a Bochner (resp. Pettis) μ-integrable, $L^\infty(G)$-valued density.

Keywords: Convolution operator, vector measure, optimal domain, Bochner-Pettis density

MSC 2000 classification: primary 43A15, 46G10, 47B10, secondary 43A77

Dedicated to the memory of V.B. Moscatelli

1 Introduction and main results

Given an infinite compact abelian group G (with dual group Γ) and $\lambda \in M(G)$ (the space of regular Borel measures on G equipped with the variation norm $\|\cdot\|_{\mathrm{var}}$) the linear operator $C_\lambda^{(p)} : L^p(G) \to L^p(G)$ of convolution with λ is bounded for every $1 \le p \le \infty$. Here $L^p(G) := L^p(\mu)$ is equipped with its usual norm $\|\cdot\|_{L^p(G)}$ (where μ is normalized Haar measure in G) and $C_\lambda^{(p)}(f) := f * \lambda \in L^p(G)$ is given by

$$f * \lambda : x \mapsto \int_G f(x-y)\, d\lambda(y), \qquad x \in G, \tag{1}$$

for each $f \in L^p(G)$, in which case $\|f * \lambda\|_{L^p(G)} \leq \|\lambda\|_{\mathrm{var}} \|f\|_{L^p(G)}$, [14, Theorem (20.12)], i.e., $\|C_\lambda^{(p)}\|_{\mathrm{op}} \leq \|\lambda\|_{\mathrm{var}}$. Certain operator ideal properties of $C_\lambda^{(p)}$ (e.g., compactness, weak compactness, complete continuity, etc.) are intimately connected to various properties of λ (e.g., $\lambda \in L^p(G)$, $\lambda \ll \mu$, $\lambda \in M_0(G) := \{\, \nu \in M(G) \mid \widehat{\nu} \in c_0(\Gamma) \,\}$, etc.).

Associated with $C_\lambda^{(p)}$ is the finitely additive, $L^p(G)$-valued set function

$$m_\lambda^{(p)} : A \mapsto C_\lambda^{(p)}(\chi_A) = \chi_A * \lambda, \qquad A \in \mathcal{B}(G), \tag{2}$$

with $\mathcal{B}(G)$ denoting the Borel σ-algebra in G. Whenever $1 \leq p < \infty$, the set function $m_\lambda^{(p)}$ is actually σ-additive (i.e., it is a vector measure) for *every* $\lambda \in M(G)$ and so the well developed theory of integration with respect to vector measures can be applied to give a finer analysis of the operators $C_\lambda^{(p)}$. It turns out that the space $L^1(m_\lambda^{(p)})$, consisting of all $m_\lambda^{(p)}$-integrable functions, contains $L^p(G)$ in the canonical way (continuously) and that the integration map $I_{m_\lambda^{(p)}} : L^1(m_\lambda^{(p)}) \to L^p(G)$, given by

$$I_{m_\lambda^{(p)}} : f \mapsto \int_G f \, dm_\lambda^{(p)}, \qquad f \in L^1(m_\lambda^{(p)}), \tag{3}$$

is an $L^p(G)$-valued extension of $C_\lambda^{(p)}$. The effects of $I_{m_\lambda^{(p)}}$ and of its operator ideal properties on $C_\lambda^{(p)}$ (hence, also on λ) have been thoroughly treated in [21], [22], [23, Chap. 7]. An important feature is that $L^1(m_\lambda^{(p)})$ is the *optimal domain* of $C_\lambda^{(p)}$ in the following sense: if $X(\mu)$ is any Banach function space over $(G, \mathcal{B}(G), \mu)$ with σ-order continuous norm such that $L^p(G) \subseteq X(\mu)$ continuously and $C_\lambda^{(p)}$ has a continuous, linear, $L^p(G)$-valued extension to $X(\mu)$, say T, then $X(\mu) \subseteq L^1(m_\lambda^{(p)})$ continuously and the restriction of $I_{m_\lambda^{(p)}}$ to $X(\mu)$ coincides with T.

Until now the case $p = \infty$ has not been treated, perhaps because it is rather different. For instance, the finitely additive, $L^\infty(G)$-valued set function $m_\lambda^{(\infty)}$ as defined by (2), namely

$$m_\lambda^{(\infty)} : A \mapsto C_\lambda^{(\infty)}(\chi_A) = \chi_A * \lambda \in L^\infty(G), \qquad A \in \mathcal{B}(G), \tag{4}$$

may *fail* to be norm σ-additive for certain $\lambda \in M(G)$. Also, the map $a \mapsto \tau_a(f)$ from G into $L^\infty(G)$, with τ_a being the translation operator (i.e., $\tau_a(f) : x \mapsto f(x - a)$ for $x \in G$) is, unlike for $L^p(G)$ with $1 \leq p < \infty$, *not* continuous for every $f \in L^\infty(G)$. Furthermore, whereas the Banach function space $L^1(G)$ is weakly sequentially complete, each Banach function space $L^p(G)$, $1 < p < \infty$, is reflexive, and all spaces $L^p(G)$, $1 \leq p < \infty$, have σ-order continuous norm, the space $L^\infty(G)$ has *none* of these useful properties. If G is metrizable, then each space $L^p(G)$, for $1 \leq p < \infty$ is separable; *not so* for $L^\infty(G)$. And so on. Despite such basic differences, our aim is to study the operators $C_\lambda^{(\infty)}$ in detail. As to be expected, the results differ significantly from those when $1 \leq p < \infty$.

To formulate some of the main results, let $\lambda \in M(G) \setminus \{0\}$. Let $W^{(\infty)}(\lambda)$ denote the class of all σ-order continuous Banach function spaces (over $(G, \mathcal{B}(G), \mu)$), briefly B.f.s., into which $L^\infty(G)$ is continuously embedded and to which $C_\lambda^{(\infty)}$ admits a continuous, linear, $L^\infty(G)$-valued extension. Not having σ-order continuous norm, $L^\infty(G)$ itself does not belong to $W^{(\infty)}(\lambda)$. Actually, it may happen that $W^{(\infty)}(\lambda) = \emptyset$; see Theorem 1 to follow. However, if λ has the property that $m_\lambda^{(\infty)}$ *is* norm σ-additive, then $m_\lambda^{(\infty)}$ and μ have the same null sets and $L^1(m_\lambda^{(\infty)}) \in W^{(\infty)}(\lambda)$. As seen by the following result, this observation is decisive.

Concerning notation, given $f \in L^\infty(G)$ we write $f \in C(G)$, where $C(G)$ is the space of continuous functions on G, if there exists $\psi \in C(G)$ with $f(x) = \psi(x)$ for μ-a.e. $x \in G$. Via the usual identification, $C(G)$ can be considered as a closed subspace of $L^\infty(G)$. Given a character $\gamma \in \Gamma$, let $(\cdot,\gamma)$ denote the function $x \mapsto (x,\gamma) := \gamma(x)$ on G. For each subset $\Delta\subseteq\Gamma$, we define $\mathcal{T}(G,\Delta) := \mathrm{span}\{\,(\cdot,\gamma) \mid \gamma \in \Delta\,\}$. If $\Delta = \Gamma$, then we use the simpler notation $\mathcal{T}(G)$ and refer to $\mathcal{T}(G)$ as the trigonometric polynomials on G. It is clear that $\mathcal{T}(G)\subseteq L^\infty(G)$ and $\|(\cdot,\gamma)\|_{L^p(G)} = 1$ whenever $1 \le p \le \infty$ and $\gamma \in \Gamma$. The space of all continuous linear operators between Banach spaces E and F is denoted by $\mathcal{L}(E,F)$ or, if $E = F$, by $\mathcal{L}(E)$. The dual Banach space $E^* := \mathcal{L}(E,\mathbb{C})$.

Theorem 1. *Let $\lambda \in M(G)\setminus\{0\}$.*

(I) *The following assertions are equivalent.*

(i) *$\lambda \ll \mu$, that is, there exists $g \in L^1(G)$ such that*

$$\lambda(A) = \int_A g\,d\mu, \qquad A \in \mathcal{B}(G). \tag{5}$$

(ii) *The set function $m_\lambda^{(\infty)} : \mathcal{B}(G) \to L^\infty(G)$ is norm σ-additive.*

(iii) *The convolution operator $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$ is compact.*

(iv) *The convolution operator $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$ is weakly compact.*

(v) *The range $R(m_\lambda^{(\infty)}) := \{\, m_\lambda^{(\infty)}(A) \mid A \in \mathcal{B}(G)\,\}$ of $m_\lambda^{(\infty)}$ is contained in the closed subspace $C(G)\subseteq L^\infty(G)$.*

(vi) *The class $W^{(\infty)}(\lambda) \neq \emptyset$.*

(II) *Let λ satisfy any one of* (i)–(vi) *in part* (I).

(i) *The space $L^1(m_\lambda^{(\infty)})$ is the largest B.f.s. in the class $W^{(\infty)}(\lambda)$. Moreover, the continuous inclusions*

$$L^\infty(G) \subsetneqq L^1(m_\lambda^{(\infty)})\subseteq L^1(G)$$

hold and we have

$$L^1(m_\lambda^{(\infty)}) = \{\, f \in L^1(G) \mid (f\chi_A) * \lambda \in C(G),\ \forall A \in \mathcal{B}(G)\,\} \tag{6}$$

with $L^1(m_\lambda^{(\infty)})$ a translation invariant subspace of $L^1(G)$.

(ii) *The integration map $I_{m_\lambda^{(\infty)}} : L^1(m_\lambda^{(\infty)}) \to L^\infty(G)$ satisfies*

$$I_{m_\lambda^{(\infty)}}(f\chi_A) := \int_A f\,dm_\lambda^{(\infty)} = (f\chi_A) * \lambda, \quad f \in L^1(m_\lambda^{(\infty)}),\ A \in \mathcal{B}(G). \tag{7}$$

Moreover, $I_{m_\lambda^{(\infty)}}$ is a continuous linear extension of $C_\lambda^{(\infty)}$, takes its values in $C(G)\subseteq L^\infty(G)$ and commutes with all translations.

(iii) *Both $\mathcal{T}(G)$ and $C(G)$ are dense in $L^1(m_\lambda^{(\infty)})$.*

(iv) *The Banach space $L^1(m_\lambda^{(\infty)})$ is separable iff G is metrizable.*

In view of Theorem 1, we will concentrate on the situation when $\lambda \ll \mu$, i.e., there exists a unique $g \in L^1(G)$ such that (5) holds, written briefly as $\lambda = g\,d\mu$. In this case we simply write $C_g^{(\infty)}$ for $C_\lambda^{(\infty)}$ and $m_g^{(\infty)}$ for $m_\lambda^{(\infty)}$.

Fix $g \in L^1(G)$. A natural question is whether or not $L^p(G)\subseteq L^1(m_g^{(\infty)})$ for some $1 \le p < \infty$, i.e., if $L^p(G) \in W_\lambda^{(\infty)}$. This is answered by Proposition 1, namely $L^p(G)\subseteq L^1(m_g^{(\infty)})$ iff $g \in L^{p^*}(G)$ iff the operator $C_g^{(\infty)} \in \mathcal{L}(L^\infty(G))$ is *p-summing*, where $\frac{1}{p^*} + \frac{1}{p} = 1$. In particular,

$L^1(m_g^{(\infty)}) = L^1(G)$ is as large as possible (cf. (6)) iff $g \in L^\infty(G)$, in which case $I_{m_g^{(\infty)}}$ coincides with the continuous convolution operator $C_g^{(1,\infty)} : L^1(G) \to L^\infty(G)$ given via $f \mapsto f * g$, for each $f \in L^1(G)$. Equivalently, the vector measure $m_g^{(\infty)}$ admits an $L^\infty(G) = L^1(G)^*$-valued Gelfand μ-density; see Theorem 2.

As well as making a detailed study of the optimal domain space $L^1(m_g^{(\infty)})$, for $g \in L^1(G)$, and of the associated integration map $I_{m_g^{(\infty)}}$, we also determine some intrinsic properties of $m_g^{(\infty)}$. Indeed, since $m_g^{(\infty)} \ll \mu$ (if $g \neq 0$), the question arises of whether $m_g^{(\infty)}$ is given by an $L^\infty(G)$-valued density relative to μ, say as a Gelfand, Pettis or Bochner μ-density; for the definition of these three kinds of vector-integrals we refer to [5], for example. The crucial point is that $L^\infty(G)$ fails to have the Radon-Nikodým property, [5, p.219]. Since $L^\infty(G)$ is a Banach lattice, this is the same as failing the weak Radon-Nikodým property, [13, Theorem 5]. Accordingly, even when $m_g^{(\infty)}$ has finite variation, there is no guarantee that it has a Bochner or Pettis μ-density. Given any Gelfand (resp. Bochner, Pettis) μ-integrable function $F : G \to L^\infty(G)$, its corresponding integral with respect to μ over $A \in \mathcal{B}(G)$, an element of $L^\infty(G)$, is denoted by

$$(w^*)\text{-}\int_A F\,d\mu \ \ (\text{ resp. } (B)\text{-}\int_A F\,d\mu,\ \ (P)\text{-}\int_A F\,d\mu\,).$$

For $g \in L^\infty(G)$, denote by $K_g : G \to L^\infty(G)$ the function $x \mapsto \tau_x(g)$, for $x \in G$.

Theorem 2. *For $g \in L^1(G) \setminus \{0\}$ the following assertions are equivalent.*

(i) $g \in L^\infty(G)$.

(ii) $L^1(m_g^{(\infty)}) = L^1(G)$.

(iii) $m_g^{(\infty)}$ *admits a Gelfand μ-density $F : G \to L^\infty(G)$, i.e.,*

$$m_g^{(\infty)}(A) = (w^*)\text{-}\int_A F\,d\mu, \qquad A \in \mathcal{B}(G).$$

(iv) *The total variation measure $|m_g^{(\infty)}|$ is finite on $\mathcal{B}(G)$.*

(v) *The convolution operator $C_g^{(\infty)}$ is 1-summing.*

In this case, K_g is a Gelfand μ-density of $m_g^{(\infty)}$ and, for each $f \in L^1(G)$, the function $fK_g : G \to L^\infty(G)$ is Gelfand μ-integrable with

$$I_{m_g^{(\infty)}}(f) = C_g^{(1,\infty)}(f) = (w^*)\text{-}\int_G fK_g\,d\mu, \quad f \in L^1(G) = L^1(m_g^{(\infty)}). \tag{8}$$

To determine when $m_g^{(\infty)}$ admits a Bochner or Pettis μ-density, recall that such densities are necessarily Gelfand μ-densities. So, by Theorem 2, we may restrict attention to $g \in L^\infty(G)$, in which case $L^1(m_g^{(\infty)}) = L^1(G)$ as isomorphic B.f.s.' and $I_{m_g^{(\infty)}} = C_g^{(1,\infty)}$.

Given $g \in L^\infty(G)$, the following result shows that $m_g^{(\infty)}$ admits a Bochner μ-density iff $g \in C(G)$. For $g \in L^\infty(G)$, conditions connecting the requirement $g \in C(G)$ with continuity of $K_g : G \to L^\infty(G)$, have been studied by various authors, e.g., [8], [9], [29].

Theorem 3. *Let $g \in L^\infty(G) \setminus \{0\}$.*

(I) *The following assertions are equivalent.*

(i) *The function $g \in C(G)$.*

(ii) $m_g^{(\infty)}$ *admits a Bochner μ-density $F : G \to L^\infty(G)$, i.e.,*

$$m_g^{(\infty)}(A) = (B)\text{-}\int_A F\,d\mu, \qquad A \in \mathcal{B}(G).$$

(iii) *The extended operator* $I_{m_g^{(\infty)}} = C_g^{(1,\infty)} : L^1(G) \to L^\infty(G)$ *is compact.*

(iv) *The function* $K_g : G \to L^\infty(G)$ *is continuous.*

(v) *There is a set* $A_0 \in \mathcal{B}(G)$ *of positive* μ*-measure such that* $K_g(A_0)$ *is separable in* $L^\infty(G)$.

(II) *If any one of* (i)–(v) *in part* (I) *holds, then* K_g *is actually a* $C(G)$*-valued Bochner* μ*-density of* $m_g^{(\infty)}$ *and*

$$I_{m_g^{(\infty)}}(f) = C_g^{(1,\infty)}(f) = (B)\text{-}\int_G fK_g\, d\mu, \quad f \in L^1(G) = L^1(m_g^{(\infty)}). \tag{9}$$

The following result determines when $m_g^{(\infty)}$ admits a Pettis μ-density, under the assumption of Martin's Axiom; see [12] for consequences of this axiom.

Theorem 4. *Assume Martin's Axiom and let* $g \in L^\infty(G) \setminus \{0\}$*. Then* $m_g^{(\infty)}$ *admits a Pettis* μ*-density iff there exists a bounded, Riemann measurable function* ψ *on* G *with* $g = \psi$ μ*-a.e. In this case, the bounded function* $K_g : G \to L^\infty(G)$ *is a Pettis* μ*-density of* $m_g^{(\infty)}$ *and*

$$I_{m_g^{(\infty)}}(f) = C_g^{(1,\infty)}(f) = (P)\text{-}\int_G fK_g\, d\mu, \quad f \in L^1(G) = L^1(m_g^{(\infty)}). \tag{10}$$

2 Preliminaries

Let G be an infinite compact abelian group and $L^0(\mu)$ denote the vector space of all $\mathbb{C}$-valued, $\mathcal{B}(G)$-measurable functions on G, where those functions which are μ-a.e. equal are identified. Equipped with the μ-a.e. pointwise order for its positive cone, $L^0(\mu)$ is a complex vector lattice, namely the complexification of the real vector lattice $\{ f \in L^0(\mu) \mid f \text{ is } \mathbb{R}\text{-valued}\}$. The subspace $\mathrm{sim}\mathcal{B}(G)$ of all $\mathbb{C}$-valued, $\mathcal{B}(G)$-simple functions on G is a complex vector sublattice of $L^0(\mu)$. An order ideal $X(\mu)$ of $L^0(\mu)$ is called a *Banach function space* (briefly, B.f.s.) based on the positive, finite measure space $(G, \mathcal{B}(G), \mu)$ if $X(\mu)$ contains $\mathrm{sim}\mathcal{B}(G)$ and if it is equipped with a *lattice norm* (i.e., $\|f\|_{X(\mu)} \le \|g\|_{X(\mu)}$ whenever $f, g \in X(\mu)$ satisfy $|f| \le |g|$) for which $X(\mu)$ is complete. Being an order ideal of $L^0(\mu)$, each B.f.s. $X(\mu)$ must contain $L^\infty(G)$ (because $\chi_G \in \mathrm{sim}\mathcal{B}(G) \subseteq X(\mu)$). Moreover, the inclusion $L^\infty(G) \subseteq X(\mu)$ turns out to be continuous, [23, Proposition 2.2(iv)]. A B.f.s. $X(\mu)$ is said to have σ*-order continuous norm* if, for every positive decreasing sequence $\{f_n\}_{n=1}^\infty$ with $\inf_{n\in\mathbb{N}} f_n = 0$ in the order of $X(\mu)$, we necessarily have $\lim_{n\to\infty} \|f_n\|_{X(\mu)} = 0$. In this case, we also say that $X(\mu)$ is a σ-order continuous B.f.s. or simply that $X(\mu)$ is σ-order continuous. For each $1 \le p < \infty$, the B.f.s. $L^p(G) = L^p(\mu) \subseteq L^0(\mu)$ is σ-order continuous but the B.f.s. $L^\infty(G)$ is not.

Let E be a Banach space (over $\mathbb{C}$) with norm $\|\cdot\|_E$. Its closed unit ball is denoted by $\mathbb{B}[E]$. The duality between E and E^* (with dual norm $\|\cdot\|_{E^*}$) is given by $\langle x, x^*\rangle := x^*(x)$, for $x \in E$, $x^* \in E^*$. The operator norm of a continuous linear operator T between Banach spaces is denoted by $\|T\|_{\mathrm{op}}$.

Let $m : \mathcal{B}(G) \to E$ be a *vector measure* (i.e., a σ-additive set function). Its variation measure $|m| : \mathcal{B}(G) \to [0, \infty]$ is defined analogous to that for a scalar measure, [5, Chap. I, Definition 1.4]. Given $x^* \in E^*$, the set function $\langle m, x^*\rangle : A \mapsto \langle m(A), x^*\rangle$ on $\mathcal{B}(G)$ is a complex measure. We say that a $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$ is *m-integrable* if

(I-1) f is $\langle m, x^*\rangle$-integrable for all $x^* \in E^*$, and

(I-2) given $A \in \mathcal{B}(G)$, there is a (unique) element $\int_A f\, dm \in E$ such that $\langle \int_A f\, dm, x^*\rangle = \int_A f\, d\langle m, x^*\rangle$ for all $x^* \in E^*$;

see [17, Chap. II], [20]. The vector space $L^1(m)$ of all m-integrable functions is endowed with the seminorm

$$\|f\|_{L^1(m)} := \sup_{x^* \in \mathbb{B}[E^*]} \int_G |f| \, d|\langle m, x^* \rangle|, \qquad f \in L^1(m), \tag{11}$$

for which $L^1(m)$ is complete and in which sim$\mathcal{B}(G)$ is dense; see [10], [17, Chap. IV]. Moreover, all bounded Borel functions are m-integrable, [20, Theorem 2.2]. A function $f \in L^1(m)$ is called *m-null* if $\|f\|_{L^1(m)} = 0$. We identify $L^1(m)$ with its quotient space modulo m-null functions. The *semivariation* $\|m\| : \mathcal{B}(G) \to [0, \infty)$ is defined by $\|m\|(A) := \|\chi_A\|_{L^1(m)}$ for $A \in \mathcal{B}(G)$. Then we have

$$\|m(A)\|_E \le \|m\|(A) \le |m|(A) \le \infty, \qquad A \in \mathcal{B}(G).$$

A set $A \in \mathcal{B}(G)$ is called *m-null* if $\|m\|(A) = 0$, which is equivalent to the condition that $\|m(B)\|_E = 0$ for all $B \in \mathcal{B}(G)$ with $B \subseteq A$. The m-null and $|m|$-null sets coincide and we have the continuous inclusion

$$L^1(|m|) \subseteq L^1(m). \tag{12}$$

Equip $L^1(m)$ with the μ-a.e. pointwise order for its positive cone. Then (11) implies that $\|\cdot\|_{L^1(m)}$ is a lattice norm. The integration operator $I_m : L^1(m) \to E$ associated with m is defined by

$$I_m : f \mapsto \int_G f \, dm, \qquad f \in L^1(m).$$

Then I_m is continuous and linear. Moreover, $\|I_m\|_{\text{op}} = 1$; [23, p.152]. We write $m \simeq \mu$ whenever m and μ have the same null sets.

Let $L^1_w(m)$ denote the vector space of all $\mathbb{C}$-valued, $\mathcal{B}(G)$-measurable functions on G satisfying (I-1). We call such functions *weakly m-integrable*. The space $L^1_w(m)$ is a Banach space with respect to the norm $\|\cdot\|_{L^1_w(m)}$ defined by the right-hand side of (11) and $L^1(m)$ is a closed subspace of $L^1_w(m)$. We refer to [23, Chap. 3] for general facts related to $L^1(m), L^1_w(m)$ and I_m.

Lemma 1. *Let $m : \mathcal{B}(G) \to E$ be a Banach-space-valued measure such that $m \simeq \mu$. Then $L^1(m)$ is a σ-order continuous B.f.s. over $(G, \mathcal{B}(G), \mu)$ and $L^\infty(G) \subsetneq L^1(m)$ continuously with the natural embedding α_∞ satisfying $\|\alpha_\infty\|_{\text{op}} = \|m\|(G)$.*

Proof. Since m and μ have the same null sets, it follows that $L^1(m)$ is a σ-order continuous B.f.s., [23, Theorem 3.7]. In view of (11), it is clear that $\|\alpha_\infty\|_{\text{op}} \le \|m\|(G)$. Actually, $\|\alpha_\infty\|_{\text{op}} = \|m\|(G)$ because $\left\|\alpha_\infty(\chi_G)\right\|_{L^1(m)} = \|m\|(G)$ and $\left\|\chi_G\right\|_{L^\infty(G)} = 1$. Since $L^\infty(G)$ does not have σ-order continuous norm, the inclusion $L^\infty(G) \subseteq L^1(m)$ is strict. QED

The following result will be required in Section 4. We omit the proof as it follows from the definitions of Bochner, Pettis and Gelfand integrals, [5, Chap. II].

Lemma 2. *(i) Let $F : G \to L^\infty(G)$ be a function.*

(a) If F is Pettis μ-integrable, then it is Gelfand μ-integrable and

$$(P)\text{-}\int_A F \, d\mu = (w^*)\text{-}\int_A F \, d\mu, \qquad A \in \mathcal{B}(G).$$

(b) If F is Bochner μ-integrable, then it is Pettis μ-integrable and

$$(B)\text{-}\int_A F \, d\mu = (P)\text{-}\int_A F \, d\mu = (w^*)\text{-}\int_A F \, d\mu, \qquad A \in \mathcal{B}(G).$$

(*ii*) *Let F and H be two $L^\infty(G)$-valued functions such that $F(x) = H(x)$, as elements of $L^\infty(G)$, for μ-a.e. $x \in G$. If F is Gelfand (resp. Bochner, Pettis) μ-integrable, then so is H and (w^*)-$\int_A F\,d\mu = (w^*)$-$\int_A H\,d\mu$ (resp. (B)-$\int_A F\,d\mu = (B)$-$\int_A H\,d\mu$, (P)-$\int_A F\,d\mu = (P)$-$\int_A H\,d\mu$) for $A \in \mathcal{B}(G)$.*

The Fourier-Stieltjes transform $\widehat{\lambda} : \Gamma \to \mathbb{C}$ of $\lambda \in M(G)$ is given by $\widehat{\lambda}(\gamma) := \int_G (x, -\gamma)\,d\lambda(x)$, for $\gamma \in \Gamma$, in which case $\widehat{\lambda} \in \ell^\infty(\Gamma)$. If $\lambda = g\,d\mu$, for $g \in L^1(G)$, then $\widehat{\lambda}$ coincides with $\widehat{g}(\gamma) = \int_G (x, -\gamma) g(x)\,d\mu(x)$, for $\gamma \in \Gamma$. The Riemann-Lebesgue Lemma then ensures that $\widehat{g} \in c_0(\Gamma)$, [15, Theorem (28.40)]. The reflection $\widetilde{\lambda} : \mathcal{B}(G) \to \mathbb{C}$ of $\lambda \in M(G)$, defined by $\widetilde{\lambda}(A) := \lambda(-A)$ for $A \in \mathcal{B}(G)$, also belongs to $M(G)$. If $\lambda = g\,d\mu$ for some $g \in L^1(G)$, then $\widetilde{\lambda} = \widetilde{g}\,d\mu$, where $\widetilde{g} \in L^1(G)$ is the reflection of g, i.e., $\widetilde{g}(x) := g(-x)$ for $x \in G$. Given $\lambda \in M(G)$, let $\operatorname{supp}(\widehat{\lambda}) := \{\, \gamma \in \Gamma \mid \widehat{\lambda}(\gamma) \neq 0 \,\}$. If $\lambda = g\,d\mu$ for some $g \in L^1(G)$, then we write $\operatorname{supp}(\widehat{g}) := \operatorname{supp}(\widehat{\lambda})$.

Lemma 3. *Let $\lambda \in M(G)$ and Γ_λ be the subgroup generated by* $\operatorname{supp}(\widehat{\lambda})$ *in Γ. Then there exists $\lambda_1 \in M(G)$ such that $\widehat{\lambda_1} = \chi_{\Gamma_\lambda}$ on Γ, i.e., $\lambda * \lambda_1 = \lambda$.*

Proof. Let $H \subseteq G$ be the annihilator of the open subgroup $\Gamma_\lambda \subseteq \Gamma$, i.e., $H := \{\, x \in G \mid (x, \gamma) = 1,\ \forall \gamma \in \Gamma_\lambda \,\}$. Normalized Haar measure λ_1 on the compact subgroup H of G is regarded as an element of $M(G)$ and satisfies $\widehat{\lambda_1} = \chi_{\Gamma_\lambda}$, [26, p.59]. So, $(\lambda_1 * \lambda)\widehat{\ } = \widehat{\lambda_1}\widehat{\lambda} = \widehat{\lambda}$ as $\operatorname{supp}(\widehat{\lambda}) \subseteq \Gamma_\lambda$, i.e., $\lambda * \lambda_1 = \lambda$. QED

Let $\lambda \in M(G)$. Since $L^1(G)^* = L^\infty(G)$, for each $h \in L^1(G)$ we can define the $\mathbb{C}$-valued, finitely additive set function

$$\langle h, m_\lambda^{(\infty)} \rangle : A \mapsto \langle h, m_\lambda^{(\infty)}(A) \rangle = \int_G h \cdot (\chi_A * \lambda)\,d\mu, \qquad A \in \mathcal{B}(G).$$

Lemma 4. *Let $\lambda \in M(G) \setminus \{0\}$.*

(i) *Given $h \in L^1(G)$, we have $h * \widetilde{\lambda} \in L^1(G)$ and*

$$\langle h, m_\lambda^{(\infty)} \rangle(A) = \int_A (h * \widetilde{\lambda})\,d\mu, \qquad A \in \mathcal{B}(G), \tag{13}$$

so that $\langle h, m_\lambda^{(\infty)} \rangle \in M(G)$. Moreover,

$$|\langle h, m_\lambda^{(\infty)} \rangle|(A) = \int_A |(h * \widetilde{\lambda})|\,d\mu, \qquad A \in \mathcal{B}(G). \tag{14}$$

(ii) *For every $\mathcal{B}(G)$-measurable function $f : G \to \mathbb{C}$, we have*

$$\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} \int_G |f|\,d\mu \le \sup_{h \in \mathbb{B}[L^1(G)]} \int_G |f|\,d|\langle h, m_\lambda^{(\infty)} \rangle|. \tag{15}$$

(iii) *Assume, in addition, that $m_\lambda^{(\infty)}$ (cf. (4)) is norm σ-additive.*

(a) *Given $f \in L^1(m_\lambda^{(\infty)})$ we have*

$$\sup_{h \in \mathbb{B}[L^1(G)]} \int_G |f|\,d|\langle h, m_\lambda^{(\infty)} \rangle| = \|f\|_{L^1(m_\lambda^{(\infty)})}. \tag{16}$$

(b) *The measures $m_\lambda^{(\infty)}$ and μ have the same null sets, i.e., $m_\lambda^{(\infty)} \simeq \mu$.*

(c) *The natural inclusion $L^1(m_\lambda^{(\infty)}) \subseteq L^1(G)$ is continuous and has norm $1/\|m_\lambda^{(\infty)}\|(G)$.*

Proof. (i) That $h * \widetilde{\lambda} \in L^1(G)$ is known; see Section 1. The definition of $\langle h, m_\lambda^{(\infty)}\rangle$ and Fubini's Theorem proves (13). Since $h * \widetilde{\lambda} \in L^1(G)$, it is clear from (13) that $\langle h, m_\lambda^{(\infty)}\rangle$ is σ-additive. Moreover, (14) is a consequence of a general fact on complex measures, [27, Theorem 6.13].

(ii) Let f be as stated. Fix $\gamma \in \Gamma$. Since $(\cdot, -\gamma) \in \mathbb{B}[L^1(G)]$ as well as $|(\cdot,-\gamma) * \widetilde{\lambda}| = |\widehat{\lambda}(\gamma)(\cdot,-\gamma)| = |\widehat{\lambda}(\gamma)| \cdot \chi_G$, it follows from (14) that

$$\begin{aligned} |\widehat{\lambda}(\gamma)| \int_G |f|\, d\mu &= \int_G |f| \cdot |(\cdot,-\gamma) * \widetilde{\lambda}|\, d\mu = \int_G |f|\, d|\langle(\cdot,-\gamma), m_\lambda^{(\infty)}\rangle| \\ &\le \sup_{h\in\mathbb{B}[L^1(G)]} \int_G |f|\, d|\langle h, m_\lambda^{(\infty)}\rangle|. \end{aligned}$$

So, (15) holds because $\gamma \in \Gamma$ is arbitrary.

(iii) (a) Since $\mathbb{B}[L^1(G)]\subseteq\mathbb{B}[L^\infty(G)^*]$ in a canonical way, the left-hand side of (16) is at most the right-hand side; see (11). To prove the reverse inequality, let $\varepsilon > 0$ and choose $s \in \mathrm{sim}\mathcal{B}(G)$ with $\sup_{x\in G} |s(x)| \le 1$ such that

$$-\varepsilon + \|f\|_{L^1(m_\lambda^{(\infty)})} < \Big\| \int_G sf\, dm_\lambda^{(\infty)} \Big\|_{L^\infty(G)}, \tag{17}$$

[23, Lemma 3.11]. Since $L^\infty(G) = L^1(G)^*$, we have

$$\begin{aligned} &\Big\| \int_G sf\, dm_\lambda^{(\infty)} \Big\|_{L^\infty(G)} = \sup_{h\in\mathbb{B}[L^1(G)]} \Big|\langle h, \int_G sf\, dm_\lambda^{(\infty)}\rangle\Big| \\ &= \sup_{h\in\mathbb{B}[L^1(G)]} \Big| \int_G sf\, d\langle h, m_\lambda^{(\infty)}\rangle \Big| \le \sup_{h\in\mathbb{B}[L^1(G)]} \int_G |sf|\, d|\langle h, m_\lambda^{(\infty)}\rangle| \\ &\le \sup_{h\in\mathbb{B}[L^1(G)]} \int_G |f|\, d|\langle h, m_\lambda^{(\infty)}\rangle|. \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, this and (17) imply that the right-hand side of (16) is at most the left-hand side, which establishes (16).

(b) If $A \in \mathcal{B}(G)$ is μ-null, then every $B \in \mathcal{B}(G)$ with $B\subseteq A$ is also μ-null and hence, $m_\lambda^{(\infty)}(B) = \chi_B * \lambda = 0$ in $L^\infty(G)$. So, A is $m_\lambda^{(\infty)}$-null. Conversely, if $A \in \mathcal{B}(G)$ is $m_\lambda^{(\infty)}$-null, then $\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)}\mu(A) \le \|m_\lambda^{(\infty)}\|(A) = 0$ via (15) and (16) with $f := \chi_A$. Since, $\|\widehat{\lambda}\|_{\ell^\infty(\Gamma)} > 0$ (as $\lambda \neq 0$), it follows that $\mu(A) = 0$.

(c) The inclusion $L^1(m_\lambda^{(\infty)})\subseteq L^1(G)$ follows from (15), (16) and (iii)(b). Moreover, since $C_\lambda^{(\infty)}$ commutes with all translations and $I_{m_\lambda^{(\infty)}} = C_\lambda^{(\infty)}$ on $\mathrm{sim}\mathcal{B}(G)$, we have, for all $a \in G$, that

$$(\tau_a \circ I_{m_\lambda^{(\infty)}})(s) = (\tau_a \circ C_\lambda^{(\infty)})(s) = (C_\lambda^{(\infty)} \circ \tau_a)(s) = (I_{m_\lambda^{(\infty)}} \circ \tau_a)(s) \tag{18}$$

and hence, that $\big\|(\tau_a \circ I_{m_\lambda^{(\infty)}})(s)\big\|_{L^\infty(G)} = \big\|(I_{m_\lambda^{(\infty)}} \circ \tau_a)(s)\big\|_{L^\infty(G)}$ for all $s \in \mathrm{sim}\mathcal{B}(G)$. In other words, $m_\lambda^{(\infty)}$ is *norm integral translation invariant*, [4, Definition 3.1]. Recalling that $L^1(m_\lambda^{(\infty)})$ is a closed subspace of $L^1_w(m_\lambda^{(\infty)})$ (cf. Section 2) we can apply Theorem 4.4 of [4] to give

$$\|f\|_{L^1(G)} \le (1/\|m_\lambda^{(\infty)}\|(G))\, \|f\|_{L^1(m_\lambda^{(\infty)})}, \; f \in L^1(m_\lambda^{(\infty)}).$$

This, together with the fact that the function $f_0 := (1/\|m_\lambda^{(\infty)}\|(G))\cdot\chi_G \in \mathbb{B}[L^1(m_\lambda^{(\infty)})]$ satisfies $\|f_0\|_{L^1(G)} = 1/\left\|m_\lambda^{(\infty)}\right\|(G)$, verifies that the natural embedding from $L^1(m_\lambda^{(\infty)})$ into $L^1(G)$ has operator norm equal to $1/\left\|m_\lambda^{(\infty)}\right\|(G)$. QED

Remark 1. (i) It follows from (13) that $m_\lambda^{(\infty)}$ is σ-additive for the weak-$*$ topology in $L^\infty(G) = L^1(G)^*$ for *every* $\lambda \in M(G)$. Norm σ-additivity of $m_\lambda^{(\infty)}$ is characterized in Theorem 1 (I).

(ii) Let $\lambda \in M(G) \setminus \{0\}$. Fix $x \in G$ and $A \in \mathcal{B}(G)$. It follows from (18) that $m_\lambda^{(\infty)}(A + x) = \tau_x(m_\lambda^{(\infty)}(A))$. Since τ_x is an isometry on $L^\infty(G)$, we have $\left\|m_\lambda^{(\infty)}(A+x)\right\|_{L^\infty(G)} = \left\|m_\lambda^{(\infty)}\right\|_{L^\infty(G)}$, i.e., $|m_\lambda^{(\infty)}|$ is translation invariant. So, whenever $m_\lambda^{(\infty)}$ has finite variation, $|m_\lambda^{(\infty)}| = |m_\lambda^{(\infty)}|(G) \cdot \mu$ is a multiple of μ.

3 Proof of Theorem 1 and further consequences

Before proving Theorem 1 we recall some facts. Given $\lambda \in M(G)$, it follows from a result of Akemann, [1, Theorem 4], that $\widetilde{\lambda} \ll \mu$ iff the convolution operator $C_{\widetilde{\lambda}}^{(1)} \in \mathcal{L}(L^1(G))$ is compact iff $C_{\widetilde{\lambda}}^{(1)}$ is weakly compact. Moreover, the adjoint operator $(C_{\widetilde{\lambda}}^{(1)})^* \in \mathcal{L}(L^\infty(G))$ equals the convolution operator $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$, [23, Lemma 7.34].

Given $1 \le p \le r \le \infty$, we denote the natural imbedding of $L^r(G)$ into $L^p(G)$ by $J^{(r,p)}$. The natural isometric imbedding of $C(G)$ onto a closed subspace of $L^\infty(G)$ is denoted by $J^{(c,\infty)}$.

Proof of Theorem 1*. Part* (I). (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv). Since $C_\lambda^{(\infty)} = (C_{\widetilde{\lambda}}^{(1)})^*$, it follows from Schauder's Theorem, [2, Theorem 16.2], (resp. Gantmacher's Theorem, [2, Theorem 17.2]), that $C_\lambda^{(\infty)}$ is compact (resp. weakly compact) iff $C_{\widetilde{\lambda}}^{(1)}$ is compact (resp. weakly compact). This, together with Akemann's result, verifies the equivalences (i) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) because $\lambda \ll \mu$ is equivalent to $\widetilde{\lambda} \ll \mu$.

(iii) $\Rightarrow$ (ii). Let $\mathcal{B}(G) A_n \downarrow \emptyset$. To prove that $\lim_{n\to\infty} m_\lambda^{(\infty)}(A_n) = 0$ in the norm, it suffices to show that $m_\lambda^{(\infty)}(A_n)$ converges to 0 in the weak-$*$ topology. This is because $m_\lambda^{(\infty)}(A_n) = C_\lambda^{(\infty)}(\chi_{A_n}) \in C_\lambda^{(\infty)}(\mathbb{B}[L^\infty(G)])$ for $n \in \mathbb{N}$ and because $C_\lambda^{(\infty)}(\mathbb{B}[L^\infty(G)])$ is, by assumption, a relatively norm compact subset and so the norm topology and the weak-$*$ topology are equivalent on it. But, for each $h \in L^1(G)$, it follows from (13) that $\lim_{n\to\infty}\langle h, m_\lambda^{(\infty)}(A_n)\rangle = \lim_{n\to\infty} \int_{A_n} (h * \widetilde{\lambda})\, d\mu = 0$.

(ii) $\Rightarrow$ (iv). Since $\mathbb{B}[L^\infty(G)]$ is contained in the closed, balanced, convex set $4\overline{\text{bco}}\{\,\chi_A \mid A \in \mathcal{B}(G)\,\}$ and $C_\lambda^{(\infty)}$ is continuous, it follows that

$$\begin{aligned} C_\lambda^{(\infty)}(\mathbb{B}[L^\infty(G)]) &\subseteq 4C_\lambda^{(\infty)}(\overline{\text{bco}}\{\,\chi_A \mid A \in \mathcal{B}(G)\,\}) \\ &\subseteq 4\overline{\text{bco}}\{\,C_\lambda^{(\infty)}(\chi_A) \mid A \in \mathcal{B}(G)\,\} \subseteq 4\overline{\text{bco}}\, R(m_\lambda^{(\infty)}). \end{aligned}$$

On the other hand, the range $R(m_\lambda^{(\infty)})$ is relatively weakly compact, [5, Chap. I, Corollary 2.7], and hence, $4\overline{\text{bco}}\, R(m_\lambda^{(\infty)})$ is weakly compact in $L^\infty(G)$ via Krein's Theorem, [18, §24, 5(4)]. Consequently, $C_\lambda^{(\infty)}$ is weakly compact.

(i) $\Rightarrow$ (v). By (i) and [14, Theorem (20.16)] we have $(\chi_A * \lambda)(x)$ exists for every $x \in G$ with $\chi_A * \lambda \in C(G)$, whenever $A \in \mathcal{B}(G)$. So, (v) holds.

(v) $\Rightarrow$ (i). It follows from (v) that the continuous linear operator $C_\lambda^{(\infty)}$ maps the dense subspace sim$\mathcal{B}(G)$ of its domain space $L^\infty(G)$ into the closed subspace $C(G)$ of its codomain space $L^\infty(G)$ and so $R(C_\lambda^{(\infty)}) \subseteq C(G)$. Let $\overline{C_\lambda^{(\infty)}} : L^\infty(G) \to C(G)$ denote $C_\lambda^{(\infty)}$ considered with $C(G)$ as its codomain space. Since $\overline{C_\lambda^{(\infty)}}(f) = f * \lambda \in C(G)$, for all $f \in L^\infty(G)$, the operator $\overline{C_\lambda^{(\infty)}}$ commutes with all translations, i.e., $\overline{C_\lambda^{(\infty)}}$ is a multiplier operator. Hence, there is $g \in L^1(G)$, [15, Theorem (35.13)], satisfying $f * \lambda = C_\lambda^{(\infty)}(f) = f * g$, for $f \in L^\infty(G)$, which implies that $\widehat{\lambda} = \widehat{g}$. The injectivity of the Fourier-Stieltjes transform, [26, p.29], now gives (5).

(i) $\Rightarrow$ (vi). By Lemma 1 we have the continuous inclusion $L^\infty(G) \subseteq L^1(m_\lambda^{(\infty)})$. To show that $I_{m_\lambda^{(\infty)}}$ is a continuous linear extension of $C_\lambda^{(\infty)}$, consider the vector measure $m_\lambda^{(1)} : \mathcal{B}(G) \to L^1(G)$; see (2). Then, $m_\lambda^{(1)} = J^{(\infty,1)} \circ m_\lambda^{(\infty)}$. Since $J^{(\infty,1)}$ is injective, we have $L^1(m_\lambda^{(\infty)}) \subseteq L^1(m_\lambda^{(1)})$ and $J^{(\infty,1)}\big(\int_A f\,dm_\lambda^{(\infty)}\big) = \int_A f\,dm_\lambda^{(1)}$, for $f \in L^1(m_\lambda^{(\infty)})$ and $A \in \mathcal{B}(G)$, [23, Lemma 3.27]. On the other hand, [23, Proposition 7.35 & Remark 7.36(ii)] imply that $L^1(m_\lambda^{(1)}) = L^1(G)$ and $I_{m_\lambda^{(1)}} = C_\lambda^{(1)}$. Thus $J^{(\infty,1)}\Big(\int_A f\,dm_\lambda^{(\infty)}\Big) = I_{m_\lambda^{(1)}}(f\chi_A) = (f\chi_A)*\lambda$ and hence, $J^{(\infty,1)}$ being the natural injection, gives $\int_A f\,dm_\lambda^{(\infty)} = (f\chi_A)*\lambda$ whenever $f \in L^1(m_\lambda^{(\infty)})$ and $A \in \mathcal{B}(G)$. Therefore, $I_{m_\lambda^{(\infty)}} : f \mapsto f * g$, for $f \in L^1(m_\lambda^{(\infty)})$, is a continuous, linear, $L^\infty(G)$-valued extension of $C_\lambda^{(\infty)}$ and hence, $L^1(m_\lambda^{(\infty)})$ belongs to the class $W^{(\infty)}(\lambda)$.

(vi) $\Rightarrow$ (ii). Choose any B.f.s $X(\mu)$ belonging to $W^{(\infty)}(\lambda)$ and a continuous linear extension $T : X(\mu) \to L^\infty(G)$ of $C_\lambda^{(\infty)}$. Let $\mathcal{B}(G)A(n) \downarrow \emptyset$. Since $\chi_{A(n)} \downarrow 0$ in the order of $X(\mu)^+$, it follows that $\chi_{A(n)} \to 0$ in the topology of the σ-order continuous B.f.s. $X(\mu)$. Therefore, $m_\lambda^{(\infty)}(A(n)) = C_\lambda^{(\infty)}(\chi_{A(n)}) = T(\chi_{A(n)}) \to 0$ in $L^\infty(G)$ as T is continuous, i.e., $m_\lambda^{(\infty)}$ is norm σ-additive.

Part (II). (i) Let $X(\mu)$ be a σ-order continuous B.f.s. belonging to $W^{(\infty)}(\lambda)$ and $T : X(\mu) \to L^\infty(G)$ be a continuous linear extension of $C_\lambda^{(\infty)}$. Since $m_\lambda^{(\infty)} \simeq \mu$, we can apply [23, Theorem 4.14] to conclude that $X(\mu) \subseteq L^1(m_\lambda^{(\infty)})$ continuously and that $I_{m_\lambda^{(\infty)}}$ extends T as $m_\lambda^{(\infty)}(A) = C_\lambda^{(\infty)}(\chi_A) = T(\chi_A)$ for $A \in \mathcal{B}(G)$. In order words, $L^1(m_\lambda^{(\infty)})$ is the *largest* space in $W^{(\infty)}(\lambda)$.

The strict, continuous inclusion $L^\infty(G) \subsetneq L^1(m_\lambda^{(\infty)})$ was established in Lemma 1 and the continuous inclusion $L^1(m_\lambda^{(\infty)}) \subseteq L^1(G)$ is Lemma 4(iii)(c).

Next we establish (6). Theorem 3.5 of [23] implies that $R(I_{m_\lambda^{(\infty)}})$ lies within the closed linear hull $\overline{\mathrm{span}} R(m_\lambda^{(\infty)})$ in $L^\infty(G)$. On the other hand, by part (I), $R(m_\lambda^{(\infty)}) \subseteq C(G)$, so that $R(I_{m_\lambda^{(\infty)}}) \subseteq C(G)$. Given $f \in L^1(m_\lambda^{(\infty)})$ and $A \in \mathcal{B}(G)$, we have already established that $I_{m_\lambda^{(\infty)}}(f\chi_A) = (f\chi_A) * \lambda$ in the proof of (i) $\Rightarrow$ (vi) in part (I) and hence, $(f\chi_A) * \lambda \in C(G)$. Thus, the left-hand side of (6) is contained in the right-hand side.

To prove the reverse containment, fix $f \in L^1(G)$ satisfying $(f\chi_A) * \lambda \in C(G)$ for all $A \in \mathcal{B}(G)$. Define a $C(G)$-valued, finitely additive set function $\eta_f : A \mapsto (f\chi_A) * \lambda$ on $\mathcal{B}(G)$. We shall show that η_f is σ-additive. To this end, recall that $\widehat{\lambda} = \widehat{g} \in c_0(\Gamma)$ from (i) of part (I), so that $\mathrm{supp}(\widehat{\lambda})$ is countable. Consequently, the subgroup Γ_λ generated by $\mathrm{supp}(\widehat{\lambda})$ in Γ is also countable. In particular, the closure $\overline{\mathcal{T}(G, \Gamma_\lambda)}$ of $\mathcal{T}(G, \Gamma_\lambda)$ in $C(G)$ is separable. Given $\gamma \in \Gamma$, the measure $(\cdot, -\gamma)\,d\mu$ belongs to $M(G) = C(G)^*$ and the function

$$A \mapsto \langle \eta_f(A), (\cdot, -\gamma)\,d\mu \rangle = \big((f\chi_A) * \lambda\big)\widehat{\ }(\gamma) = \widehat{\lambda}(\gamma) \int_A f \cdot (\cdot, -\gamma)\,d\mu, \quad A \in \mathcal{B}(G),$$

is σ-additive. Since $\{\,(\cdot, -\gamma)\,d\mu \mid \gamma \in \Gamma\,\}$ is a total subset of $M(G) = C(G)^*$ (by injectivity of the Fourier-Stieltjes transform) and since $R(\eta_f)$ lies in the separable subspace $\overline{\mathcal{T}(G, \Gamma_\lambda)}$ of $C(G)$, it follows from [5, Chap. 1, Corollary 3.7] that η_f is σ-additive. To show that $f \in L^1(m_\lambda^{(\infty)})$, let $A(n) := \{\, x \in G \mid |f(x)| \le n \,\}$ for $n \in \mathbb{N}$. Then the bounded functions $f\chi_{A(n)}$, for $n \in \mathbb{N}$, necessarily $m_\lambda^{(\infty)}$-integrable, converge pointwise to f. Moreover, the σ-additivity of η_f yields, for every $A \in \mathcal{B}(G)$, that

$$\lim_{n\to\infty} \int_A f\chi_{A(n)}\, dm_\lambda^{(\infty)} = \lim_{n\to\infty} \big(f\chi_{A\cap A(n)}\big) * \lambda = \lim_{n\to\infty} \eta_f(A\cap A(n)) = \eta_f(A),$$

exists in the norm of $C(G)$. So, $f \in L^1(m_\lambda^{(\infty)})$ and $\int_A f\, dm_\lambda^{(\infty)} = \eta_f(A)$ for $A \in \mathcal{B}(G)$, [23, Theorem 3.5], and hence, the equality (6) is established.

(ii) The identity (7) was established in the proof of (i) $\Rightarrow$ (vi) in part (I). That $I_{m_\lambda^{(\infty)}}$ commutes with all translations is then an easy consequence of (7).

(iii) As noted in the proof of Lemma 4(iii)(c), $m_\lambda^{(\infty)}$ is norm integral translation invariant. So, Corollary 3.9 in [4] ensures $\mathcal{T}(G)$ is dense in $L^1(m_\lambda^{(\infty)})$. Since $\mathcal{T}(G)\subseteq C(G)\subseteq L^1(m_\lambda^{(\infty)})$, it follows $C(G)$ is also dense in $L^1(m_\lambda^{(\infty)})$.

(iv) Suppose that $L^1(m_\lambda^{(\infty)})$ is separable. Select a countable dense subset P of $L^1(m_\lambda^{(\infty)})$. Since $L^1(m_\lambda^{(\infty)})\subseteq L^1(G)$ continuously (see (i) of part (II)), the set P is also dense in $L^1(G)$. We claim that $\Gamma = \bigcup_{f\in P} \operatorname{supp}(\widehat{f})$. In fact, Γ clearly contains the right-side. To show the reverse containment, let $\gamma \in \Gamma$. Choose a sequence $\{f_n\}_{n=1}^\infty$ from P converging to $\varphi := (\cdot,\gamma)$ in $L^1(G)$. Then $\lim_{n\to\infty} \widehat{f_n}(\gamma) = \widehat{\varphi}(\gamma) = 1$, so that $\widehat{f_n}(\gamma) \neq 0$ for some $n \in \mathbb{N}$, i.e., $\gamma \in \operatorname{supp}(\widehat{f_n})$. This establishes the identity $\Gamma = \bigcup_{f\in P} \operatorname{supp}(\widehat{f})$. Consequently, Γ is *countable* because each $\widehat{f_n} \in c_0(\Gamma)$ and hence, has countable support. Via [26, Theorem 2.2.6], the group G is metrizable.

Conversely, suppose that G is metrizable. Then, $C(G)$ is separable, [26, A16, p.251]. Recall that $C(G)$ is dense in $L^1(m_\lambda^{(\infty)})$ via (iii) of part (II). So, we can conclude that $L^1(m_\lambda^{(\infty)})$ is separable. QED

Remark 2. For any $\lambda \in M(G) \setminus \{0\}$, each of the following conditions is also equivalent to any one of (i)–(vi) in part (I) of Theorem 1. Recall that a continuous linear operator between Banach spaces is *completely continuous* if it maps weakly convergent sequences to norm convergent sequences, [6, p.49].

(vii) *The operator* $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$ *is completely continuous.*

(viii) $R(m_\lambda^{(\infty)})$ *is relatively compact in* $L^\infty(G)$.

(ix) $R(m_\lambda^{(\infty)})$ *is relatively weakly compact in* $L^\infty(G)$.

(x) $R(m_\lambda^{(\infty)})$ *is a separable subset of* $L^\infty(G)$.

In fact, a Banach-space-valued, continuous linear operator defined on $L^\infty(G)$ is weakly compact if and only if it is completely continuous because $L^\infty(G)$ is an AM-space and hence, has both the Dunford-Pettis property and the reciprocal Dunford-Pettis property, [2, Theorem 19.6 and p.347]. This establishes (iv) $\Leftrightarrow$ (vii). The implications (iii) $\Rightarrow$ (viii) $\Rightarrow$ (ix) are clear. Moreover, we can establish (ix) $\Rightarrow$ (iv) as in the proof of (ii) $\Rightarrow$ (iv) in part (I). The implication (viii) $\Rightarrow$ (x) is clear. Finally, given (x), we can derive (ii) similar to the proof of (6). Indeed, the subset $\{\,(\cdot,-\gamma) \mid \gamma \in \Gamma\,\}\subseteq L^1(G)\subseteq L^\infty(G)^*$ is a total subset of $L^\infty(G)^*$ via injectivity of the Fourier transform. Furthermore, given $\gamma \in \Gamma$, the finitely additive set function $A \mapsto \langle(\cdot,-\gamma), m_\lambda^{(\infty)}(A)\rangle = \widehat{\lambda}(\gamma)\int_A (\cdot,-\gamma)\, d\mu$ on $\mathcal{B}(G)$ is σ-additive. According to (x), $m_\lambda^{(\infty)}$ takes its values in a closed *separable* subspace X of $L^\infty(G)$. Since X cannot contain an isomorphic copy of ℓ^∞, we can apply [5, Chap. I, Corollary 3.7] to obtain (ii). QED

In view of Theorem 1, we will concentrate on those $\lambda \in M(G)$ for which $m_\lambda^{(\infty)}$ is norm σ-additive. Given such a λ, there is $g \in L^1(G)$ with $\lambda = g\, d\mu$ (cf. Theorem 1 (I)) and

$$I_{m_g^{(\infty)}}(f) = f * g, \qquad f \in L^1(m_g^{(\infty)}). \tag{19}$$

For $1 \le p < \infty$, we first determine exactly when $L^p(G) \subseteq L^1(m_g^{(\infty)})$, i.e., when $L^p(G) \in W^{(\infty)}(\lambda)$.

Proposition 1. *Let $1 \le p < \infty$. The following assertions for a function $g \in L^1(G) \setminus \{0\}$ are equivalent.*

(i) $g \in L^{p^*}(G)$.

(ii) $L^p(G) \subseteq L^1(m_g^{(\infty)})$.

(iii) $L^p(G) * g \subseteq C(G)$.

(iv) $L^p(G) * g \subseteq L^\infty(G)$.

(v) *The convolution operator $C_g^{(\infty)}$ is p-summing.*

In this case, $I_{m_g^{(\infty)}}$ is a continuous linear extension of the convolution operator $C_g^{(p,\infty)} \in \mathcal{L}(L^p(G), L^\infty(G))$.

Proof. (i) ⇒ (ii). By (i) we have $L^p(G) * g \subseteq C(G) \subseteq L^\infty(G)$ continuously; see [14, Theorem (20.16)] and its proof. So, $C_g^{(p,\infty)}$ is a continuous linear extension of $C_g^{(\infty)}$ and hence, its domain space $L^p(G)$ is necessarily contained in the optimal domain $L^1(m_g^{(\infty)})$ via Theorem 1 (II)(i). That is, (ii) holds.

(ii) ⇒ (iii). This implication is valid via (6).

(iii) ⇒ (iv). Clear.

(iv) ⇒ (v). Let us denote by $\overline{C_g^{(p,\infty)}}$ the linear operator of convolution with g (it exists by (iv)) from $L^p(G)$ into $L^\infty(G)$. A closed graph argument ensures the continuity of $\overline{C_g^{(p,\infty)}}$. Since $C_g^{(\infty)} = \overline{C_g^{(p,\infty)}} \circ J^{(\infty,p)}$ and $J^{(\infty,p)}$ is p-summing, [6, Example 2.9(d)], also $C_g^{(\infty)}$ is p-summing.

(v) ⇒ (i). The restriction $C_g^{(\infty)}|_{C(G)}$ of $C_g^{(\infty)}$ to the closed subspace $C(G)$ of $L^\infty(G)$ is also p-summing, [6, p.37]. By the Theorem on p.56 of [6] (see also [25, Lemma 4.3]) applied to the p-summing operator $C_g^{(\infty)} : C(G) \to C(G)$, i.e., to $C_g^{(\infty)}|_{C(G)}$ considered with codomain $C(G) \subseteq L^\infty(G)$, there is $K > 0$ such that

$$\left\|C_g^{(\infty)}(f)\right\|_{L^\infty(G)} = \left\|C_g^{(\infty)}(f)\right\|_{C(G)} \le K \|f\|_{L^p(G)}, \quad f \in C(G) \subseteq L^p(G).$$

This enables us to extend $C_g^{(\infty)}|_{C(G)}$ to a unique continuous linear operator $T : L^p(G) \to L^\infty(G)$. Such an extension T satisfies $T \circ \tau_a = \tau_a \circ T$ for each $a \in G$ (i.e., T is a *multiplier operator* from $L^p(G)$ into $L^\infty(G)$) because $(C_g^{(\infty)} \circ \tau_a)(f) = (\tau_a \circ C_g^{(\infty)})(f)$ for every $f \in C(G)$. Therefore, $T = C_h^{(p,\infty)}$ for some $h \in L^{p^*}(G)$; see [15, Theorem (35.12)], [19, Theorem 3.3.1]. Now, $\widehat{g}(\gamma)(\cdot,\gamma) = C_g^{(\infty)}((\cdot,\gamma)) = T((\cdot,\gamma)) = C_h^{(p,\infty)}((\cdot,\gamma)) = \widehat{h}(\gamma)(\cdot,\gamma)$ and hence, $\widehat{g}(\gamma) = \widehat{h}(\gamma)$ for all $\gamma \in \Gamma$. So, $g = h \in L^{p^*}(G)$. QED

Remark 3. (i) In the notation of Proposition 1, if $g \in L^{p^*}(G)$, then the inclusion $L^p(G) \subseteq L^1(m_g^{(\infty)})$ is actually continuous. To see this, fix $f \in L^p(G)$. Via (14) and (16), with $\lambda := g\,d\mu$, and Hölder's inequality we have

$$\begin{aligned} \|f\|_{L^1(m_g^{(\infty)})} &= \sup_{h \in \mathbb{B}[L^1(G)]} \int_G |f| \cdot |h * \widetilde{g}|\,d\mu \\ &\le \sup_{h \in \mathbb{B}[L^1(G)]} \|f\|_{L^p(G)} \|h * \widetilde{g}\|_{L^{p^*}(G)} \le \|f\|_{L^p(G)} \|g\|_{L^{p^*}(G)} \end{aligned} \tag{20}$$

because, for each $h \in L^1(G)$, it follows from [14, Theorem (20.12)] that

$$\|h * \widetilde{g}\|_{L^{p^*}(G)} \le \|h\|_{L^1(G)} \|\widetilde{g}\|_{L^{p^*}(G)} = \|h\|_{L^1(G)} \|g\|_{L^{p^*}(G)}.$$

So, the operator norm of the inclusion $L^p(G)\subseteq L^1(m_g^{(\infty)})$ is at most $\|g\|_{L^{p^*}(G)}$.

(ii) Let $1 \le p < \infty$ and $\lambda \in M(G) \setminus \{0\}$. Then $C_\lambda^{(\infty)}$ is p-summing iff $\lambda = g\,d\mu$ for some $g \in L^{p^*}(G)$. Indeed, if $\lambda = g\,d\mu$ with $g \in L^{p^*}(G)$, then $C_\lambda^{(\infty)} = C_g^{(\infty)}$ is p-summing by Proposition 1. Conversely, if $C_\lambda^{(\infty)}$ is p-summing, then it is weakly compact, [6, Theorem 2.17], and hence, $\lambda = g\,d\mu$ for some $g \in L^1(G)$; see Theorem 1. Then Proposition 1 implies that $g \in L^{p^*}(G)$.

Corollary 1. *Let $g \in L^1(G) \setminus \{0\}$.*

(i) *The function $g \in L^\infty(G)$ iff $L^1(m_g^{(\infty)}) = L^1(G)$.*

(ii) *There is no $p \in (1,\infty)$ satisfying $L^p(G) = L^1(m_g^{(\infty)})$.*

Proof. (i). Always $L^1(m_g^{(\infty)})\subseteq L^1(G)$ by Lemma 4(iii)(c). On the other hand, for $p := 1$ we see from Proposition 1 that $g \in L^\infty(G)$ iff $L^1(G)\subseteq L^1(m_g^{(\infty)})$.

(ii). Assume, on the contrary, that $L^p(G) = L^1(m_g^{(\infty)})$ for some $p \in (1,\infty)$, in which case $g \in L^{p^*}(G)$ by Proposition 1. Since the identity map from $L^p(G)$ onto $L^1(m_g^{(\infty)})$ is continuous (cf. Remark 3), the Open Mapping Theorem ensures that $L^p(G)$ and $L^1(m_g^{(\infty)})$ are isomorphic Banach spaces. So, $I_{m_g^{(\infty)}} = C_g^{(p,\infty)}$ is the continuous convolution operator (via $g \in L^{p^*}(G)$) of $L^p(G)$ into $L^\infty(G)$; see Theorem 1 (II)(ii). Observe that $C_g^{(p,\infty)} = \big(C_{\widetilde{g}}^{(1,p^*)}\big)^*$, where $C_{\widetilde{g}}^{(1,p^*)} : L^1(G) \to L^{p^*}(G)$ is compact, [23, Theorem 7.50]. According to Schauder's Theorem, also $I_{m_g^{(\infty)}} = C_g^{(p,\infty)}$ is compact. Then $L^1(m_g^{(\infty)}) = L^1(|m_g^{(\infty)}|)$ and $m_g^{(\infty)}$ has *finite variation*, [23, Theorem 7.50]. By Remark 1(ii), $L^1(|m_g^{(\infty)}|) = L^1(G)$, i.e., $L^p(G) = L^1(G)$, which implies that $L^\infty(G) = L^{p^*}(G)$. Since μ is *non-atomic*, [23, Lemma 7.97], it follows that $L^{p^*}(G)$ is infinite-dimensional; by a result of Grothendieck, [5, p.178], this is a contradiction. So, no such $p \in (1,\infty)$ exists. QED

Given $1 \le p < \infty$, we denote by $\Pi_p(L^\infty(G))$ the vector space of all *p-summing operators* from $L^\infty(G)$ into itself, [6, p.31].

Corollary 2. *Let $g \in L^1(G) \setminus \{0\}$.*

(i) *The following conditions are equivalent.*

  (a) $g \in \big(\bigcap_{1\le q<\infty} L^q(G)\big) \setminus L^\infty(G)$.

  (b) $\bigcup_{1<p<\infty} L^p(G) \subsetneqq L^1(m_g^{(\infty)}) \subsetneqq L^1(G)$.

  (c) $C_g^{(\infty)} \in \big(\bigcap_{1<p<\infty} \Pi_p(L^\infty(G))\big) \setminus \Pi_1(L^\infty(G))$.

(ii) *Given $1 < r < \infty$, the following conditions are equivalent.*

  (a) $g \in \big(\bigcap_{1\le q<r^*} L^q(G)\big) \setminus L^{r^*}(G)$.

  (b) $\bigcup_{r<p\le\infty} L^p(G) \subsetneqq L^1(m_g^{(\infty)})$ *and* $L^r(G) \nsubseteq L^1(m_g^{(\infty)})$.

  (c) $C_g^{(\infty)} \in \big(\bigcap_{r<p<\infty} \Pi_p(L^\infty(G))\big) \setminus \Pi_r(L^\infty(G))$.

(iii) *Given $1 < r < \infty$, the following conditions are equivalent.*

  (a) $g \in L^{r^*}(G) \setminus \big(\bigcup_{r^*<q<\infty} L^q(G)\big)$.

  (b) $L^r(G)\subseteq L^1(m_g^{(\infty)})$ *and* $L^p(G) \nsubseteq L^1(m_g^{(\infty)})$ *for each* $1 \le p < r$.

  (c) $C_g^{(\infty)} \in \Pi_r(L^\infty(G)) \setminus \big(\bigcup_{1\le p<r} \Pi_p(L^\infty(G))\big)$.

The proof of Corollary 2 is a routine application of the repeated use of various equivalences in Proposition 1, together with Corollary 1. The only point of a different nature is to check, in part (i)(b), that the inclusion $\bigcup_{1<p\le\infty} L^p(G)\subseteq L^1(m_g^{(\infty)})$ cannot be an equality. But, for each $1 < p \le \infty$, the natural inclusion $L^p(G)\subseteq L^1(m_g^{(\infty)})$ is injective and continuous (by Remark 3(i)). Moreover, the Banach spaces $L^p(G)$, $1 < p \le \infty$, are all distinct (as μ is non-atomic) and satisfy $L^p(G)\subseteq L^q(G)$ whenever $q \le p$. Consequently, for the Banach space $L^1(m_g^{(\infty)})$, the equality $L^1(m_g^{(\infty)}) = \bigcup_{1<p\le\infty} L^p(G)$ is impossible. A similar argument applies to part (ii)(b).

It is also possible to give detailed information about the surjectivity and injectivity of the extended operator $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$.

Proposition 2. *Let* $g \in L^1(G) \setminus \{0\}$.

(i) $R(I_{m_g^{(\infty)}}) \subsetneqq C(G)$.

(ii) *The following assertions are equivalent.*

(a) $R(I_{m_g^{(\infty)}})$ *is dense in* $C(G)$.

(b) $\operatorname{supp}(\widehat{g}) = \Gamma$.

(c) *The integration map* $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$ *is injective.*

If any one of (a)–(c) *holds in part* (ii)*, then* $R(C_g^{(\infty)}) \subsetneqq R(I_{m_g^{(\infty)}})$.

The previous result, for $1 \le p < \infty$, occurs in [21, Theorem 1.3] and is based on Lemma 5.1 of [21]. A careful examination of the proof of both Lemma 5.1 and Theorem 1.3 of [21] shows that they can be adapted to the present case where $p := \infty$. Note that part (c) of Lemma 5.1 in [21] is *not* used in the proof of Theorem 1.3 in [21].

Let us indicate the modifications needed to establish Proposition 2. First, (6) and (7) from Section 1 show that $R(I_{m_g^{(\infty)}})$ lies in the closed subspace $C(G)$ of $L^\infty(G)$ and hence, also its closure $\overline{R(I_{m_g^{(\infty)}})}$ (formed in $L^\infty(G)$) belongs to $C(G)$. To formulate the required analogue (for $p := \infty$) of Lemma 5.1 in [21] (without part (c)) we only need to replace $L^p(G)$ in its statement with $C(G)$. The proof of parts (a), (b) of this analogue then carry over easily, after noting in the proof of (iii) $\Rightarrow$ (i) in Lemma 5.1(b) of [21] that the series $\sum_{n=1}^\infty n^{-2}(\cdot, \gamma_n)$ specifying the function h given there is now absolutely convergent in $C(G)$. Moreover, the proof of part (d) of Lemma 5.1 in [21] is also easily adapted, provided that $\mathcal{T}(G)$ is dense in $L^1(m_g^{(\infty)})$ which is the case (cf. Theorem 1 (II)(iii)). Of course, the formula (cf. (7)) $I_{m_g^{(\infty)}}(f) = f * g$, for $f \in L^1(m_g^{(\infty)})$, is often needed.

The proof of Proposition 2 above now follows the lines of that of Theorem 1.3 in [21], modulo the following additional points. To verify part (i) of Proposition 2 one now needs the fact that $C(G)$ is infinite-dimensional; this follows from $L^1(G)$ being infinite-dimensional (as μ is non-atomic) and that $C(G)$ is dense in $L^1(G)$, [26, E8, p.268]. For the proof of (a) $\Leftrightarrow$ (b) in part (ii) of Proposition 2, the density of $\mathcal{T}(G)$ in $C(G)$ is required, [26, p.24]. Finally, to establish $R(C_g^{(\infty)}) \subsetneqq R(I_{m_g^{(\infty)}})$ in part (ii) of Proposition 2, given that g satisfies one of (a)–(c), it is necessary to know that $L^\infty(G) \subsetneqq L^1(m_g^{(\infty)})$; this is the case via Theorem 1 (II)(i).

4 Bochner, Gelfand and Pettis μ-densities.

The aim of this final section is to present the proofs of Theorems 2–4. We begin with a technical result. Given a function $F : G \to L^\infty(G)$ and $h \in L^1(G)$, we denote the scalarvalued function $x \mapsto \langle h, F(x)\rangle$, for $x \in G$, by $\langle h, F\rangle$, where $\langle h, F(x)\rangle := \int_G h(y)F(x)(y)\,d\mu(y)$ for $x \in G$.

Lemma 5. *Let $g \in L^1(G)$. Suppose that the vector measure $m_g^{(\infty)} : \mathcal{B}(G) \to L^\infty(G)$ admits a Gelfand (resp. Bochner, Pettis) μ-density. Then $g \in L^\infty(G)$ and the $L^\infty(G)$-valued function $K_g : x \mapsto \tau_x(g)$ on G is also a Gelfand (resp. Bochner, Pettis) μ-density of $m_g^{(\infty)}$.*

Proof. The subgroup Γ_g of Γ generated by the countable subset $\mathrm{supp}(\widehat{g}) \subseteq \Gamma$ is also countable. Lemma 3 applied to $\lambda := g\,d\mu$ allows us to find $\lambda_1 \in M(G)$ satisfying $\widehat{\lambda_1} = \chi_{\Gamma_g}$ on Γ and $g * \lambda_1 = g$.

Consider first the case when $m_g^{(\infty)}$ admits a Gelfand μ-density, say $F : G \to L^\infty(G)$. Define $H : G \to L^\infty(G)$ by $H := C_{\lambda_1}^{(\infty)} \circ F$. For each $f \in L^1(G)$ we have $\langle f, H\rangle = \langle C_{\tilde{\lambda}_1}^{(1)}(f), F\rangle$ and so $\langle f, H\rangle \in L^1(G)$, i.e., H is Gelfand μ-integrable, [5, p.53]. Given $A \in \mathcal{B}(G)$ the function $\varphi_A := C_{\lambda_1}^{(\infty)}\big((w^*)\text{-}\int_A F\,d\mu\big) \in L^\infty(G)$ satisfies $\langle f, \varphi_A\rangle = \int_A \langle f, H\rangle\,d\mu$, for $f \in L^1(G)$, and so $(w^*)\text{-}\int_A H\,d\mu = \varphi_A$. Moreover, for $A \in \mathcal{B}(G)$ we have

$$\begin{aligned}(w^*)\text{-}\int_A H\,d\mu = \varphi_A = \lambda_1 * \Big((w^*)\text{-}\int_A F\,d\mu\Big) = \lambda_1 * m_g^{(\infty)}(A) \\ = \lambda_1 * (\chi_A * g) = \chi_A * g = m_g^{(\infty)}(A),\end{aligned} \tag{21}$$

via the identity $g * \lambda_1 = g$. Thus, H is also a Gelfand μ-density of $m_g^{(\infty)}$.

Next we claim there exists a set $A(g) \in \mathcal{B}(G)$ with $\mu(A(g)) = 1$ such that

$$\tau_x(g) = H(x), \qquad x \in A(g), \tag{22}$$

with equality as elements of $L^1(G)$. To see this fix $\gamma \in \Gamma_g$ and observe that $[\tau_x(g)]\widehat{\ }(\gamma) = \widehat{g}(\gamma)(x, -\gamma)$ for $x \in G$. Accordingly,

$$\begin{aligned}\int_A [\tau_x(g)]\widehat{\ }(\gamma)\,d\mu(x) &= \widehat{g}(\gamma)\widehat{\chi}_A(\gamma) = (\chi_A * g)\widehat{\ }(\gamma) = \langle (\cdot, -\gamma), m_g^{(\infty)}(A)\rangle \\ &= \Big\langle (\cdot, -\gamma), (w^*)\text{-}\int_A H\,d\mu\Big\rangle = \int_A \langle (\cdot, -\gamma), H(x)\rangle\,d\mu(x) \\ &= \int_A [H(x)]\widehat{\ }(\gamma)\,d\mu(x),\end{aligned}$$

for every $A \in \mathcal{B}(G)$. So, there is $A_\gamma \in \mathcal{B}(G)$ with $\mu(A_\gamma) = 1$ such that $[\tau_x(g)]\widehat{\ }(\gamma) = [H(x)]\widehat{\ }(\gamma)$ for all $x \in A_\gamma$. As Γ_g is countable, the Borel set $A(g) := \bigcap_{\gamma \in \Gamma_g} A_\gamma$ satisfies $\mu(A(g)) = 1$ and we have

$$[\tau_x(g)]\widehat{\ }(\gamma) = [H(x)]\widehat{\ }(\gamma), \qquad x \in A(g),\ \gamma \in \Gamma_g. \tag{23}$$

On the other hand, for $x \in A(g)$, it follows that $[\tau_x(g)]\widehat{\ }(\gamma) = \widehat{g}(\gamma)(x, -\gamma) = 0$ and also that

$$[H(x)]\widehat{\ }(\gamma) = [\lambda_1 * F(x)]\widehat{\ }(\gamma) = \widehat{\lambda_1}(\gamma)[F(x)]\widehat{\ }(\gamma) = \chi_{\Gamma_g}(\gamma)[F(x)]\widehat{\ }(\gamma) = 0$$

whenever $\gamma \in (\Gamma \setminus \Gamma_g) \subseteq (\Gamma \setminus \mathrm{supp}(\widehat{g}))$. This and (23) imply, given $x \in A(g)$, that $[\tau_x(g)]\widehat{\ }(\gamma) = [H(x)]\widehat{\ }(\gamma)$ for all $\gamma \in \Gamma$, i.e., $\tau_x(g) = H(x)$ as elements of $L^1(G)$. So, (22) is established.

Since $\mu(A(g)) = 1$, the set $A(g) \neq \emptyset$ and so we can select $a \in A(g)$. Then, by (22) with $x := a$, we have $g = (\tau_{-a} \circ \tau_a)(g) = \tau_{-a}(H(a))$. Since $H(a) \in L^\infty(G)$, also $\tau_{-a}(H(a)) \in L^\infty(G)$, i.e, $g \in L^\infty(G)$. This enables us to consider the $L^\infty(G)$-valued function K_g on G. Then (22) means that

$$K_g(x) = H(x), \qquad \mu\text{-a.e. } x \in G. \tag{24}$$

This, with (21) and Lemma 2(ii), imply K_g is a Gelfand μ-density of $m_g^{(\infty)}$.

Next assume that F is a Bochner μ-density of $m_g^{(\infty)}$. Since $C_{\lambda_1}^{(\infty)} \in \mathcal{L}(L^\infty(G))$, it is routine to verify from the definition of Bochner integrals, [5, Chap. II, §2], that $H := C_{\lambda_1}^{(\infty)} \circ F$ is also Bochner μ-integrable. In particular, H is also Gelfand μ-integrable; see Lemma 2(i). Moreover, (B)-$\int_A H\,d\mu = (w^*)$-$\int_A H\,d\mu = m_g^{(\infty)}(A)$, for $A \in \mathcal{B}(G)$; see (21) and Lemma 2(i). So, H is also a Bochner μ-density of $m_g^{(\infty)}$. Now (24) and Lemma 2(ii) ensure K_g is a Bochner μ-density of $m_g^{(\infty)}$.

Finally suppose that F is a Pettis μ-density of $m_g^{(\infty)}$. Since $C_{\lambda_1}^{(\infty)} \in \mathcal{L}(L^\infty(G))$, it is again routine to verify from the definition of Pettis integrals, [5, Chap. II, §3], that $H := C_{\lambda_1}^{(\infty)} \circ F$ is also Pettis μ-integrable. By Lemma 2(i) we see that H is also Gelfand μ-integrable. Moreover, (P)-$\int_A H\,d\mu = (w^*)$-$\int_A H\,d\mu = m_g^{(\infty)}(A)$, for $A \in \mathcal{B}(G)$; see (21) and Lemma 2(i). Then K_g is a Pettis μ-density of $m_g^{(\infty)}$ by (24) and Lemma 2(ii). QED

Given $h \in L^1(G)$ and $g \in L^\infty(G)$ we have

$$\langle h, K_g(x)\rangle = \int_G h(y)g(y-x)\,d\mu(y) = (h * \widetilde{g})(x), \qquad x \in G.$$

This and [14, Corollary (20.14) & Theorem (20.16)] imply the next result.

Lemma 6. *Let $h \in L^1(G)$ and $g \in L^\infty(G)$. Then $\langle h, K_g\rangle = h * \widetilde{g} \in C(G)$ and*

$$\|\langle h, K_g\rangle\|_{L^\infty(G)} = \|h * \widetilde{g}\|_{L^\infty(G)} \le \|h\|_{L^1(G)} \|g\|_{L^\infty(G)}. \tag{25}$$

We know from Corollary 1 that $g \in L^1(G)$ belongs to $L^\infty(G)$ iff $L^1(m_g^{(\infty)}) = L^1(G)$. Moreover, $L^1(m_g^{(\infty)}) = L^1(G)$ are isomorphic Banach spaces because

$$\left\|m_g^{(\infty)}\right\|(G) \cdot \|f\|_{L^1(G)} \le \|f\|_{L^1(m_g^{(\infty)})} \le \|g\|_{L^\infty(G)} \|f\|_{L^1(G)}, \tag{26}$$

for every $f \in L^1(G) = L^1(m_g^{(\infty)})$. Indeed, the left inequality has been established in Lemma 4 (iii)(c). On the other hand, the estimates in (20) for $p := 1$ yield $\|f\|_{L^1(m_g^{(\infty)})} \le \|g\|_{L^\infty(G)} \cdot \|f\|_{L^1(G)}$ which is the right inequality in (26). So, we can write $I_{m_g^{(\infty)}} = C_g^{(1,g)}$ as elements of $\mathcal{L}(L^1(G), L^\infty(G))$.

Proof of Theorem 2. (i) $\Leftrightarrow$ (ii). See Corollary 1(i).

(i) $\Leftrightarrow$ (v). This follows from Proposition 1 with $p := 1$.

(i) $\Rightarrow$ (iii). Because of (i) we have, via Lemma 6, that $\langle h, K_g\rangle = h * \widetilde{g} \in L^1(G)$ for $h \in L^1(G)$, i.e., K_g is Gelfand μ-integrable. With $\lambda := g\,d\mu$, (13) yields

$$\langle h, m_g^{(\infty)}(A)\rangle = \int_A \langle h, K_g\rangle\,d\mu = \langle h, (w^*)\text{-}\int_A K_g\,d\mu\rangle, \qquad h \in L^1(G),\ A \in \mathcal{B}(G),$$

which implies that K_g is a Gelfand μ-density of $m_g^{(\infty)}$. So, (iii) holds.

(iii) $\Rightarrow$ (i). See Lemma 5.

(i) $\Rightarrow$ (iv). Condition (i) implies that

$$\left\|m_g^{(\infty)}(A)\right\|_{L^\infty(G)} = \left\|\chi_A * g\right\|_{L^\infty(G)} \le \|g\|_{L^\infty(G)} \left\|\chi_A\right\|_{L^1(G)} = \|g\|_{L^\infty(G)}\,\mu(A),$$

for each $A \in \mathcal{B}(G)$, [14, Corollary (20.14)]. Thus, $|m_g^{(\infty)}|(A) \le \|g\|_{L^\infty(G)}\,\mu(A)$ for $A \in \mathcal{B}(G)$. This establishes (iv).

(iv) $\Rightarrow$ (ii). By Remark 1(ii) we have that $|m_g^{(\infty)}|$ is a positive multiple of μ. So, $L^1(G) = L^1(|m_g^{(\infty)}|) \subseteq L^1(m_g^{(\infty)}) \subseteq L^1(G)$; for the two inclusions we refer to (12) and Lemma 4(iii)(c). So, (ii) holds.

This completes the proof of the mutual equivalence of (i)–(v).

So, assume now that any one of (i)–(v) hold. In the proof of (i) ⇒ (iii) it was established that K_g is a Gelfand μ-density of $m_g^{(\infty)}$. Consequently,

$$I_{m_g^{(\infty)}}(s) = (w^*)\text{-}\int_G sK_g\,d\mu, \qquad s \in \operatorname{sim}\mathcal{B}(G). \tag{27}$$

Fix $f \in L^1(G)$. If $h \in L^1(G)$, then $\langle h, K_g\rangle \in C(G)$ gives $\langle h, fK_g\rangle = f\langle h, K_g\rangle \in L^1(G)$. Hence, fK_g is Gelfand μ-integrable. Moreover,

$$\|\langle h, fK_g\rangle\|_{L^1(G)} \le \|f\|_{L^1(G)}\,\|\langle h, K_g\rangle\|_{L^\infty(G)} \le \|f\|_{L^1(G)}\,\|g\|_{L^\infty(G)}\,\|h\|_{L^1(G)}\,;$$

see (25). It follows that

$$\begin{aligned}\Big\|(w^*)\text{-}\int_G fK_g\,d\mu\Big\|_{L^\infty(G)} &= \sup_{h\in\mathbb{B}[L^1(G)]} \big|\langle h, (w^*)\text{-}\int_G fK_g\,d\mu\rangle\big| \\ &= \sup_{h\in\mathbb{B}[L^1(G)]} \Big|\int_G \langle h, fK_g\rangle\,d\mu\Big| \le \sup_{h\in\mathbb{B}[L^1(G)]} \|\langle h, fK_g\rangle\|_{L^1(G)} \le \|g\|_{L^\infty(G)}\,\|f\|_{L^1(G)}\,.\end{aligned}$$

Consequently, the $L^\infty(G)$-valued, linear operator $T : f \mapsto (w^*)\text{-}\int_G fK_g\,d\mu$ on $L^1(G)$ is continuous. Via (27) we see that $I_{m_g^{(\infty)}} = T$ on the dense subspace $\operatorname{sim}\mathcal{B}(G)$ of $L^1(G)$ and hence, $I_{m_g^{(\infty)}} = T$ on $L^1(G)$. So, (8) holds.

This completes the proof of Theorem 2. QED

Remark 4. Whenever $g \in L^\infty(G)$ we point out that $I_{m_g^{(\infty)}} : L^1(G) \to L^\infty(G)$ is necessarily completely continuous. This is the case because $\{\, I_{m_g^{(\infty)}}(\chi_A) \mid A \in \mathcal{B}(G)\,\} = R(m_g^{(\infty)})$ is relatively compact in $L^\infty(G)$ (cf. Remark 2) and because the domain space of $I_{m_g^{(\infty)}}$ is $L^1(G)$, [23, Corollary 2.42]. QED

Proof of Theorem 3. *Part* (I). Recall from Theorem 2 that necessarily $L^1(m_g^{(\infty)}) = L^1(G)$.

(i) ⇔ (iv). See [8, Theorem].

(iv) ⇒ (iii). By compactness of G the continuous function K_g is Bochner μ-integrable and has compact range in $L^\infty(G)$. Since fK_g is clearly strongly μ-measurable, [5, Chap. II, §1], and $\|f(x)K_g(x)\|_{L^\infty(G)} \le M|f(x)|$ for $x \in G$ and some constant $M > 0$ the function fK_g is also Bochner μ-integrable, for each $f \in L^1(G)$. It follows from [5, Chap. III, Theorem 2.2] that the $L^\infty(G)$-valued linear operator $T : f \mapsto (B)\text{-}\int_G fK_g\,d\mu$ on $L^1(G)$ is compact. On the other hand, $T = I_{m_g^{(\infty)}}$ via (8) and Lemma 2(i), which thereby establishes (iii).

(iii) ⇒ (ii). By [5, Chap. III, Theorem 2.2] there exists a bounded, Bochner μ-integrable function $F : G \to L^\infty(G)$ such that $I_{m_g^{(\infty)}}(f) = (B)\text{-}\int_G fF\,d\mu$ for $f \in L^1(G)$. Substituting $f := \chi_A$, for $A \in \mathcal{B}(G)$, gives (ii).

(ii) ⇒ (v). It follows from Lemma 5 that K_g is also a Bochner μ-density of $m_g^{(\infty)}$. In particular, K_g is strongly μ-measurable and so there exists $A_0 \in \mathcal{B}(G)$ with $\mu(A_0) = 1$ such that $K_g(A_0)$ is norm separable in $L^\infty(G)$, [5, Chap. II, Theorem 1.2].

(v) ⇒ (i). Assume first that (v) holds with $A_0 := G$, so that $K_g(G)$ is contained in a *separable*, closed subspace Y of $L^\infty(G)$. Let $Y_1^* := \{\, \xi_h \mid h \in L^1(G)\,\}$ denote the vector subspace of Y^* defined by $\xi_h : f \mapsto \int_G fh\,d\mu$, for $f \in Y$, as h varies in $L^1(G)$. Given $h \in L^1(G)$, it follows that $\langle \xi_h, K_g\rangle = \langle h, K_g\rangle$ is $\mathcal{B}(G)$-measurable on G because $\langle h, K_g\rangle \in C(G)$; see Lemma 6. Since Y_1^* is total in Y^* and since $K_g(G)$ lies in the separable Banach space Y (necessarily a Suslin space), we have as two consequences of [30, Theorem 1 & Remark] that

K_g is the pointwise norm-limit (everywhere on G) of a sequence of Y-valued, $\mathcal{B}(G)$-simple functions (hence, being bounded, is also Bochner μ-integrable) and that $K_g^{-1}(U) \in \mathcal{B}(G)$ for every Borel subset U of Y. Of course, K_g is then also Bochner μ-integrable as an $L^\infty(G)$-valued function and satisfies $K_g^{-1}(B) \in \mathcal{B}(G)$ for every Borel subset B of $L^\infty(G)$. The first mentioned consequence together with (8) and Lemma 4(i) give

$$I_{m_g^{(\infty)}}(f) = (w^*)\text{-}\int_G fK_g\,d\mu = (B)\text{-}\int_G fK_g\,d\mu, \qquad f \in L^1(G). \tag{28}$$

Since $K_g(G)$ lies in the separable Banach space Y, it follows from the second consequence and [28, Theorem 5, p.26] that K_g is *Lusin μ-measurable*. Accordingly, there is a compact set $W \subseteq G$, with $\mu(W) > 0$, on which the Y-valued function K_g is continuous, [28, Definition 9, p.25]. In particular, $K_g(W)$ is compact in Y and hence, also compact in $L^\infty(G)$. Via (28) and [5, Chap. III, Theorem 2.2], the restriction of $I_{m_g^{(\infty)}}$ to the complemented subspace $L^1(W) := \{ f \in L^1(G) \mid f\chi_W = f \}$, denoted by S_W, is an $L^\infty(G)$-valued, compact operator. On the other hand, Theorem 1 (II) implies that $R(S_W) \subseteq R(I_{m_g^{(\infty)}}) \subseteq C(G)$ so that $S_W = J^{(c,\infty)} \circ T_W$, where $T_W : L^1(W) \to C(G)$ is the compact operator denoting S_W when interpreted with $C(G)$ as its codomain space. By [5, Chap. III, Theorem 2.2], there is a bounded, Bochner μ-integrable function $F : W \to C(G)$ satisfying $T_W(f) = (B)\text{-}\int_W fF\,d\mu$, for $f \in L^1(W)$. Hence, (28) yields

$$(B)\text{-}\int_W fK_g\,d\mu = S_W(f) = (J^{(c,\infty)} \circ T_W)(f) = (B)\text{-}\int_W f \cdot (J^{(c,\infty)} \circ F)\,d\mu,$$

for every $f \in L^1(W)$. Consequently, $K_g(x) = (J^{(c,\infty)} \circ F)(x)$ for μ-a.e. $x \in W$. Select such an $x \in W$, possible as $W \neq \emptyset$ because of $\mu(W) > 0$, to obtain $g = \tau_{-x}(K_g(x)) = (\tau_{-x} \circ J^{(c,\infty)} \circ F)(x) = J^{(c,\infty)}((\tau_{-x} \circ F)(x))$. This yields $g \in C(G)$ because $x \in W$ implies that $(\tau_{-x} \circ F)(x) = \tau_{-x}(F(x)) \in C(G)$ as $F(x) \in C(G)$.

Consider now the case when A_0 in (v) is a proper Borel subset of G with $\mu(A_0) > 0$. The argument for the case when $A_0 = G$ can be adapted to deduce that the restriction $K_g|_{A_0} : A_0 \to L^\infty(G)$ is Lusin μ-measurable when A_0 is equipped with the relative topology induced from G. So, there is a compact subset $W_0 \subseteq A_0$ on which K_g is continuous and we can again conclude that $g = J^{(c,\infty)}((\tau_{-x} \circ F_0)(x))$ for some $x \in W_0$ and some $C(G)$-valued, Bochner μ-integrable function F_0 on W_0. Again we have that $g \in C(G)$.

Part (II). That K_g is a $C(G)$-valued, Bochner μ-density of $m_g^{(\infty)}$ was established in the proof of part (I). Furthermore, (9) has been verified in (28). This completes the proof of Theorem 3. QED

At this stage we recall some notions concerning measurability. Let $\overline{\mathcal{B}(G)}^\mu$ denote the *completion* of $\mathcal{B}(G)$ with respect to μ; elements in $\overline{\mathcal{B}(G)}^\mu$ are called Haar measurable sets. The extension of μ to the σ-algebra $\overline{\mathcal{B}(G)}^\mu$ is denoted by $\overline{\mu}$, [16, p.155], and is still called Haar measure. A Borel set of μ-measure zero is called μ-null. Similarly, an element of $\overline{\mathcal{B}(G)}^\mu$ is called $\overline{\mu}$-null if it has $\overline{\mu}$-measure zero. A basic fact is that if $A \in \overline{\mathcal{B}(G)}^\mu$ is $\overline{\mu}$-null, then there is $C \in \mathcal{B}(G)$ such that $A \subseteq C$ and $\mu(C) = 0$, i.e., C is μ-null. A scalar function $f : G \to \mathbb{C}$ is called *μ-measurable* if it is $\overline{\mathcal{B}(G)}^\mu$-measurable, i.e., $f^{-1}(U) \in \overline{\mathcal{B}(G)}^\mu$ for each $U \in \mathcal{B}(\mathbb{C})$. Equivalently, there is a μ-null set $A \in \mathcal{B}(G)$ such that the restriction $f|_{G \setminus A} : G \setminus A \to \mathbb{C}$ is $\mathcal{B}(G \setminus A)$-measurable, where $\mathcal{B}(G \setminus A) := \{ C \cap (G \setminus A) \mid C \in \mathcal{B}(G) \}$. There is no difference between the integration of $\mathcal{B}(G)$-measurable scalar functions and μ-measurable scalar functions, [16, p.186].

Consider a Banach-space-valued function $F : G \to E$. Then F is said to be *scalarly μ-measurable* if, for each $\xi \in E^*$, the scalar function $x \mapsto \langle F(x), \xi \rangle$ on G is μ-measurable. On the hand, F is called *μ-measurable* if $F^{-1}(B) \in \overline{\mathcal{B}(G)}^\mu$ for each Borel set $B \subseteq E$, where E is considered with its norm topology. This is in agreement with the case when $E := \mathbb{C}$.

Remark 5. (I) Each of the following conditions is equivalent to (i)–(v) of Theorem 3 (I).

(vi) *There is a bounded, Bochner μ-integrable function $F : G \to L^\infty(G)$ with*

$$I_{m_g^{(\infty)}}(f) = C_g^{(1,\infty)}(f) = (B)\text{-}\int_G fF\,d\mu, \qquad f \in L^1(G).$$

(vii) *The integration map $I_{m_g^{(\infty)}} : L^1(G) \to L^\infty(G)$ is weakly compact.*

(viii) *The function $K_g : G \to L^\infty(G)$ is continuous when $L^\infty(G)$ is equipped with its weak topology $\sigma(L^\infty(G), L^\infty(G)^*)$.*

(ix) *The function $K_g : G \to L^\infty(G)$ is μ-measurable.*

(x) *The function $K_g : G \to L^\infty(G)$ is Lusin μ-measurable.*

Indeed, we have (i) $\Rightarrow$ (vi) or, to be precise, (9) implies (vi). Clearly (vi) $\Rightarrow$ (ii) and the implication (iii) $\Rightarrow$ (vii) is obvious. Recalling that weakly compact subsets of $L^\infty(G)$ are norm separable, [5, Chap. VIII, Theorem 4.13], we have (viii) $\Rightarrow$ (v). Since $R(m_g^{(\infty)}) \subseteq L^\infty(G)$ is relatively weakly compact, it is norm separable and hence, $I_{m_g^{(\infty)}}$ has separable range in $L^\infty(G)$. So, (vii) $\Rightarrow$ (vi) by [5, Chap. III, Lemma 2.9]. Clearly (iv) $\Rightarrow$ (viii) and also (iv) $\Rightarrow$ (ix). The equivalence (ix) $\Leftrightarrow$ (x) is a special case of [11, Theorem 2B]. Finally, (x) $\Rightarrow$ (i) has been shown in the proof of (v) $\Rightarrow$ (i).

(II) As already noted, (i) $\Leftrightarrow$ (iv) is a special case of [8, Theorem]. Moreover, (iv) $\Rightarrow$ (i) and (x) $\Rightarrow$ (i) occur in Corollary 3 and the discussion prior to it in [9]. The equivalence (i) $\Leftrightarrow$ (ix) is in [29, Theorem 21]; its proof there also verifies (x) $\Rightarrow$ (iv) $\Rightarrow$ (i). Our arguments in the proof of Theorem 3 (I) are based on the methods of this paper and differ from those in [9], [29]. QED

The standing assumption in Theorem 3 is that $g \in L^\infty(G) \setminus \{0\}$. Beginning with the weaker requirement that $g \in L^1(G) \setminus \{0\}$, we can still obtain (i) $\Leftrightarrow$ (iii). A precise statement is:

Corollary 3. *A function $g \in L^1(G)$ belongs to $C(G)$ iff the integration map $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$ is compact.*

Proof. If $g \in C(G)$, then $I_{m_g^{(\infty)}}$ is compact by Theorem 3 (I). Conversely, assume that $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$ is compact. Then $m_g^{(\infty)}$ has finite variation, [23, Proposition 3.48], and hence, $g \in L^\infty(G)$ via Theorem 2. Then Theorem 3 (I) gives $g \in C(G)$. QED

In connection with Theorem 3 we note that for $g \in L^1(G)$, and not just for $g \in L^\infty(G)$, it is still the case that (i) $\Leftrightarrow$ (ii) holds, i.e., $g \in C(G)$ iff $m_g^{(\infty)}$ admits a Bochner μ-density. Indeed, the existence of a Bochner μ-density already implies that $g \in L^\infty(G)$ via Lemma 5.

In Theorem 1, Remark 2 and Remark 3(ii) we have determined, for $\lambda \in M(G)$, exactly when $C_\lambda^{(\infty)}$ is completely continuous, compact, weakly compact or p-summing for $1 \le p < \infty$. A further contribution in this direction is as follows. Part of the proof follows ideas from the proof of Theorem 4.7 in [25].

Proposition 3. *Let $\lambda \in M(G)$. Then the convolution operator $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$ is nuclear iff $\lambda = g\,d\mu$ for some $g \in C(G)$.*

Proof. Of course, we may (and do) assume that $\lambda \neq 0$.

Suppose that $C_\lambda^{(\infty)}$ is nuclear. Then $C_\lambda^{(\infty)}$ is also 1-summing, [31, III. F Proposition 22], and so there is $g \in L^\infty(G)$ satisfying $\lambda = g\,d\mu$; see Remark 3(ii) with $p := 1$. Furthermore, $|m_g^{(\infty)}|$ is finite (cf. Theorem 2) and there is $c > 0$ such that $|m_g^{(\infty)}| = c\mu$ (cf. Remark 1(ii)). Of course, $L^1(m_g^{(\infty)}) = L^1(G)$; see Theorem 2.

Given $\xi \in L^\infty(G)^*$, the scalar measure $\langle m_g^{(\infty)}, \xi\rangle$ is a regular Borel measure (as $\langle m_g^{(\infty)}, \xi\rangle \ll \mu$) and, for $f \in C(G)$, we have via (8) that

$$\int_G f\, d\langle m_g^{(\infty)}, \xi\rangle = \langle I_{m_g^{(\infty)}}(f), \xi\rangle = \langle \big(C_g^{(\infty)} \circ J^{(c,\infty)}\big)(f), \xi\rangle.$$

Thus, $m_g^{(\infty)}$ is the *representing measure* of the nuclear operator $C_g^{(\infty)} \circ J^{(c,\infty)} = C_\lambda^{(\infty)} \circ J^{(c,\infty)} \in \mathcal{L}(C(G), L^\infty(G))$ in the sense of [5, Chap. VI, Definition 2.2]. Accordingly, the vector measure $m_g^{(\infty)}$ admits a Bochner density relative to $|m_g^{(\infty)}|$, say $F : G \to L^\infty(G)$, such that

$$m_g^{(\infty)}(A) = (B)\text{-}\int_A F\, d|m_g^{(\infty)}| = (B)\text{-}\int_A cF\, d\mu, \qquad A \in \mathcal{B}(G),$$

[5, Chap. VI, Theorem 4.4]. So, cF is a Bochner μ-density of $m_g^{(\infty)}$ and hence, $g \in C(G)$ via Theorem 2.

Conversely, let $\lambda = g\, d\mu$ for some $g \in C(G) \setminus \{0\}$. By Theorem 3, the function $K_g : G \to L^\infty(G)$ is a Bochner μ-density of $m_g^{(\infty)}$. Select a sequence $\{\varphi_n\}_{n=1}^\infty \subseteq L^\infty(G)$ and a sequence $\{A(n)\}_{n=1}^\infty \subseteq \mathcal{B}(G)$, not necessarily pairwise disjoint, such that

(a) for μ-a.e. $x \in G$ we have $K_g(x) = \sum_{n=1}^\infty \varphi_n \chi_{A(n)}(x)$ with the series absolutely convergent in $L^\infty(G)$, i.e., $\sum_{n=1}^\infty \|\varphi_n\|_{L^\infty(G)} \chi_{A(n)}(x) < \infty$, and

(b) $\int_G \|K_g(x)\|_{L^\infty(G)}\, d\mu(x) \le \sum_{n=1}^\infty \|\varphi_n\|_{L^\infty(G)}\, \mu(A(n)) < \infty$, [5, Chap. VI, Lemma 4.3].

Given $f \in L^\infty(G) \subseteq L^1(G)$, we have from (a), (b) and (9) that

$$\begin{aligned} C_g^{(\infty)}(f) &= I_{m_g^{(\infty)}}(f) = (B)\text{-}\int_G f K_g\, d\mu \\ &= (B)\text{-}\int_G f(x)\Big(\sum_{n=1}^\infty \varphi_n \chi_{A(n)}(x)\Big)\, d\mu(x) = \sum_{n=1}^\infty \Big(\int_G f\chi_{A(n)}\, d\mu\Big)\varphi_n \\ &= \sum_{n=1}^\infty \langle \chi_{A(n)}, f\rangle \varphi_n. \end{aligned}$$

As $\sum_{n=1}^\infty \left\|\chi_{A(n)}\right\|_{L^1(G)} \|\varphi_n\|_{L^\infty(G)} < \infty$ (cf. (b)), the operator $C_g^{(\infty)}$ is nuclear. QED

A function $\psi : G \to \mathbb{C}$ is called *Riemann-measurable* if ψ is continuous at each point in a set of full μ-measure, [29, p.39]. A typical non-trivial example of a Riemann-measurable function is the characteristic function of any non-μ-null Borel subset of G whose boundary is μ-null. Let $R^\infty(G)$ denote the subspace of $L^\infty(G)$ consisting of all bounded, Riemann-measurable functions on G. To be precise, to say that $g \in L^\infty(G)$ belongs to $R^\infty(G)$ means that g is μ-a.e. equal to some bounded, Riemann-measurable function on G. Then, $R^\infty(G)$ is a closed subspace of $L^\infty(G)$ and

$$C(G) \subseteq R^\infty(G) \subseteq L^\infty(G). \tag{29}$$

The following result is a special case of [29, Theorem 16].

Lemma 7. *Suppose that Martin's Axiom holds. A function $g \in L^\infty(G)$ belongs to $R^\infty(G)$ iff $K_g : G \to L^\infty(G)$ is scalarly μ-measurable.*

A scalarly μ-measurable function $F : G \to L^\infty(G)$ is called *Pettis μ-integrable* if $\langle F, \xi\rangle \in L^1(G)$ for each $\xi \in L^\infty(G)^*$ and if, for every $A \in \mathcal{B}(G)$, there is a vector $(P)\text{-}\int_A F\, d\mu \in L^\infty(G)$ satisfying $\langle (P)\text{-}\int_A F\, d\mu, \xi\rangle = \int_A \langle F, \xi\rangle\, d\mu$, for $\xi \in L^\infty(G)^*$.

Proof of Theorem 4. Assume that $m_g^{(\infty)}$ admits a Pettis μ-density. By Lemma 5, also K_g is a Pettis μ-density of $m_g^{(\infty)}$. In particular, K_g is scalarly μ-measurable and hence, $g \in R^\infty(G)$; see Lemma 7.

Conversely, suppose that $g = \psi$ μ-a.e. for some $\psi \in R^\infty(G)$. Then K_g is scalarly μ-measurable, again via Lemma 7. Since $K_g(x) = K_\psi(x)$ as elements of $L^\infty(G)$, for each $x \in G$, we may as well assume that g itself is a bounded, Riemann measurable function. Suppose first that g is $\mathbb{R}$-valued, in which case also $m_g^{(\infty)}(A) = g * \chi_A$ is $\mathbb{R}$-valued, for $A \in \mathcal{B}(G)$. Fix $A \in \mathcal{B}(G)$. Let ξ be any *positive* linear functional belonging to the dual Banach lattice $L^\infty(G)^*$. Then

$$\langle m_g^{(\infty)}(A), \xi\rangle \le \int_A \langle K_g(x), \xi\rangle \, d\mu(x). \tag{30}$$

To see this, fix $\varepsilon > 0$ and select $\mathbb{R}$-valued continuous functions g_1 and g_2 on G such that $g_1 \le g \le g_2$ (pointwise on G) and $\int_G (g_2 - g_1)\, d\mu < \varepsilon$, [29, Lemma 2]. Since $g_1 \in C(G)$, we have from Theorem 3 (I) applied to g_1 that K_{g_1} is a Bochner μ-density of $m_{g_1}^{(\infty)}$ and hence, is also a Pettis μ-density of $m_{g_1}^{(\infty)}$; see Lemma 2(i). So,

$$\langle m_{g_1}^{(\infty)}(A), \xi\rangle = \langle (P)\text{-}\int_A K_{g_1}\, d\mu, \xi\rangle = \int_A \langle K_{g_1}(x), \xi\rangle\, d\mu(x) \le \int_A \langle K_g(x), \xi\rangle \, d\mu(x),$$

where we need that $g_1 \le g$ implies $K_{g_1}(x) \le K_g(x)$, for $x \in G$. Moreover,

$$\begin{aligned}\left\|(g * \chi_A) - (g_1 * \chi_A)\right\|_{L^\infty(G)} &\le \|g - g_1\|_{L^1(G)} \left\|\chi_A\right\|_{L^\infty(G)} \\ &\le \int_G (g - g_1)\, d\mu \le \int_G (g_2 - g_1)\, d\mu < \varepsilon.\end{aligned}$$

Accordingly, we have that

$$\begin{aligned}\langle m_g^{(\infty)}(A), \xi\rangle &= \langle (g * \chi_A) - (g_1 * \chi_A), \xi\rangle + \langle g_1 * \chi_A, \xi\rangle \\ &\le \left\|(g * \chi_A) - (g_1 * \chi_A)\right\|_{L^\infty(G)} \|\xi\|_{L^\infty(G)^*} + \langle m_{g_1}^{(\infty)}(A), \xi\rangle \\ &\le \varepsilon \|\xi\|_{L^\infty(G)^*} + \int_A \langle K_g(x), \xi\rangle\, d\mu(x),\end{aligned}$$

from which (30) follows because $\varepsilon > 0$ is arbitrary. Replacing g_1 with g_2, a similar argument yields $\langle m_g^{(\infty)}(A), \xi\rangle \ge \int_A \langle K_g(x), \xi\rangle\, d\mu(x)$. This inequality and (30) imply that

$$\langle m_g^{(\infty)}(A), \xi\rangle = \int_A \langle K_g(x), \xi\rangle\, d\mu(x). \tag{31}$$

The same identity is valid for an arbitrary $\xi \in L^\infty(G)^*$ because $L^\infty(G)$ is the complexification of the real Banach lattice $L^\infty_{\mathbb{R}}(G)$, where $L^\infty_{\mathbb{R}}(G)$ consists of all $\mathbb{R}$-valued functions in $L^\infty(G)$, and because each continuous linear functional on $L^\infty_{\mathbb{R}}(G)$ is the difference of two positive continuous linear functionals.

Suppose now that g is $\mathbb{C}$-valued. Then its real part $\mathrm{Re}(g)$ and its imaginary part $\mathrm{Im}(g)$ are also bounded and Riemann-measurable. Moreover, $m_g^{(\infty)}(A) = m_{\mathrm{Re}(g)}^{(\infty)}(A) + \mathrm{i} m_{\mathrm{Im}(g)}^{(\infty)}(A)$ and $K_g(x) = K_{\mathrm{Re}(g)}(x) + \mathrm{i}K_{\mathrm{Im}(g)}(x)$, for $x \in G$. Since (31) holds with $\mathrm{Re}(g)$ and $\mathrm{Im}(g)$ in place of g, whenever $\xi \in L^\infty(G)^*$, it follows that (31) is also valid for g. Thus, K_g is a Pettis μ-density of $m_g^{(\infty)}$.

It remains to establish (10), under the assumption that $g \in R^\infty(G)$. Observe that $\|K_g(x)\|_{L^\infty(G)} = \|g\|_{L^\infty(G)}$ for all $x \in G$. Therefore, given $\xi \in L^\infty(G)^*$, the scalar function

$\langle K_g, \xi\rangle$ on G is bounded and satisfies $\sup_{x\in G} |\langle K_g(x), \xi\rangle| \le \|g\|_{L^\infty(G)} \|\xi\|_{L^\infty(G)^*}$. So, for each $f \in L^1(G)$, we have

$$\int_G |\langle fK_g, \xi\rangle|\, d\mu = \int_G |f(x)| \cdot |\langle K_g(x), \xi\rangle|\, d\mu(x) \le \|f\|_{L^1(G)} \|g\|_{L^\infty(G)} \|\xi\|_{L^\infty(G)^*}$$

and hence, the linear functional $\xi_g : f \mapsto \int_G \langle fK_g, \xi\rangle\, d\mu$ on $L^1(G)$ is continuous. Since K_g is a Pettis μ-density of $m_g^{(\infty)}$ it follows, given $s \in \mathrm{sim}\mathcal{B}(G)$, that the function sK_g is also Pettis μ-integrable and satisfies

$$\langle I_{m_g^{(\infty)}}(s), \xi\rangle = \langle (P)\text{-}\int_G sK_g\, d\mu, \xi\rangle = \int_G \langle sK_g, \xi\rangle\, d\mu = \langle s, \xi_g\rangle.$$

So, the continuous linear functionals $\langle I_{m_g^{(\infty)}}(\cdot), \xi\rangle$ and ξ_g, both defined on $L^1(G)$, coincide on the dense subspace $\mathrm{sim}\mathcal{B}(G)$ and hence, must be equal. In other words, $\langle I_{m_g^{(\infty)}}(f), \xi\rangle = \xi_g(f) = \int_G \langle fK_g, \xi\rangle\, d\mu$ for $f \in L^1(G)$. Thus, fK_g is Pettis μ-integrable and (10) holds because $\xi \in L^\infty(G)^*$ is arbitrarily fixed. The proof of Theorem 4 is thereby complete. *QED*

Remark 6. (i) The inclusions in (29) can be strict. Indeed, consider the circle group $\mathbb{T}$, which we identify with the interval $[-\pi, \pi)$ in the usual way. Then $\chi_{[-\pi,0]} \in R^\infty(\mathbb{T})$ but, there is no function in $C(\mathbb{T})$ which coincides with $\chi_{[-\pi,0]}$ μ-a.e. Accordingly, $C(\mathbb{T}) \subsetneqq R^\infty(\mathbb{T})$.

According to [7, Theorem], the closed subspace $R^\infty(\mathbb{T})$ of $L^\infty(\mathbb{T})$ contains a complemented copy of c_0 and so is not a Grothendieck space whereas $L^\infty(\mathbb{T})$ is known to be a Grothendieck space, [2, Theorem 13.13]. So, $R^\infty(\mathbb{T}) \subsetneqq L^\infty(\mathbb{T})$.

(ii) The inclusion $C(G) \subsetneqq L^\infty(G)$ is proper for every infinite compact abelian group G, [15, Lemma (37.3)]. *QED*

For $g \in L^\infty(G)$, Theorem 3 and Remark 5 show that the integration map $I_{m_g^{(\infty)}}$ is compact iff it is weakly compact iff $g \in C(G)$; see also Corollary 3. We conclude with a result characterizing further operator ideal properties of $I_{m_g^{(\infty)}}$.

Proposition 4. *For $g \in L^1(G)$, the following assertions are equivalent.*

(i) *$\widehat{g} \in \ell^1(\Gamma)$, that is, $\sum_{\gamma\in\Gamma} |\widehat{g}(\gamma)| < \infty$.*

(ii) *The integration map $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$ is nuclear.*

(iii) *The integration map $I_{m_g^{(\infty)}} : L^1(m_g^{(\infty)}) \to L^\infty(G)$ is 1-summing.*

(iv) *The Fourier series $\sum_{\gamma\in\Gamma} \widehat{g}(\gamma)(\cdot, \gamma)$ of g is unconditionally convergent in $L^\infty(G)$.*

Proof. (i) $\Rightarrow$ (ii). Since g coincides with its absolutely convergent Fourier series (in $C(G)$), we see that $g \in C(G) \subseteq L^\infty(G)$. In particular, $L^1(m_g^{(\infty)}) = L^1(G)$ via Theorem 2. Then we can write

$$I_{m_g^{(\infty)}}(f) = f * g = \sum_{\gamma\in\Gamma} \widehat{g}(\gamma) \langle f, (\cdot, \gamma)\rangle (\cdot, \gamma), \qquad f \in L^1(G),$$

with the series absolutely convergent in $C(G) \subseteq L^\infty(G)$. As $\widehat{g} \in \ell^1(\Gamma)$, we have $\mathrm{supp}(\widehat{g})$ is countable. So, (ii) is valid, [6, Proposition 5.23], [31, pp.216–217].

(ii) $\Rightarrow$ (iii). See [31, III.F Proposition 22].

(iii) $\Rightarrow$ (i). According to Example 2.61 and Proposition 3.74 in [23] the integration map $I_{m_g^{(\infty)}}$ is necessarily 1-concave and hence, $L^1(m_g^{(\infty)}) = L^1(|m_g^{(\infty)}|)$. By Remark 1(ii) we then have $L^1(m_g^{(\infty)}) = L^1(G)$ and so $g \in L^\infty(G)$ with $I_{m_g^{(\infty)}} = C_g^{(1,\infty)}$; see Theorem 2. Since the convolution operator $C_g^{(1,\infty)} = I_{m_g^{(\infty)}}$ is 2-summing (as it is 1-summing, [6, Inclusion Theorem 2.8]), it factors through a Hilbert space, [6, Corollary 2.16]. So, we can obtain (i) by adapting the proof of Lemma 4.12 in [24].

(i) $\Rightarrow$ (iv). Clear as $\|(\cdot,\gamma)\|_{L^\infty(G)} = 1$, for each $\gamma \in \Gamma$.

(iv) $\Rightarrow$ (i). The inclusion map $J^{(\infty,1)} : L^\infty(G) \to L^1(G)$ is 1-summing, [6, Example 2.9(d)], equivalently, it is *absolutely summing*, [6, pp.34–35], and hence, maps unconditionally convergent series to absolutely convergent series, [6, p.15]. So, recalling that $\mathrm{supp}(\widehat{g})$ is countable, we have from (iv) that

$$\sum_{\gamma\in\Gamma} |\widehat{g}(\gamma)| = \sum_{\gamma\in\Gamma} \left\| J^{(\infty,1)}\big(\widehat{g}(\gamma)(\cdot,\gamma)\big)\right\|_{L^1(G)} < \infty,$$

i.e., (i) holds. QED

We note that (i) $\Leftrightarrow$ (iv) in Proposition 4 is stated in [3, Theorem 1].

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